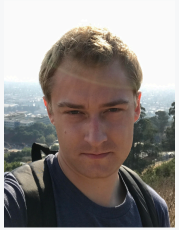
The Tractability Border of Reachability in Simple Vector Addition Systems with States

Henry Sinclair-Banks

Based on work with Dmitry Chistikov, Wojciech Czerwiński, Filip Mazowiecki, Łukasz Orlikowski, and Karol Węgrzycki to appear in FOCS'24.





Highlights'24: Vector Addition Systems

19th September 2024

LaBRI, Bordeaux, France

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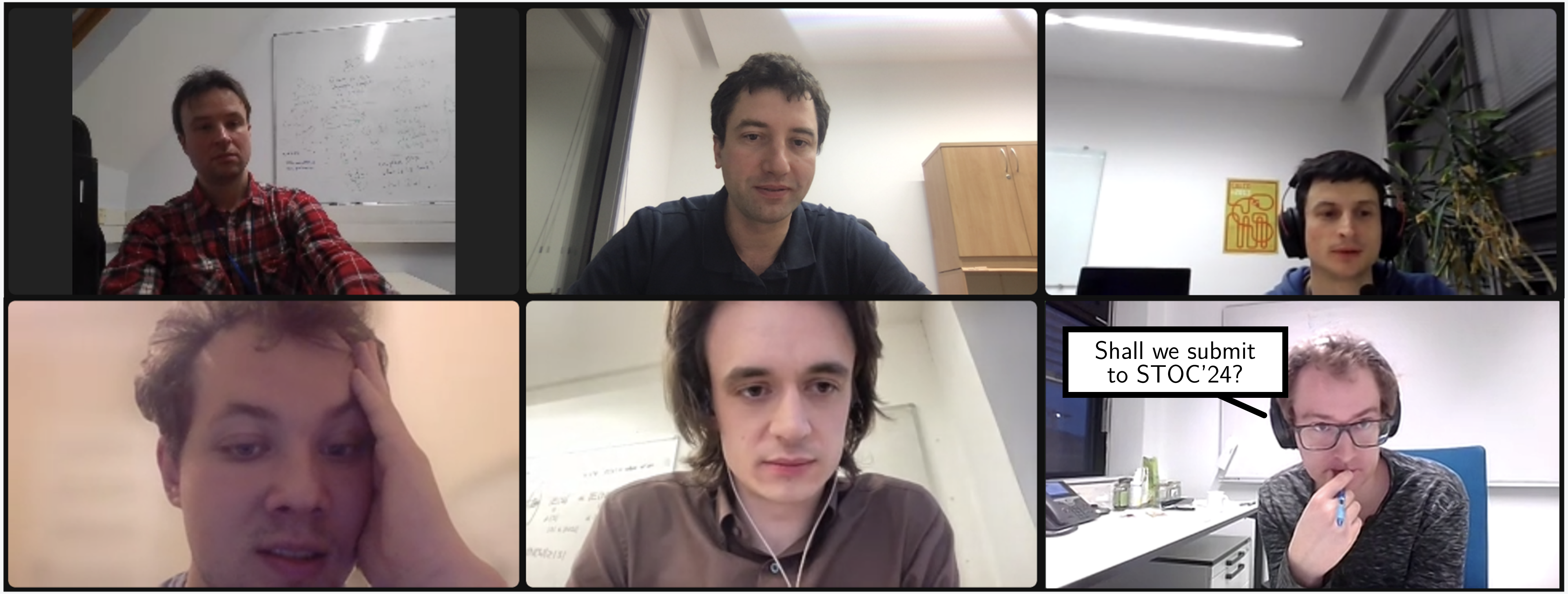

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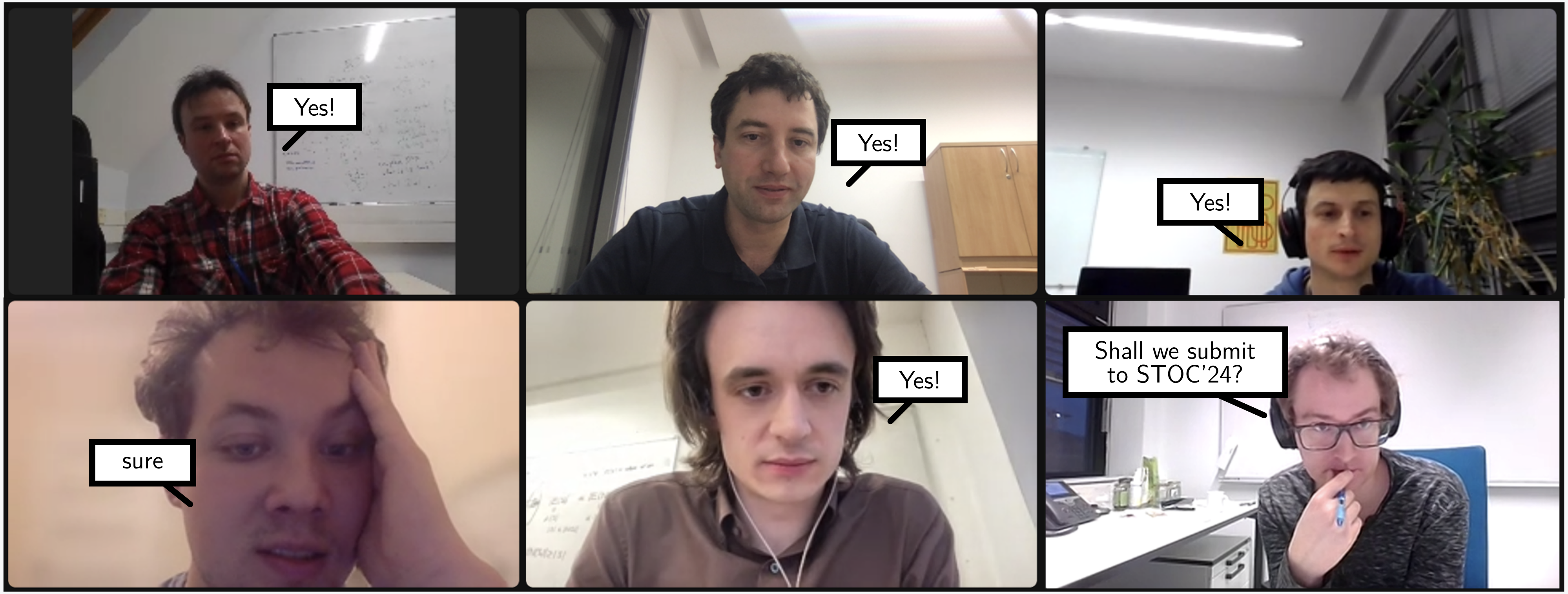

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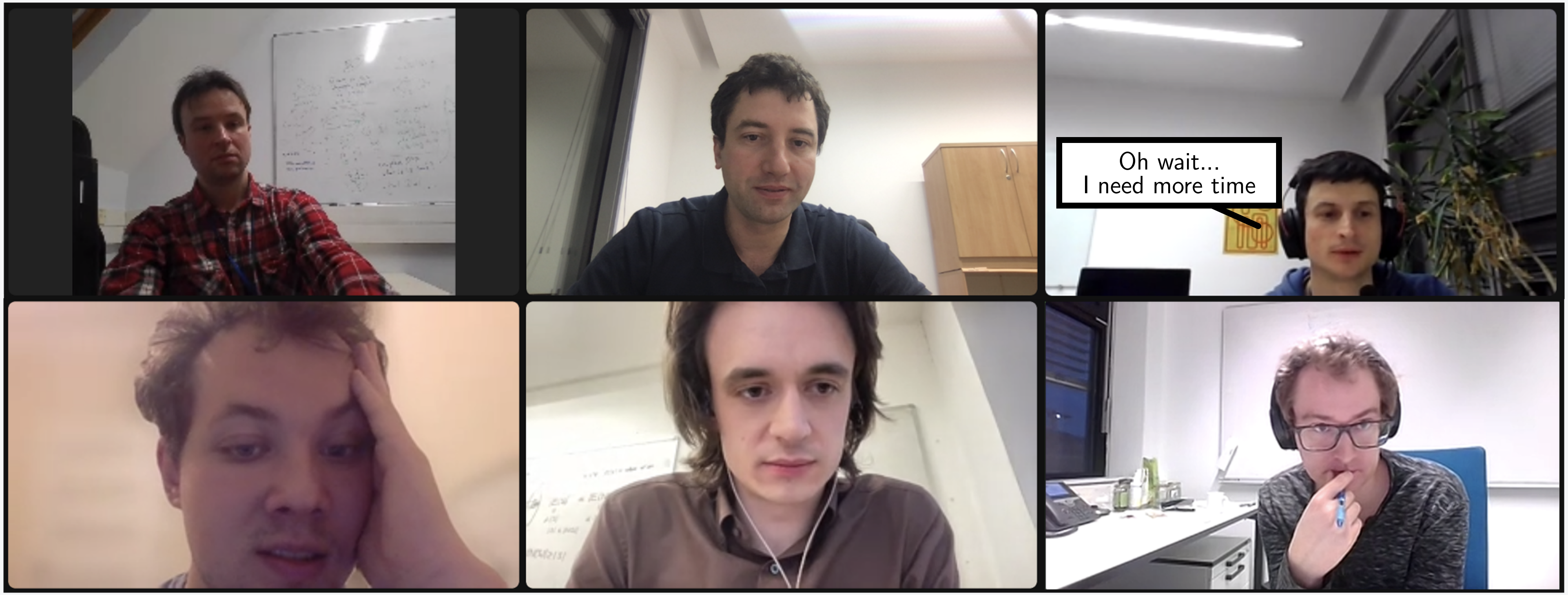

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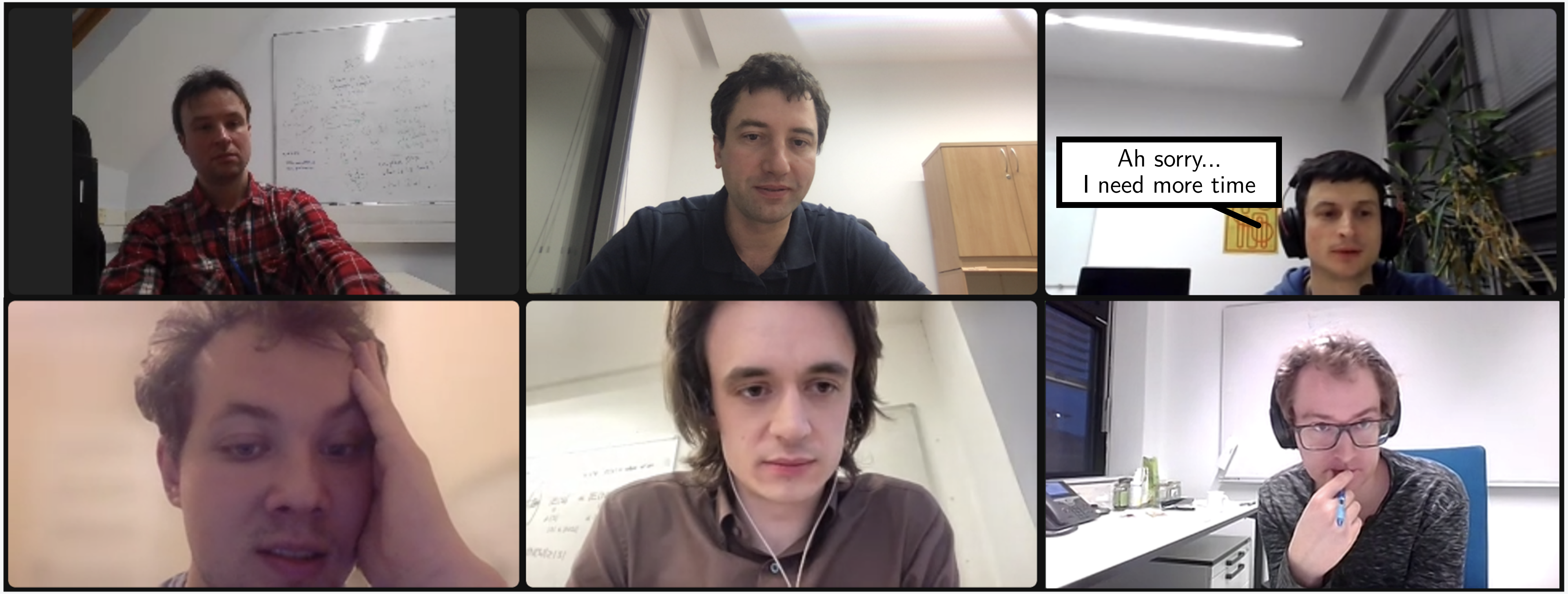

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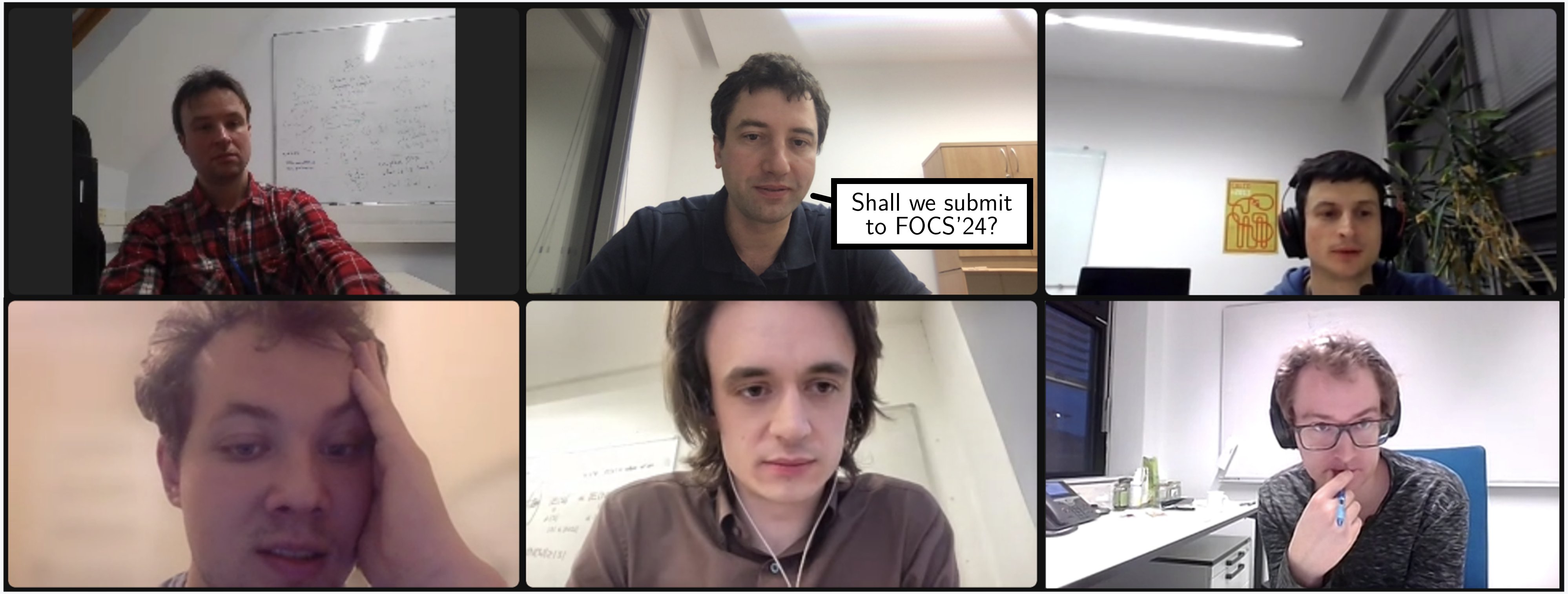

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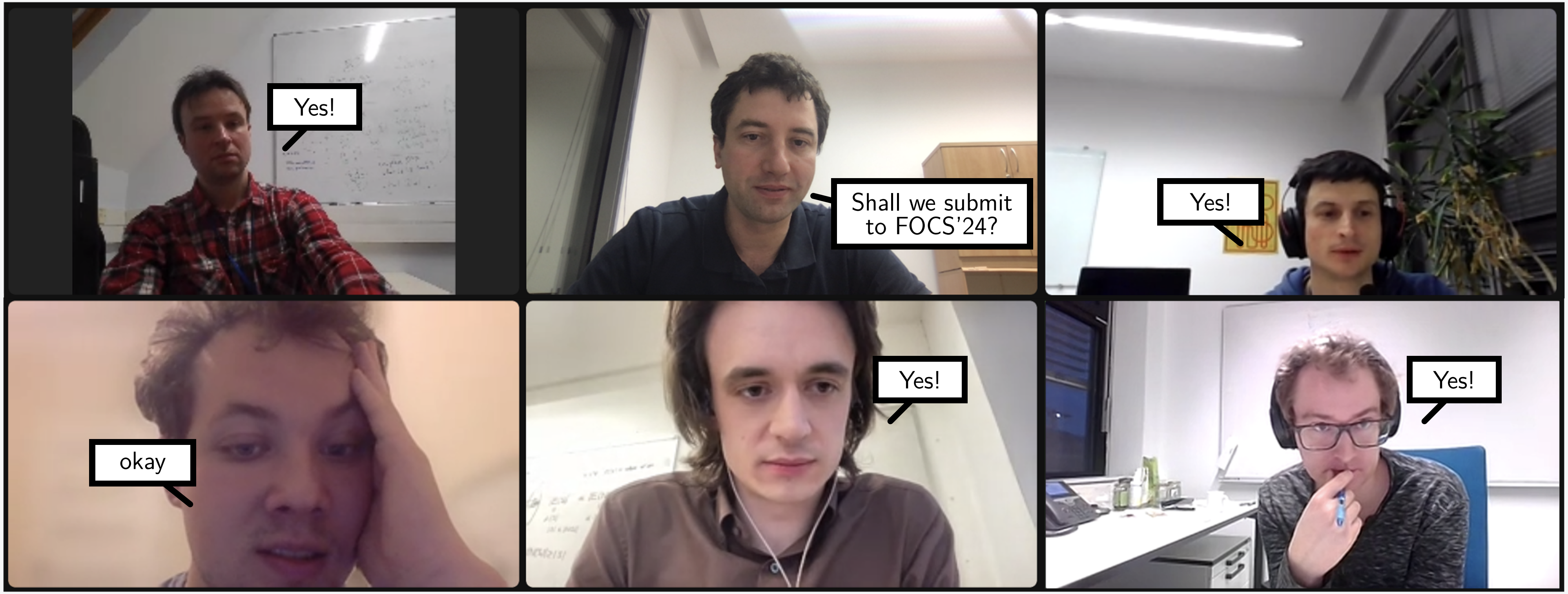

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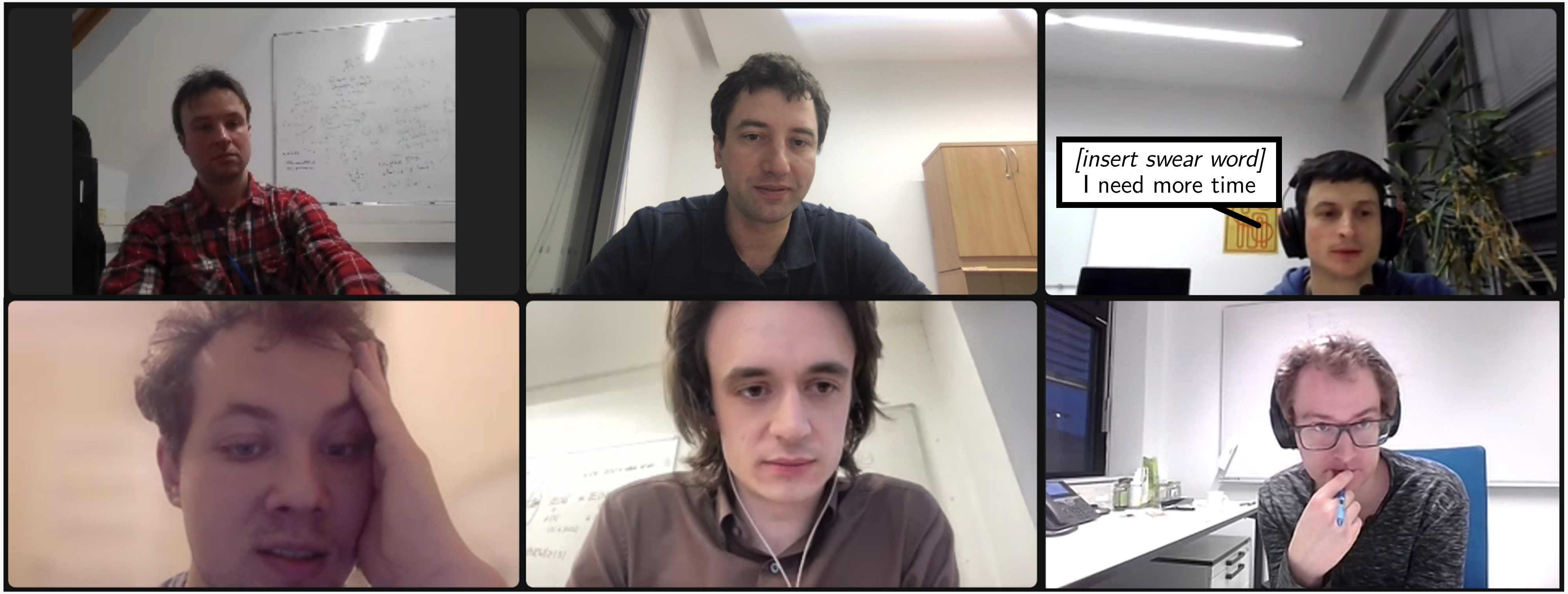

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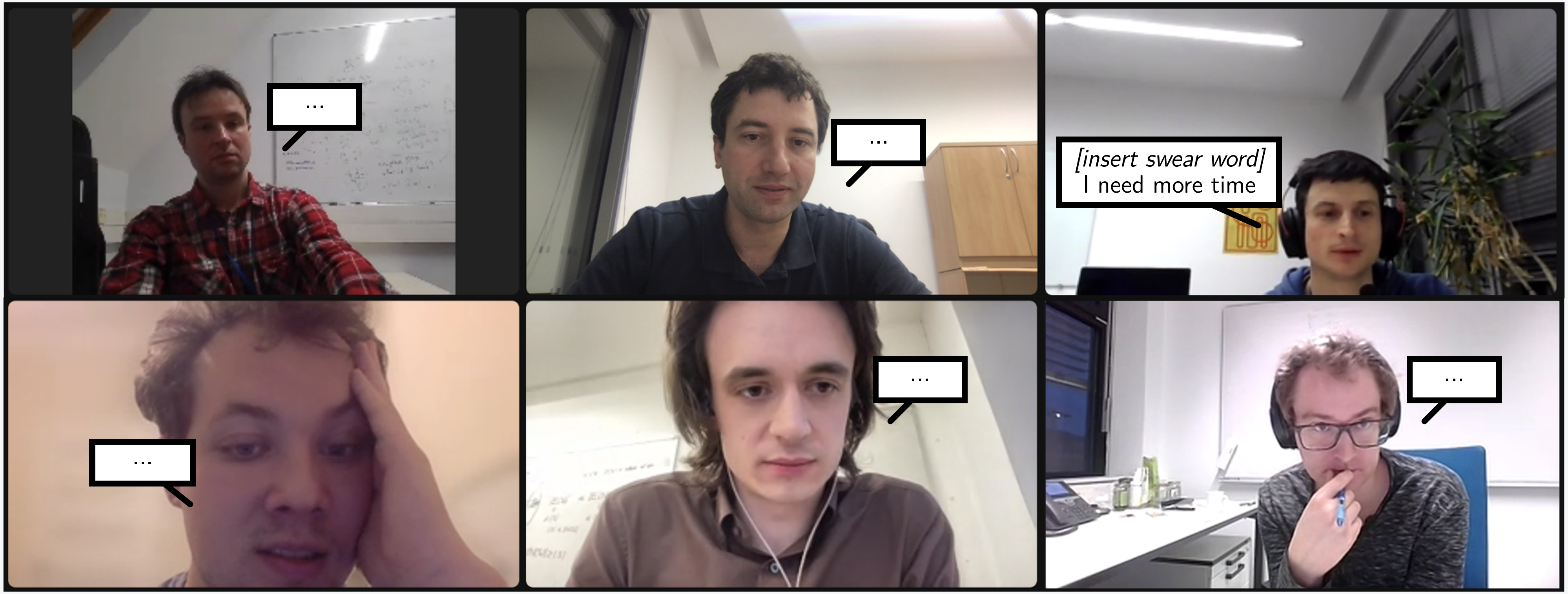

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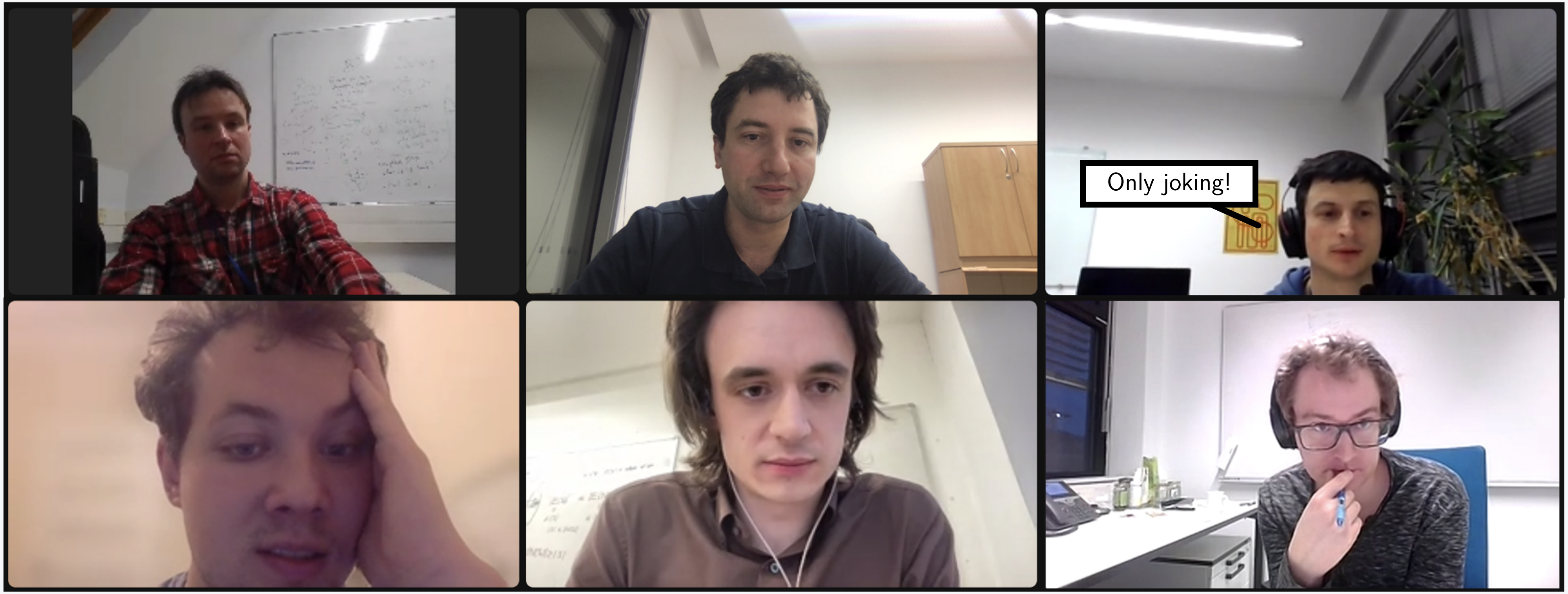

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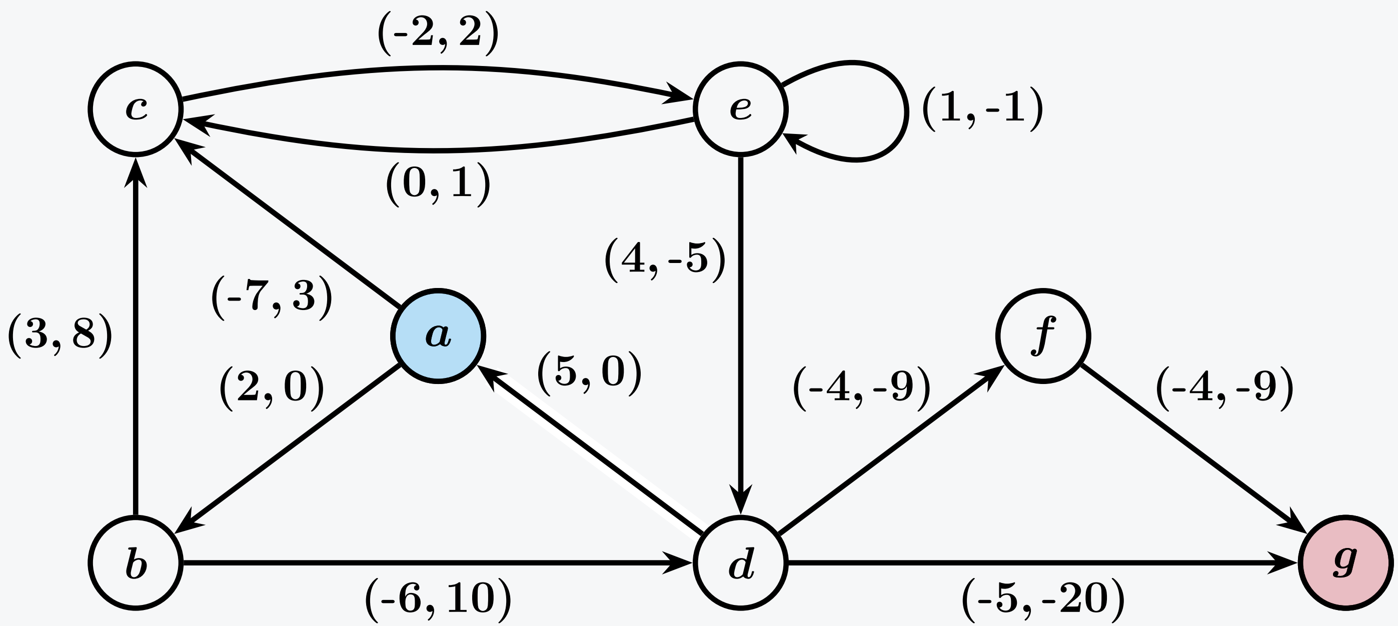
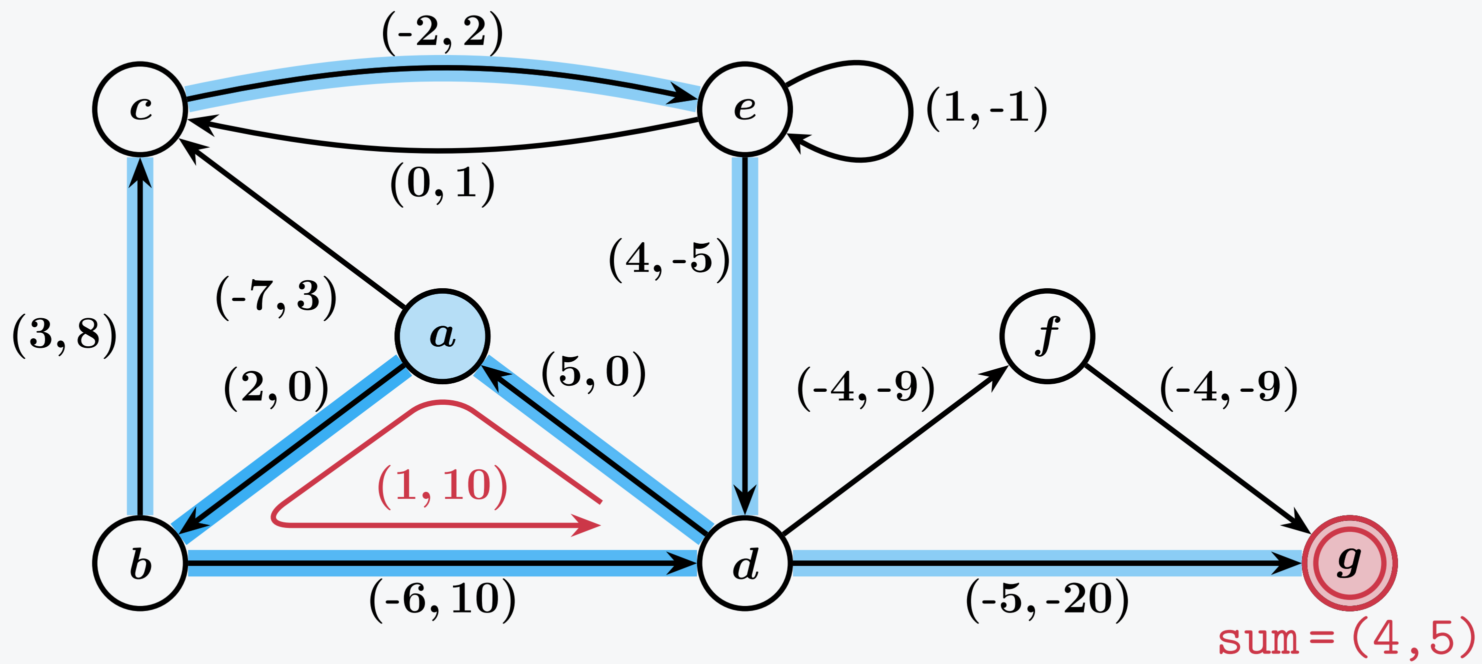
Reachability in 2-Dimensional VASS



Does there exist a run from a with counter values (0,0) to g with counter values (4,5) ?

(the counters must remain nonnegative at all times)

Reachability in 2-Dimensional VASS

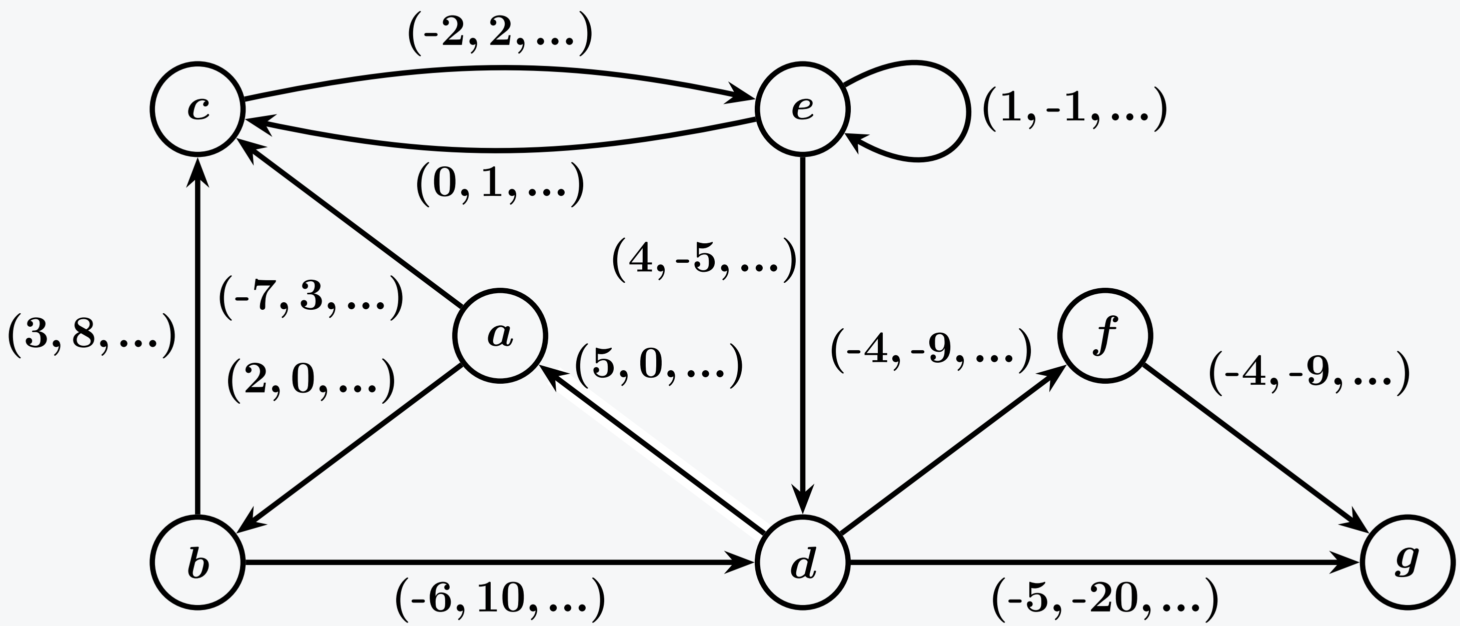


Does there exist a run from a with counter values `(0,0)` to g with counter values `(4,5)` ?

(the counters must remain nonnegative at all times)

YES!

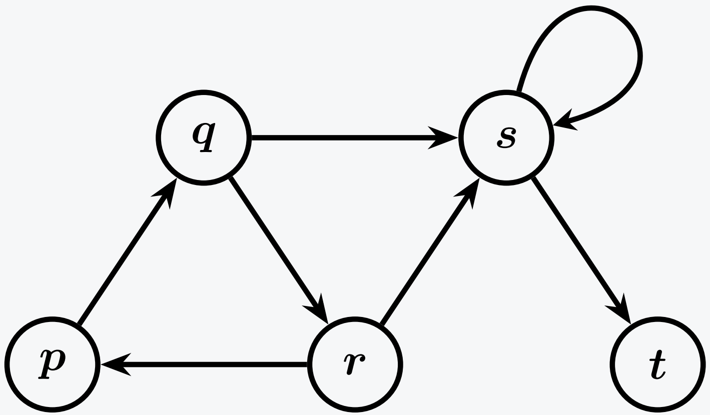
Reachability in VASS



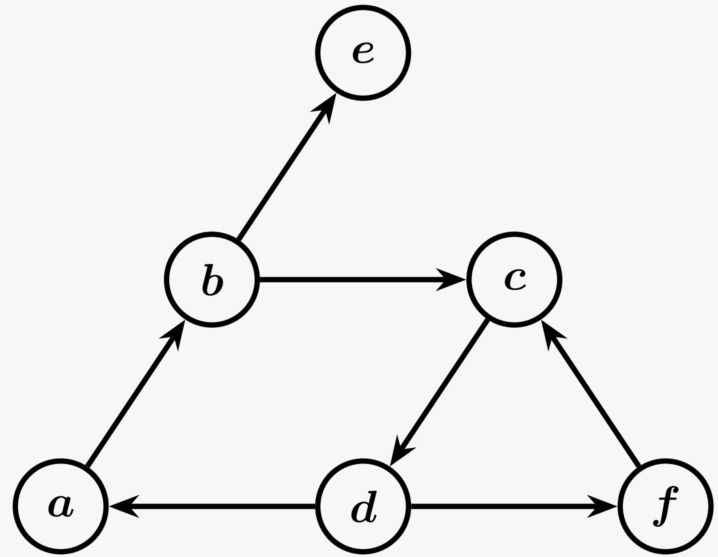
Reachability problem: does there exist a run from $p(\mathbf{u})$ to $q(\mathbf{v})$?

What is a Flat VASS?

Definition (Flat)**.** For every state $q \in Q$, there is at most one simple cycle that contains q.



Flat :)



Not flat :(

Reachability in Flat VASS

Definition (Flat)**.** For every state $q \in Q$, there is at most one simple cycle that contains q.

Theorem. Reachability in flat VASS is in NP (even with binary encoding). [Fribourg and Olsén '97]
[Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

Theorem. Reachability in binary flat 1-VASS is NP-hard. [Rosier and Yen '85]

Theorem. Reachability in unary (flat) 1-VASS and 2-VASS is in NL. [Valiant and Paterson '73]
[Englert, Lazić, and Totzke '16]

Theorem. Reachability in unary flat d-VASS is NP-hard for $d = 7$.
[Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

... for $d = 5$. [Dubiak '20]

... for $d = 4$. [Czerwiński and Orlikowski '22]

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What is the complexity of reachability in unary flat $\mathbf{3}$-VASS?

[Blondin, Finkel, Göller, Haase, and McKenzie '15]

[Englert, Lazić, and Totzke '16]

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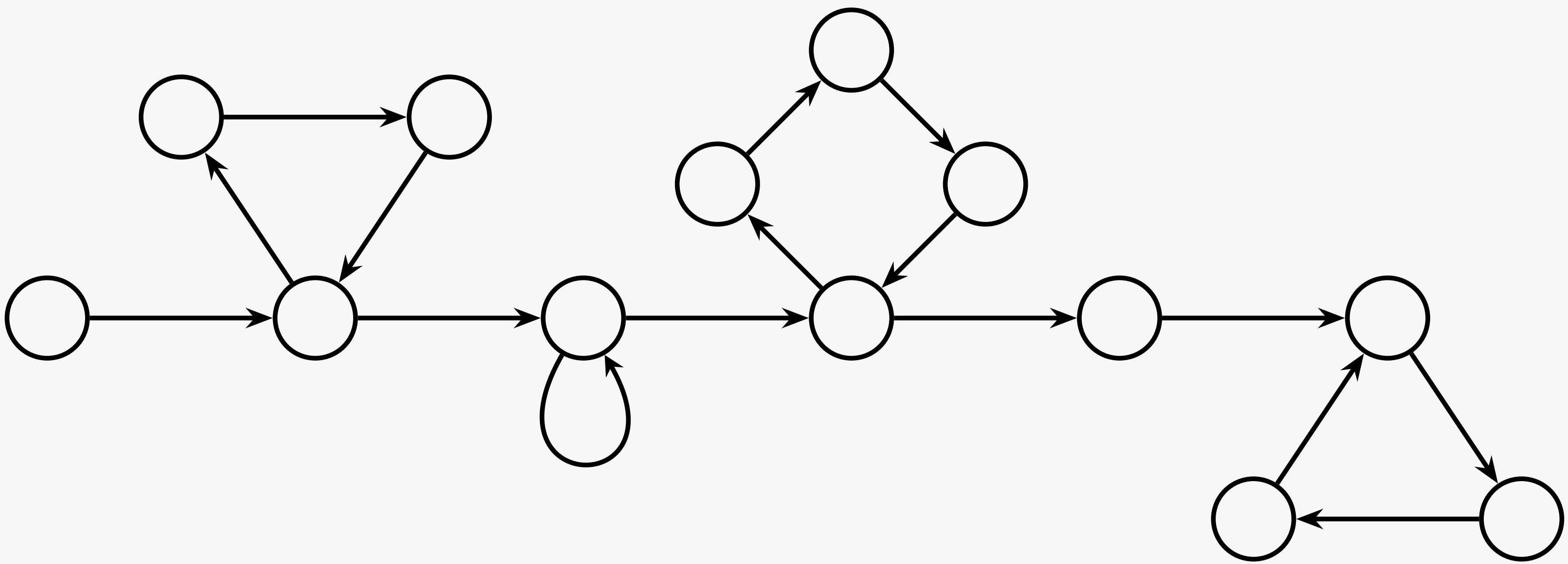
~~Flat VASS~~ Linear Path Schemes

Definition (LPS)**.** A VASS where the states and transitions form a simple path between disjoint cycles.



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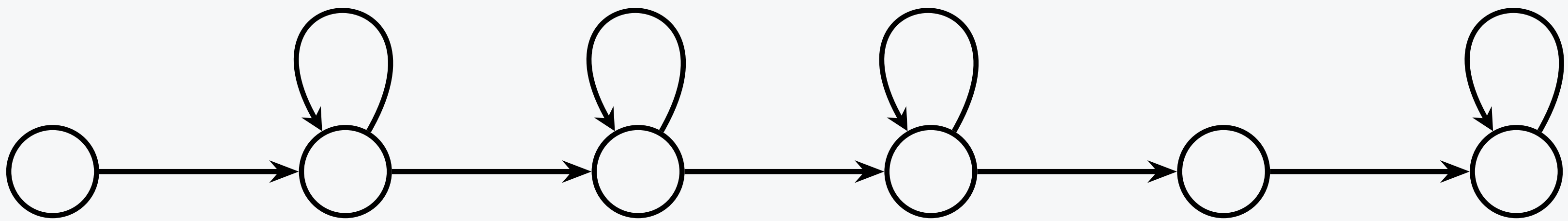
Definition (LPS)**.** A VASS where the states and transitions form a simple path between disjoint cycles.

Definition (SLPS)**.** A *Simple* LPS has cycles of length one ("self-loops").



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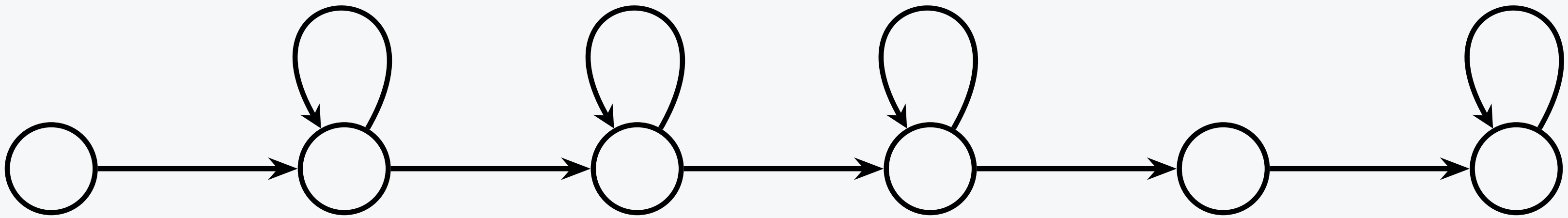

Open problem. For $d \geq 3$, is reachability in unary d-dimensional linear path schemes in P?

[Englert, Lazić, and Totzke '16] [Leroux '21]

Theorem. Reachability in unary 3-dimensional *simple* linear path schemes is NP-hard. [This paper]

Reachability in Unary 3-SLPS is NP-hard

Proof idea:

1) Use "Chinese remainder encoding" of a SAT instance with k variables.

Let $n \in \mathbb{N}$ represent an assignment and let $p_1, \ldots, p_k$ be the first k primes.

Encoding: $x_i =$ true $\iff n \equiv 1 \bmod p_i$ and $x_i =$ false $\iff n \equiv 0 \bmod p_i$.

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2) Turn the SAT instance into a conjunction of non-divisibility checks.

Example: $x_1 \vee \neg x_2 \vee x_3$; need to check $n \equiv 1 \bmod 2$ or $n \equiv 0 \bmod 3$ or $n \equiv 1 \bmod 5$.

Only falsified when $n \equiv 10 \bmod 2{\cdot}3{\cdot}5$, so check $2{\cdot}3{\cdot}5 \nmid n - 10$.

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3) Devise a unary 2-SLPS with zero tests that can check non-divisibility.

Come and ask me for the construction – I promise its very neat!

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4) Simulate the zero tests with a dedicated third counter. We use the "controlling counter technique".

[Czerwiński and Orlikowski '21]

The Tractability Border of Reachability in Simple Vector Addition Systems with States

Theorem. Reachability in unary 3-SLPS is NP-complete.

Theorem. Reachability in unary *ultraflat* 4-VASS is NP-complete.

Theorem. Reachability in *unitary* inverse-Ackermann-dimensional SLPS is NP-complete.

Theorem. Reachability in unary 2-SLPS with *binary encoded initial and target configurations* is in P.

Thank You!

LaBRI

Presented by Henry Sinclair-Banks

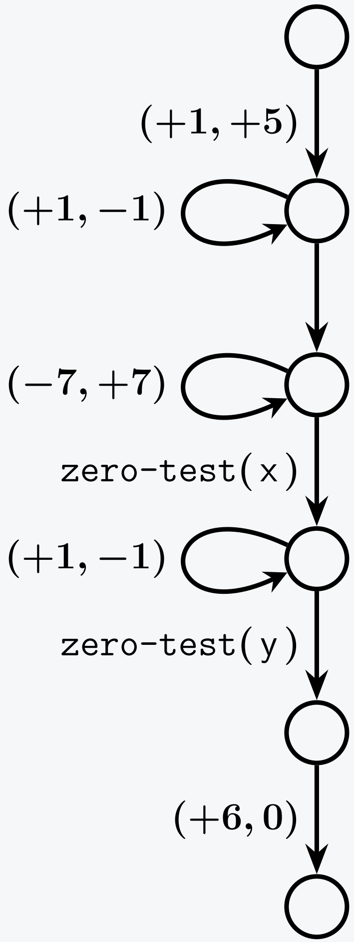
Highlights'24 in LaBRI, Bordeaux, France

Non-Divisibility Testing Simple Linear Path Schemes

Suppose we want to perform a non-divisibility test $v \nmid 7$.

Let's construct a 2-SLPS with zero tests that:

- starts with x $= v$, y $= 0$,
- can only be passed if $v \nmid 7$, and
- ends with x $= v$, y $= 0$.



Non-Divisibility Testing Simple Linear Path Schemes

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- can only be passed if $\boldsymbol{v} \nmid \mathbf{7}$, and
- ends with x $= \boldsymbol{v}$, y $= \mathbf{0}$.

(i) Choose $\boldsymbol{r} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}$ ...

(ii) ... such that $\mathbf{7} \mid \boldsymbol{v} + \boldsymbol{r}$.

(iii) Restore x $= \boldsymbol{v}$, y $= \mathbf{0}$.