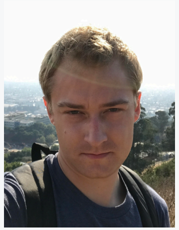
The Tractability Border of Reachability in Simple Vector Addition Systems with States

Henry Sinclair-Banks

Based on work with Dmitry Chistikov, Wojciech Czerwiński, Filip Mazowiecki, Łukasz Orlikowski, and Karol Węgrzycki to appear in FOCS'24.





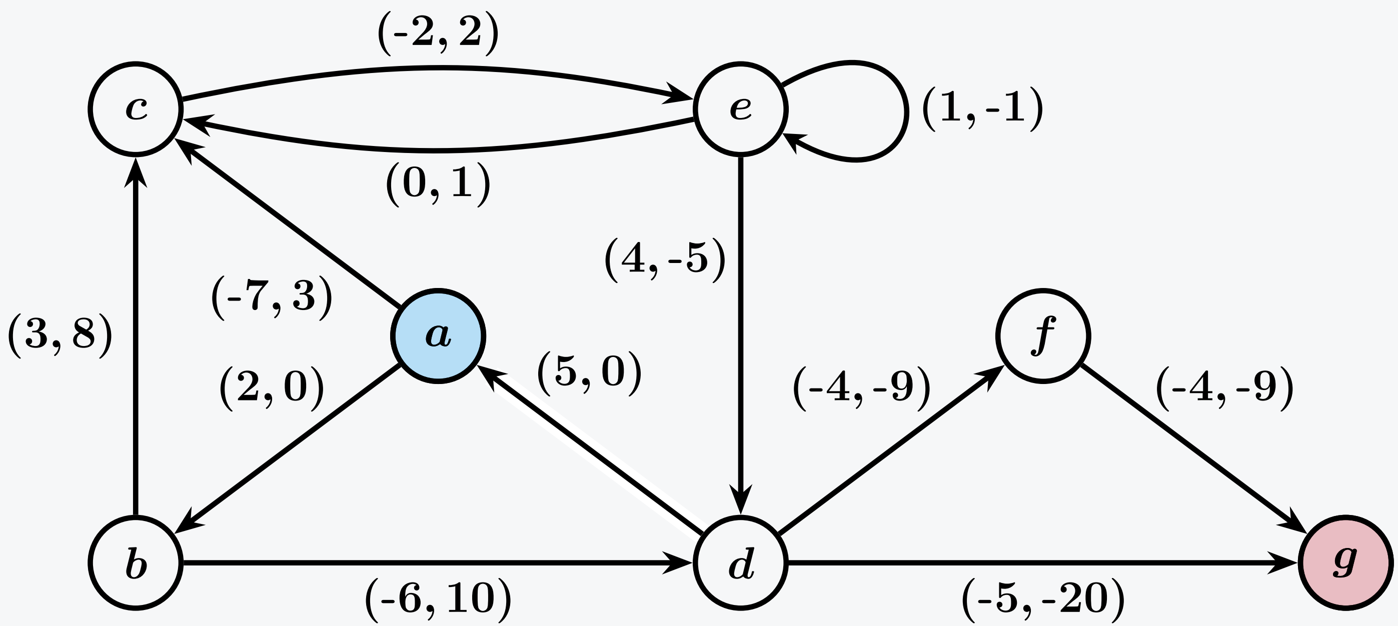



Algorithms & Complexity Seminar

27th August 2024

KIT, Karlsruhe, Germany

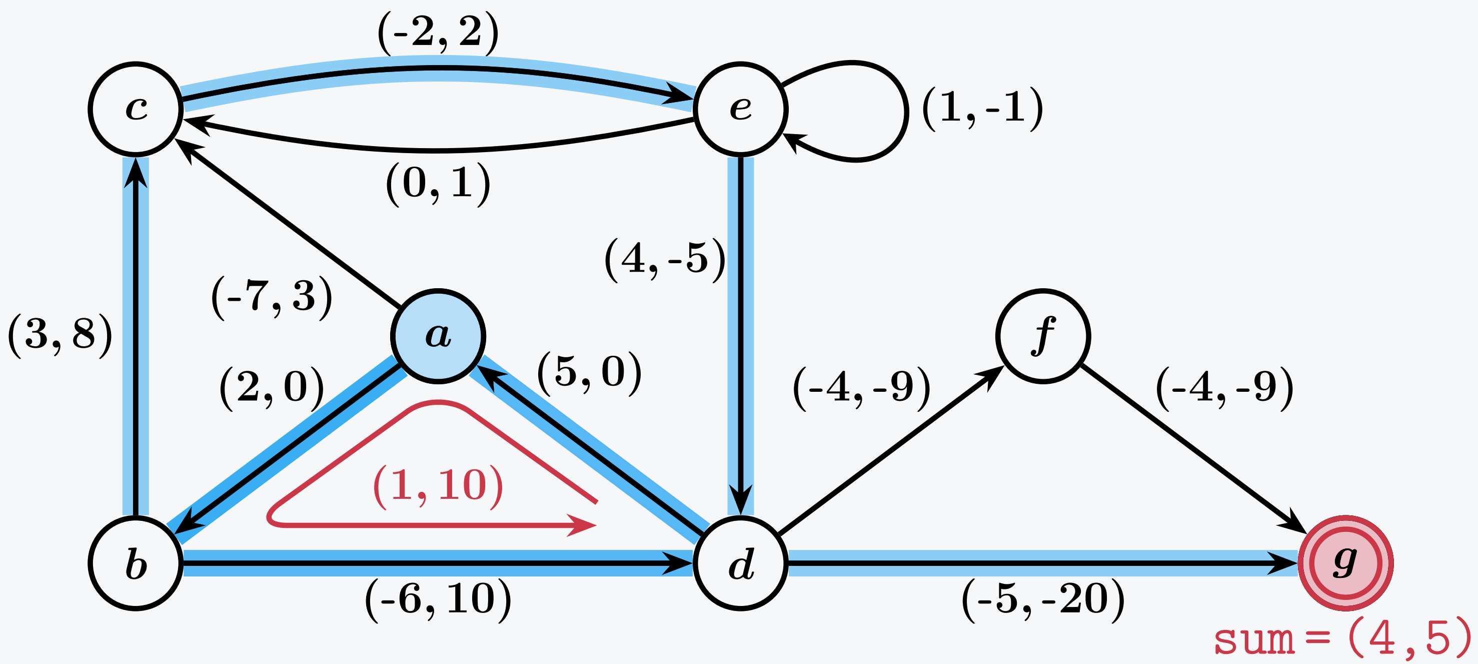
Reachability in 2-Dimensional VASS



Does there exist a run from a with counter values (0,0) to g with counter values (4,5) ?

(the counters must remain nonnegative at all times)

Reachability in 2-Dimensional VASS

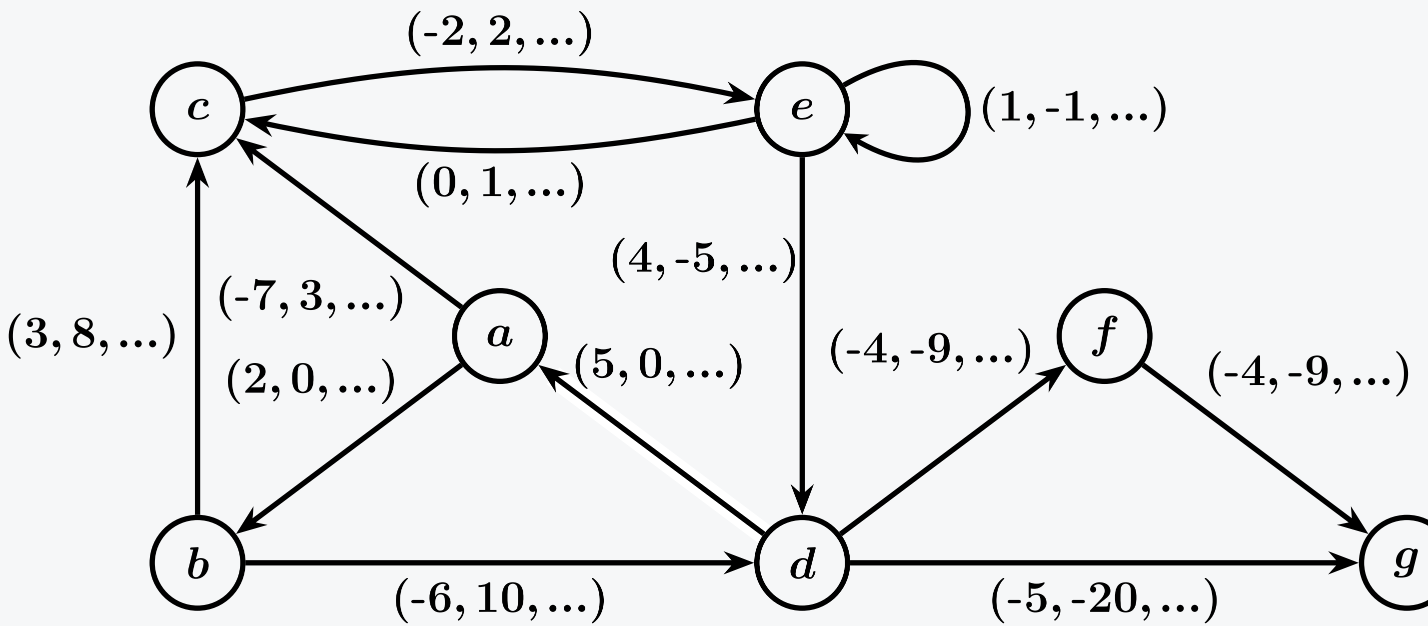


Does there exist a run from a with counter values (0,0) to g with counter values (4,5) ?

YES!

(the counters must remain nonnegative at all times)

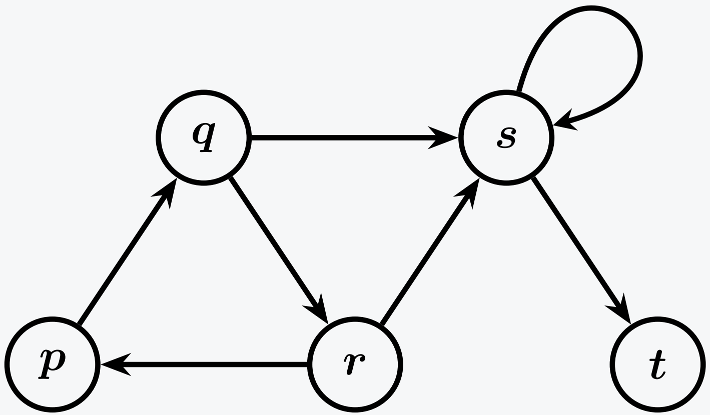
Reachability in VASS



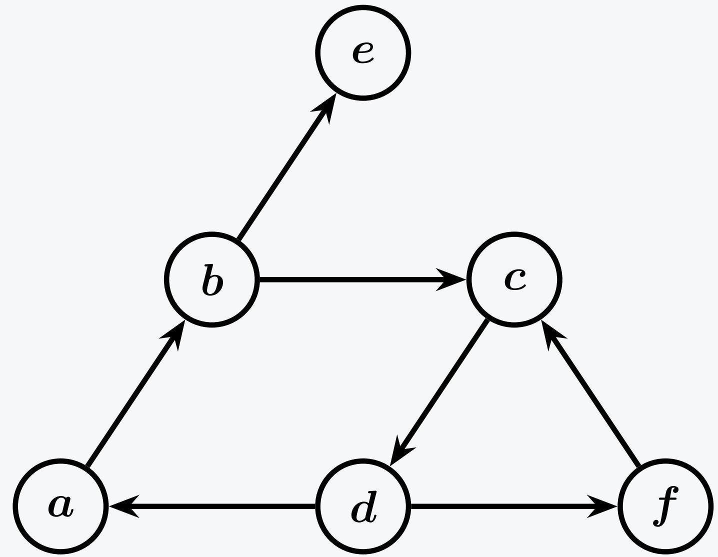
Reachability problem: does there exist a run from $p(\mathbf{u})$ to $q(\mathbf{v})$?

Flat VASS

Definition (Flat)**.** For every state $q \in Q$, there is at most one simple cycle that contains q.



Flat :)



Not flat :(

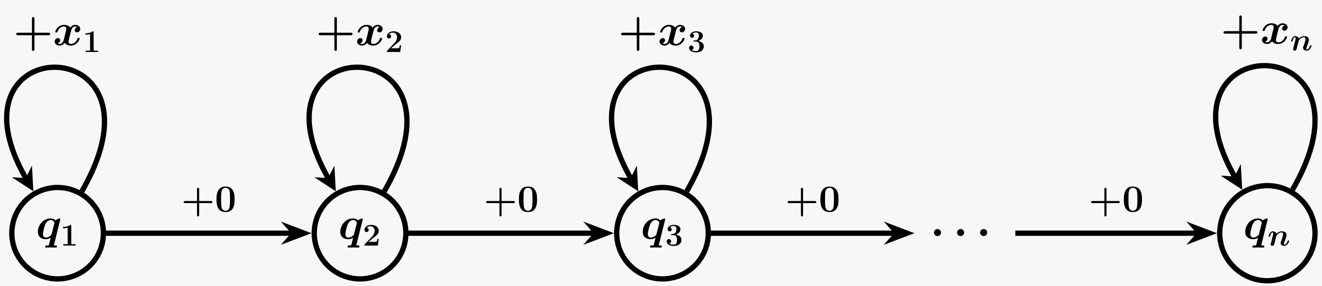
Flat VASS

Definition (Flat). For every state $q \in Q$, there is at most one simple cycle that contains q.

Theorem. Reachability in flat VASS is in NP (even with binary encoding). [Fribourg and Olsén '97] [Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

Theorem. Reachability in binary flat 1-VASS is NP-hard. [Rosier and Yen '85]

Proof sketch. Let $(\{x_1, \ldots, x_n\}, t)$ be an instance of subset sum (with multiplicities).



There exist $k_1, \ldots, k_n$ such that $t = \Sigma k_i \cdot x_i$ if and only if there is a run from $q_1(0)$ to $q_n(t)$.

Flat VASS

Definition (Flat)**.** For every state $\boldsymbol{q} \in \boldsymbol{Q}$, there is at most one simple cycle that contains $\boldsymbol{q}$.

Theorem. Reachability in flat VASS is in NP (even with binary encoding). [Fribourg and Olsén '97]

[Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

Theorem. Reachability in binary flat 1-VASS is NP-hard. [Rosier and Yen '85]

What is the complexity of reachability in unary flat VASS?

[Blondin, Finkel, Göller, Haase, and McKenzie '15]

[Englert, Lazić, and Totzke '16]

Complexity of Reachability in Flat VASS

Dimension	Binary encoding	Unary encoding
1	NP-complete [Rosier and Yen '85]	NL-complete [Valiant and Paterson '73]
2	NP-complete	NL-complete [Englert, Lazić, and Totzke '16]
	⋮	

Complexity of Reachability in Flat VASS

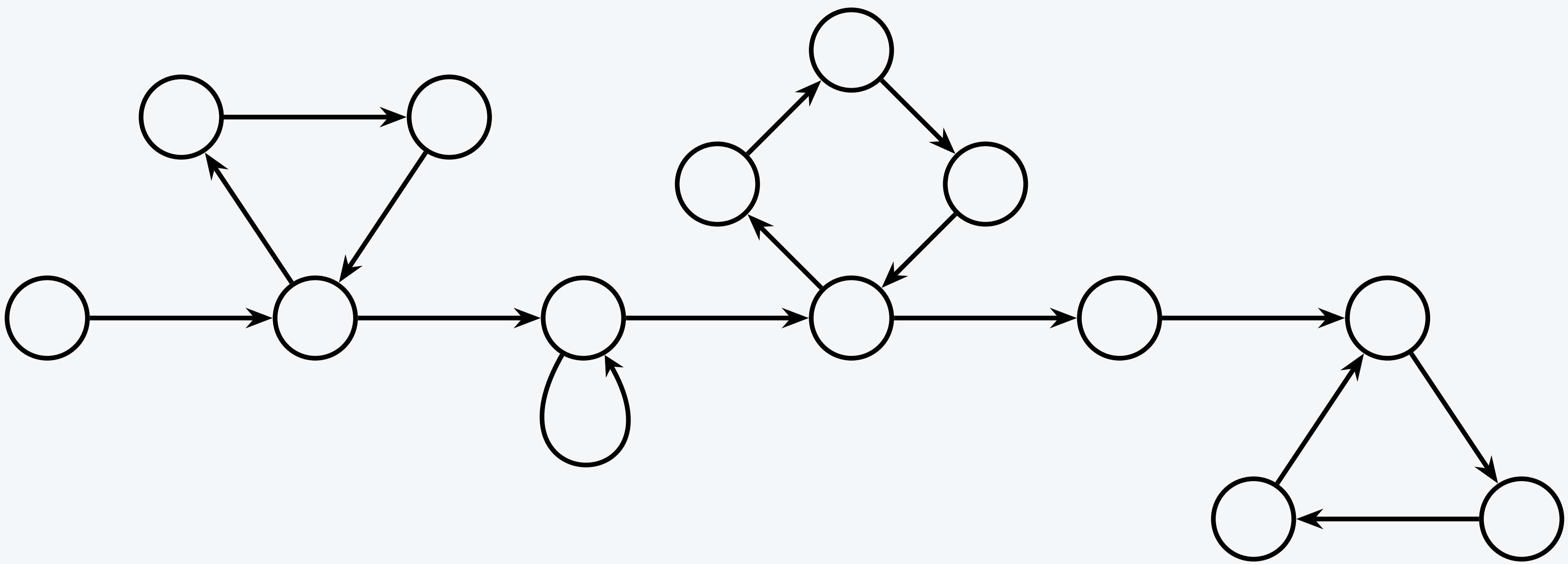
Dimension	Binary encoding	Unary encoding
1	NP-complete [Rosier and Yen '85]	NL-complete [Valiant and Paterson '73]
2	NP-complete	NL-complete [Englert, Lazić, and Totzke '16]
	⋮	
4		NP-complete [Czerwiński and Orlikowski '22]
5		NP-complete [Dubiak '20]
6		NP-complete
7		NP-complete [Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

Complexity of Reachability in Flat VASS

Dimension	Binary encoding	Unary encoding
1	NP-complete [Rosier and Yen '85]	NL-complete [Valiant and Paterson '73]
2	NP-complete	NL-complete [Englert, Lazić, and Totzke '16]
3		NP-complete This presentation!
4	⋮	NP-complete [Czerwiński and Orlikowski '22]
5		NP-complete [Dubiak '20]
6		NP-complete
7		NP-complete [Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

~~Flat VASS~~ Linear Path Schemes

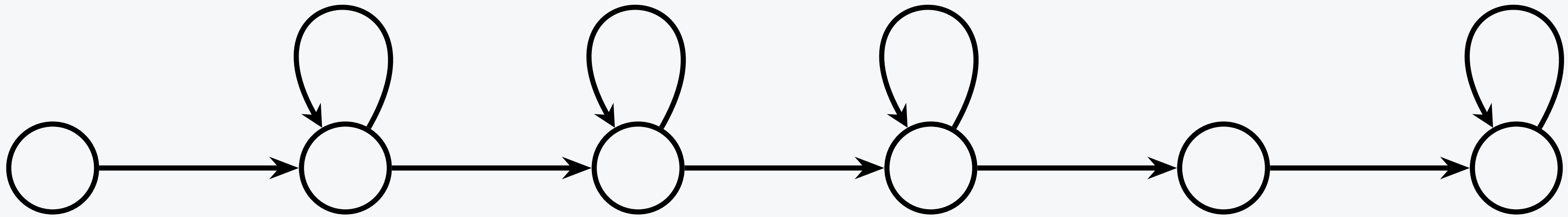
Definition (LPS)**.** A VASS where the states and transitions form a simple path between disjoint cycles.



~~Flat VASS~~ Linear Path Schemes

Definition (LPS)**.** A VASS where the states and transitions form a simple path between disjoint cycles.

Definition (SLPS)**.** A *Simple* LPS has cycles of length one (“self-loops”).



For $d \geq 3$, is reachability in unary d-dimensional linear path schemes in P?

[Englert, Lazić, and Totzke ’16]

[Leroux ’21]

Main Contribution

Theorem. Reachability in unary 3-SLPS is NP-complete.

Proof approach. Reduce from 3-SAT. Suppose φ has k variables $(x_1, \ldots, x_k)$ and m clauses.

1) Use "Chinese remainder encoding" for SAT.

- Let $p_1, \ldots, p_k$ be the first k primes.
- Let $n \in \mathbb{N}$ such that $n \equiv 0 \bmod p_i \iff x_i$ is false and $n \equiv 1 \bmod p_i \iff x_i$ is true.

2) Use a conjunction of non-divisibility tests to verify that n represents a valid assignment.

- To verify that $n \equiv 0 \bmod p_i$ OR $n \equiv 1 \bmod p_i$, check $p_i \mid n$ OR $p_i \mid n - 1$.
- Instead, check $p_i \nmid n - 2$ AND $p_i \nmid n - 3$ AND $\cdots$ AND $p_i \nmid n - (p_i - 1)$.

3) Again, use a conjunction of non-divisibility tests to verify that n represents a satisfying assignment.

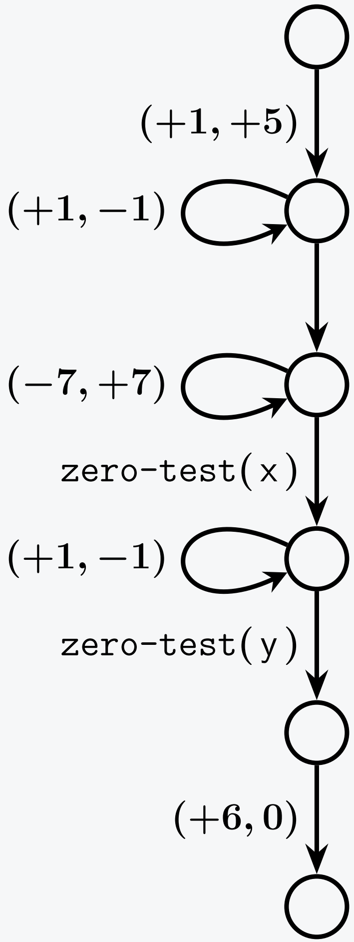
- A clause $x_1 \vee \neg x_2 \vee x_3$ is satisfied if $n \equiv 1 \bmod 2$ OR $n \equiv 0 \bmod 3$ OR $n \equiv 1 \bmod 5$.
- Instead, check $2 \cdot 3 \cdot 5 \nmid n - 10$.

Non-Divisibility Testing Simple Linear Path Schemes

Suppose we want to perform a non-divisibility test $v \nmid 7$.

Let's construct a 2-SLPS with zero tests that:

- starts with x $= v$, y $= 0$,
- can only be passed if $v \nmid 7$, and
- ends with x $= v$, y $= 0$.



Non-Divisibility Testing Simple Linear Path Schemes

Suppose we want to perform a non-divisibility test $v \nmid 7$.

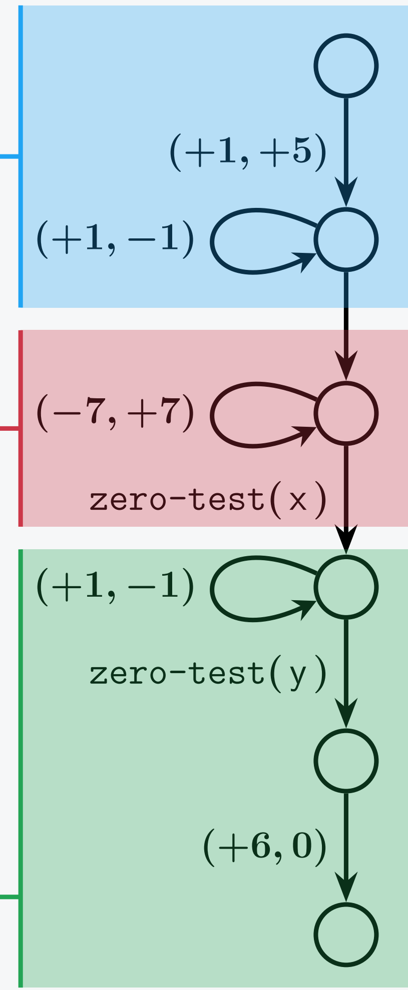
Let's construct a 2-SLPS with zero tests that:

- starts with x $= v$, y $= 0$,
- can only be passed if $v \nmid 7$, and
- ends with x $= v$, y $= 0$.

(i) Choose $r \in \{1, 2, 3, 4, 5, 6\}$ ...

(ii) ... such that $7 \mid v + r$.

(iii) Restore x $= v$, y $= 0$.



Simulating Zero Tests

Lemma 2.2 (Controlling Counter Technique). *Let $\mathcal{Z}$ be a d-VASS with zero tests and let $s(\mathbf{x}), t(\mathbf{y})$ be two configurations. Suppose $\mathcal{Z}$ has the property that on any accepting run from $s(\mathbf{x})$ to $t(\mathbf{y})$, at most m zero tests are performed on each counter. Then there exists a $(d+1)$-VASS $\mathcal{V}$ and two configurations $s'(\mathbf{0}), t'(\mathbf{y}')$ such that:*

(1) $s(\mathbf{x}) \xrightarrow{*}_{\mathcal{Z}} t(\mathbf{y})$ *if and only if* $s'(\mathbf{0}) \xrightarrow{*}_{\mathcal{V}} t'(\mathbf{y}')$,

(2) $\mathcal{V}$ *can be constructed in* $\mathcal{O}((size(\mathcal{Z}) + \|x\|) \cdot (m+1)^d)$ *time, and*

(3) $\|\mathbf{y}'\| \leq \|\mathbf{y}\|$.

Moreover, if $\mathcal{Z}$ is a flat VASS or a (simple) linear path scheme in which no zero-testing transition lies on a cycle, then $\mathcal{V}$ can be assumed to be a flat VASS or a (simple) linear path scheme, respectively.

[Czerwiński and Orlikowski '21] [Chistikov, Czerwiński, Mazowiecki, Orlikowski, S., and Węgrzycki '24]

Takeway message: A "small" number of zero tests can be simulated by an additional counter.

Recap – Proof of Main Theorem

Theorem. Reachability in unary 3-SLPS is NP-complete.

Proof sketch: Recall that reachability in (binary) flat VASS is in NP. For NP-hardness:

- Reduce from 3-SAT and use Chinese remainder encoding.

- Obtain an equivalent conjunction of non-divisibility tests.

- For each non-divisibility test, construct the corresponding unary 2-SLPS with zero tests.

- Prepend a $\mathsf{x} + \mathbf{1}$ self-loop to allow the assignment value $\mathsf{x} = \boldsymbol{v}$ to be guessed.

- Use the controlling counter technique to obtain unary 3-SLPS for the SAT instance.

The Tractability Border of Reachability in Simple Vector Addition Systems with States

Theorem. Reachability in unary 3-SLPS is NP-complete.

Theorem. Reachability in unary *ultraflat* 4-VASS is NP-complete.

Theorem. Reachability in *unitary* inverse-Ackermann-dimensional SLPS is NP-complete.

Theorem. Reachability in unary 2-SLPS with *binary encoded initial and target configurations* is in P.

Thank You!

Karlsruher Institut für Technologie

Presented by Henry Sinclair-Banks, University of Warwick, UK

KIT, Karlsruhe, Germany

henry.sinclair-banks.com