Infinite Automata 2025/26

Lecture Notes 11

Henry Sinclair-Banks

Definition 11.1. Unboundedness of VASS (problem).

Input. A VASS V and an initial configuration $(p, \mathbf{u})$

Question. Is the set of configurations that can be reached from $(p, \mathbf{u})$ in V infinite?

Theorem 11.2. Unboundedness in VASS is in EXPSPACE (regardless of whether the problem is encoded in unary or binary). Moreover, if the dimension d is fixed, unboundedness in unary-encoded and binary-encoded d-VASS is in NL and PSPACE, respectively.

Like for coverability (in Lecture 10), we will work on VAS instead of VASS. Recall from Exercise 6.2 that a d-VASS can be simulated by a $(d+3)$-VAS. We shall fix our attention on a d-VAS $V \subset \mathbb{Z}^d$, an initial configuration $\mathbf{s} \in \mathbb{N}^d$, and a target configuration $\mathbf{t} \in \mathbb{N}^d$. We use $\|\mathbf{v}\| = \max\{1, \|\mathbf{v}\|_\infty\}$ to denote a function that is not technically a norm (we want the size of a vector to always be at least 1). Let $n = \sum_{\mathbf{v} \in V} \|\mathbf{v}\| + \|\mathbf{t}\|$. Notice that the size of $\mathbf{s}$ does not contribute to the value of n.

To prove Theorem 11.2, we will establish the existence "short" runs for unboundedness. First, notice that unboundedness is captured by the following kinds of runs.

Definition 11.3. *Self-covering run.* Let $\pi = (\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_\ell)$ be a $\mathbb{Z}$-run (i.e. $\mathbf{c}_j \in \mathbb{Z}^d$ for all j). We say that π is *self-covering* if there exists $j_1, j_2 \in \{0, 1, \ldots, \ell\}$ such that $j_1 < j_2$ and $\mathbf{c}_{j_1} < \mathbf{c}_{j_2}$.

To be clear, if $(\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_\ell)$ is a self-covering run such that, for every $j \in \{0, 1, \ldots, \ell\}$, $\mathbf{c}_j \in \mathbb{N}^d$, then V is unbounded from $\mathbf{c}_0$. This is because from $\mathbf{c}_0$, there is a reachable configuration $\mathbf{c}_{j_1}$ from which there is another reachable configuration $\mathbf{c}_{j_2}$ such that $\mathbf{c}_{j_2} > \mathbf{c}_{j_1}$. This means there is a cycle with effect $\mathbf{c}_{j_2} - \mathbf{c}_{j_1} > \mathbf{0}$ which can be iterated indefinitely, every time a different configuration is reached and hence the set of configurations that can be reached from $\mathbf{c}_0$ is infinite.

Second, we shall comment on the "shortness" of runs that witness unboundedness. Like for coverability, we will prove that if unboundedness holds, then there are self-covering runs that are at most doubly-exponential in length. More precisely, using Claims 11.6 and 11.7 (which themselves use Lemma 11.4), we will prove that unboundedness can be witnessed by runs of length at most $n^{d^{\mathcal{O}(d)}}$ at the end of these notes. Assuming this result, Theorem 11.2 immediately follows: there exists a non-deterministic algorithm for unboundedness that uses $d^{\mathcal{O}(d)} \cdot \log(n)$ space. Hence unboundedness in VASS in NEXPSPACE, and by Satich's theorem, NEXPSPACE = EXPSPACE. When the dimension d of the VASS is fixed, then $d^{\mathcal{O}(d)}$ is just some (potentially large) constant. This means that there is a non-deterministic $\mathcal{O}(\log(n))$-space algorithm for unboundedness in unary-encoded d-VASS. Accordingly, when the input is encoded in unary, the problem is in NL, and when the input is encoded in binary, the problem is in NPSPACE. Again, thanks to Savitch's theorem, NPSPACE = PSPACE. Before proving that unboundedness can be witnessed by runs of length at most $n^{d^{\mathcal{O}(d)}}$, we will provide some preliminary definitions and an important lemma (Lemma 11.4).

As indicated in Definition 11.3, we will be working with $\mathbb{Z}$-runs. We will also work on runs that are nonnegative ($\mathbb{N}$) on some counters and potentially negative ($\mathbb{Z}$) on other counters. We call $\mathbf{c} \in \mathbb{Z}^d$ an *i-pseudoconfiguration* if $\mathbf{c} \in \mathbb{N}^i \times \mathbb{Z}^{d-i}$. We call a $\mathbb{Z}$-run $(\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_\ell)$ an *i-pseudorun* if, for every $j \in \{0, 1, \ldots, \ell\}$, $\mathbf{c}_j$ is an i-pseudoconfiguration. Let $B \in \mathbb{N}$; we call an i-pseudorun $(\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_\ell)$ *B-bounded* if, for every $j \in \{0, 1, \ldots, \ell\}$, $\mathbf{c}_j \in [0, B]^i \times \mathbb{Z}^{d-i}$.

Lemma 11.4. Let $0 \le i \le d$, $\mathbf{c} \in \mathbb{Z}^k$, and $B \in \mathbb{N}$ such that there is a B-bounded self-covering i-pseudorun from $\mathbf{c}$. Then there is a B-bounded self-covering i-pseudorun from $\mathbf{c}$ of length at most $(nB)^{d^C}$ for some constant $C \in \mathbb{N}$.

Proof. Consider a minimal length B-bounded self-covering i-pseudorun from $\mathbf{c}$; we shall write the run as

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\ell, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$$

where $\mathbf{c} = \mathbf{v}_1$ and $\mathbf{w}_1 < \mathbf{w}_m$. For a vector $\mathbf{c} \in \mathbb{Z}^d$, we write $\pi(\mathbf{c}) \in \mathbb{Z}^i$ to denote the projection of $\mathbf{c}$ to the first i coordinates; precisely, $(\pi(\mathbf{c}))[j] = \mathbf{c}[j]$ for $1 \le j \le i$. In the case $i = 0$, then $\pi(\mathbf{c}) = \varepsilon \in \mathbb{Z}^0$. Thanks to the minimality of the length of the above self-covering run, $\pi(\mathbf{v}_1), \ldots, \pi(\mathbf{v}_\ell)$ must be distinct; hence $\ell \le B^d$.

Now we will outline the remainder of the proof. To get a bound on m, we will see that the sequence $\pi(\mathbf{w}_1), \ldots, \pi(\mathbf{w}_m)$ can be rearranged to consist, essentially, of a sequence of bounded length, together with a collection of (possibly a very large number) of simple cycles of bounded length appended to parts of the bounded length sequence. We will then use Pottier's lemma (Lemma 7.2) to obtain an upper bound on the number of simple cycles needed to create a B-bounded self-covering i-pseudorun from $\mathbf{c} = \mathbf{v}_1$.

Let $s = (\mathbf{u}_1, \ldots, \mathbf{u}_k)$ be a sequence of vectors in V. If $\mathbf{w} \in \mathbb{Z}^d$, then we define the pseudorun

$$Q(\mathbf{w}, s) \coloneqq (\mathbf{w},\ \mathbf{w} + \mathbf{u}_1,\ \ldots,\ \mathbf{w} + \mathbf{u}_1 + \cdots + \mathbf{u}_k).$$

We overload the our notation and define, for $\mathbf{x} \in \mathbb{Z}^i$,

$$Q(\mathbf{x}, s) \coloneqq (\mathbf{x},\ \mathbf{x} + \pi(\mathbf{u}_1),\ \ldots,\ \mathbf{x} + \pi(\mathbf{u}_1) + \cdots + \pi(\mathbf{u}_k)).$$

Suppose that $\mathbf{x} \in \mathbb{Z}^i$ and $s \in V^*$ such that $Q(\mathbf{x}, s) = (\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_k)$ is B-bounded, $\mathbf{x} = \mathbf{x}_k$, and $\mathbf{x}_{j_1} \neq \mathbf{x}_{j_2}$ for any $1 \le j_1 < j_2 \le k$, then we call $Q(\mathbf{x}, s)$ a *simple loop*. We use $\sum s \in \mathbb{Z}^d$ as short hand to denote $\mathbf{u}_1 + \ldots + \mathbf{u}_k$. We call $\sum s$ the effect of the simple loop and note that the effect can only be non-zero on the later $d - i$ coordinates because $\mathbf{x}_k - \mathbf{x} = \mathbf{0}$ on the first i coordinates.

Note that $(\pi(\mathbf{w}_1), \pi(\mathbf{w}_2) \ldots, \pi(\mathbf{w}_m))$ is the projection of a B-bounded i-pseudorun from $\mathbf{w}_1$. If $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is a simple loop occurring as an infix of $(\pi(\mathbf{w}_1), \ldots, \pi(\mathbf{w}_m))$, then if $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is removed, the remaining sequence will still be the projection of a B-bounded i-pseudorun from $\mathbf{w}_1$. If this process of removing simple loops is repeated carefully, one eventually obtains a "short" sequence which is the projection of a B-bounded i-pseudorun from $\mathbf{w}_1$ and a collection of simple loops whose combined effect is $\mathbf{e} \in \{0\}^i \times \mathbb{Z}^{d-i}$. The difference between the last and first vectors of the obtained sequence plus $\mathbf{e}$ is equal to $\mathbf{w}_m - \mathbf{w}_0 > \mathbf{0}$. Using Pottier's lemma (Lemma 7.2), we will be able to show that $\mathbf{e}$ can be obtained by using a "small" number of cycles (instead of the potentially large number of cycles removed from the original run). We then reverse the process and carefully reinsert simple loops back into the run in order to obtain a run that is a projection of B-bounded i-pseudorun from $\mathbf{w}_1$ and ends at $\pi(\mathbf{w}_m)$.

Now, let $s_1 = (\mathbf{u}_1, \ldots, \mathbf{u}_{m-1}) \in V^*$ be the sequence of transitions such that, for every $j \in \{1, \ldots, m-1\}$, $\mathbf{w}_{j+1} - \mathbf{w}_j = \mathbf{u}_j$. Clearly, $\sum \mathbf{u}_j = \mathbf{w}_m - \mathbf{w}_1 > \mathbf{0} \in \mathbb{Z}^d$ and $Q(\mathbf{w}_1, s_1)$ is an B-bounded i-pseudorun from $\mathbf{w}_1$. Let $\mathbf{e}_1 = \mathbf{0} \in \mathbb{Z}^d$. We will define a sequence $(s_1, \mathbf{e}_1, s_2, \mathbf{e}_2, \ldots)$ such that for each j:

(1) $Q(\pi(\mathbf{w}_1), s_j)$ and $Q(\pi(\mathbf{w}_1, s_1))$ contain the same set of vectors (perhaps with different multiplicities), in particular, the projection of s_j is a B-bounded i-pseudorun from $\pi(\mathbf{w}_1)$;

(2) $\mathbf{e}_j \in \mathbb{Z}^d$ and $\sum s_j + \mathbf{e}_j = \sum s_1$; and

(3) $\mathbf{e}_j$ can be expressed as a nonnegative linear combination of loop effects for vectors occurring in $Q(\pi(\mathbf{w}_1), s_j)$.

First, s_1 and $\mathbf{e}_1$ are already defined and clearly satisfy conditions (1), (2), and (3). Now, we shall inductive define s_j and $\mathbf{e}_j$. Assume now that s_j and $\mathbf{e}_j$ are defined and satisfy conditions (1), (2), and (3). If the length of s_j is less than $((B+1)^d+1)^2$, then this construction is defined to half; so assume that $s_j = (\mathbf{u}'_1, \ldots, \mathbf{u}'_k)$ where $k \ge (B^d + 1)^2$, $Q(\pi(\mathbf{w}_1), s_j) = (\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1})$. Now, consider the first $((B+1)^d+1)^2$ members of $(\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1})$ being divided up into $(B+1)^d+1$ many blocks each consisting of $(B+1)^d+1$ many members. By the pigeonhole principle, it is true that every block must contain a vector which is repeated twice as $(B+1)^d + 1 > (B+1)^i$ which is the number of possible (projected) vectors in $[0, B]^i$. Moreover, again by the pigeonhole principle, it is also true that there exists a block that contains vectors which all occur in earlier blocks. Let $\mathbf{x}_{j_1}$ and $\mathbf{x}_{j_2}$ be two vectors inside this block consisting of previously observed vectors such that $a < b$ and $\mathbf{x}_a = \mathbf{x}_b$. Clearly, the infix $(\mathbf{x}_a, \mathbf{x}_{a+1}, \ldots, \mathbf{x}_b)$ forms a simple loop. Moreover, its removal does not change the set of vectors that are present in run (because all vectors in this block are also observed in earlier blocks). Define $s_{j+1} = (\mathbf{u}'_1, \ldots, \mathbf{u}'_{a-1}, \mathbf{u}'_b, \ldots, \mathbf{u}'_{k+1})$; clearly $Q(\pi(\mathbf{w}_1), s_{j+1}) = (\mathbf{x}_1, \ldots, \mathbf{x}_{a-1}, \mathbf{x}_b, \ldots, \mathbf{x}_{k+1})$ which has the same set of vectors as $Q(\mathbf{w}_1, s_j)$. We also define $\mathbf{e}_{j+1} = \mathbf{e}_j + (\mathbf{x}_a + \mathbf{x}_{a+1} + \ldots + \mathbf{x}_{b-1})$. It is true that $\sum s_{j+1} + \mathbf{e}_{j+1} = \sum s_j = \sum s_1$. It is also true that $\mathbf{u}'_a + \mathbf{u}'_{a+1} + \ldots + \mathbf{u}'_{b-1}$ is the effect of a simple loop starting at $\mathbf{x}_a$. Hence s_{j+1} and $\mathbf{e}_{j+1}$ satisfy conditions (1), (2), and (3).

Now, suppose this construction reaches $(s_1, \mathbf{e}_1, \ldots, s_j, \mathbf{e}_j)$ where s_j has length $k < ((B+1)^d+1)^2$. We know that $\sum s_j + \mathbf{e}_j = \sum s_1 > \mathbf{0} \in \mathbb{Z}^d$. In fact, let $\mathbf{p} = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^d$ be a vector such that $\sum s_1 \geq \mathbf{p}$. Let $L \subseteq \mathbb{Z}^d$ be the set simple loop effects occurring in $Q(\mathbf{w}_1, s_j)$ and let A be the matrix with d rows, whose columns are the elements of L. Let $\mathbf{b} \in \mathbb{Z}^d$ be the (column) vector equal to $\mathbf{p} - \sum s_j$. Since $\mathbf{e}_j = \mathbf{p} - \sum s_j = \mathbf{b}$ and $\mathbf{e}_j$ is a nonnegative linear combination of elements from L (simple loops effects), we know that $A\mathbf{y} \geq \mathbf{b}$ has a nonnegative integer solution.

The effect of a simple loop is the sum of at most B^d many transition vectors from V, so each component of every vector in L (i.e. every coefficient in A) has absolute value at most $B^d \cdot n$. Therefore, $|L| \leq (2nB^d + 1)^d$; this means A has this many columns. Each component of $\mathbf{b}$ has absolute value at most $n((B+1)^d + 1)^2$. Now, by Lemma 7.2 (Pottier's lemma) we deduce that there is a minimal $\mathbf{y}_0 \in \mathbb{N}^{|L|}$ such that $A\mathbf{y}_0 \geq \mathbf{b}$ such that

$$\begin{aligned}
\|\mathbf{y}_0\|_1 &\leq (1 + n((B+1)^d + 1)^2 \cdot nB^d)^d \\
&\leq (1 + n(4B)^{2d} \cdot nB^d)^d \\
&\leq (2n(4B)^{3d})^d \\
&\leq (nB)^{d^C},
\end{aligned}$$

for some constant $C \in \mathbb{N}$.

Now, let $s'_1 = s_j$ and $\mathbf{e}'_1 = A\mathbf{y}_0$; $\mathbf{e}'_1 \geq \mathbf{p} - \sum s'_1$. This means that $\mathbf{e}'_1 + \sum s_j > \mathbf{0}$. Moreover, $\mathbf{e}'_1$ can be expressed as the sum of $\|\mathbf{y}_0\|_1$ simple loop effects of simple loops. Suppose $s'_1 = (\mathbf{u}_1, \ldots, \mathbf{u}_k)$ and $Q(\pi(\mathbf{w}), s'_1) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k+1})$. Let $t \in V^*$ such that $Q(\mathbf{x}_{j_1}, t)$ is a simple loop starting from $\mathbf{x}_{j_1}$, for some $1 \leq j_1 \leq k+1$ and such that $\mathbf{e}'_1 - \sum t$ can be expressed as a sum of $\|\mathbf{y}_0\|_1 - 1$ vectors that are effects of simple loops in $Q(\pi(\mathbf{w}_1), s'_1)$. Let $\mathbf{e}'_2 = \mathbf{e}'_1 - \sum t$ and s'_2 be the sequence in V^* that is obtained by inserting the simple loop t in s'_1 at the j_1-st position. By repeating this process, we can reinsert all $\|\mathbf{y}_0\|_1$ simple loops back to eventually obtain a sequence s' and a vector $\mathbf{e}'$ such that $\mathbf{e}' = \mathbf{0} \in \mathbb{Z}^d$, $\sum s' > \mathbf{0} \in \mathbb{Z}^d$, and $Q(\pi(\mathbf{w}_1), s')$ is a B-bounded self-covering run.

This construction takes $\|\mathbf{y}_0\|_1 \leq (nB)^{d^C}$ many stages, each increasing the length of the run by at most B^d. So $\mathbf{v}_1, \ldots, \mathbf{v}_\ell, Q(\mathbf{w}_1, s')$ is a B-bounded self-covering i-pseudorun from $\mathbf{v}_1 = \mathbf{c}$ in V. Since $\ell \leq B^d$ and $|s'| \leq |s'_1| + B^d \cdot (nB)^{d^C}$, in total we know that this B-bounded self-covering i-pseudorun has length at most $\ell + |s'| \leq (nB)^{d^C}$ for some constant C. □

Definition 11.5. Let $0 \leq i \leq d$. For every $\mathbf{c} \in \mathbb{Z}^d$, we define $f(i, \mathbf{c})$ to be the length of the shortest B-bounded self-covering i-pseudorun from $\mathbf{c}$ if one exists; and if no such run exists, then $f(i, \mathbf{c})$ is defined to be 0. We also define $g(i) \coloneqq \max\{f(i, \mathbf{c}) : \mathbf{c} \in \mathbb{Z}^d\}$.

In the following two claims, C is the constant in Lemma 11.4.

Claim 11.6. $g(0) \leq (2n)^{d^C}$.

Proof. Let $\mathbf{c} \in \mathbb{Z}^d$ be a vector such that there exists a self-covering $\mathbb{Z}$-run in V from $\mathbf{c}$. This run is trivially a 2-bounded self-covering 0-pseudorun from $\mathbf{c}$. Lemma 11.4 therefore tells us that there is a run of length at most $(2n)^{d^C}$. □

Claim 11.7. $g(i+1) \le (n^2 \cdot g(i))^{d^C}$, for all $1 \le i < d$.

Proof. Let $\mathbf{c} \in \mathbb{Z}^d$ be a vector such that there is a self-covering $(i+1)$-pseudorun in V from $\mathbf{c}$.

Case 1. There exists an $(n \cdot g(i))$-bounded self-covering $(i+1)$-pseudorun in V from $\mathbf{c}$. Then by Lemma 11.4, there must exist a self-covering $(i+1)$-pseudorun in V from $\mathbf{c}$ of length at most $(n^2 \cdot g(i))^{d^C}$.

Case 2. Otherwise, there does not exist an $(n \cdot g(i))$-bounded self-covering $(i+1)$-pseudorun in V from $\mathbf{c}$. Let $(\mathbf{c} = \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ be a self-covering $(i+1)$-pseudorun from $\mathbf{c}$; we know that there exists $\ell < m$ such that $\mathbf{w}_\ell < \mathbf{w}_m$ (for self-covering) and let $1 \le j < m$ be the least index such that $\mathbf{w}_j$ is not $(n \cdot g(i))$-bounded on the first $(i+1)$ coordinates (i.e. there exists $i' \le i+1$ such that $\mathbf{w}_j[i'] > n \cdot g(i)$). Let $(\mathbf{c} = \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ be the shortest run in V such that $\mathbf{v}_k[i'] = \mathbf{w}_j[i']$, for every $1 \le i' \le i+1$, and such that $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1})$ is an $(n \cdot g(i))$-bounded $(i+1)$-pseudorun. By the minimality of the length of this pseudorun, we know that $\mathbf{v}_{j_1}$ and $\mathbf{v}_{j_2}$ cannot have the same first $i+1$ coordinates for any $1 \le j_1 < j_2 \le k-1$. Hence, $k-1 \le (n \cdot g(i))^{i+1}$. Without loss of generality, suppose that $\mathbf{v}_k[i+1] = \mathbf{w}_j[i+1] > n \cdot g(i)$.

For $1 \le h < m$, let $\mathbf{u}_h \in V$ be the transition vector such that $\mathbf{u}_h = \mathbf{w}_{h+1} - \mathbf{w}_h$. Let $s \in V^*$ be the path for which $s = (\mathbf{u}_j, \mathbf{u}_{j+1}, \ldots, \mathbf{u}_{m-1}, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}, \ldots, \mathbf{u}_m)$. Then $Q(\mathbf{v}_k, s)$ is an $(i+1)$-pseudorun, so in particular it is a self-covering i-pseudorun from $\mathbf{v}_k$. Let q be an self-covering i-pseudorun from $\mathbf{v}_k$ of length at most $g(i)$. Since $\mathbf{v}_k[i+1] > n \cdot g(i)$, and $\|\mathbf{u}\| \le n$ for all $\mathbf{u} \in V$, q is in fact an $(i+1)$-pseudorun (as the $(i+1)$-st counter value starts at a large enough value that it does not decrease below n). So, $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, q)$ is a self-covering $(i+1)$-pseudorun (from $\mathbf{v}_1 = \mathbf{c}$) of length at most $(n \cdot g(i))^{i+1} + g(i) \le (n^2 \cdot g(i))^{n^C}$. ☐

Finally, as indicated after the statement of Theorem 11.2, we will show how to use Claim 11.6 and Claim 11.7 to prove that unboundedness is witnessed by self-covering runs of length at most $n^{d^{\mathcal{O}(d)}}$. If there is a self-covering d-pseudorun (i.e. a "real self-covering run") from $\mathbf{s}$, then we know that there is a self-covering run from $\mathbf{s}$ of length at most $g(d)$.

We know that

$$g(d) \le (n^2 \cdot g(d-1))^{d^C} = n^{2d^C} \cdot (g(d-1))^{d^C},$$

and

$$(g(d-1))^{d^C} \le \left((n^2 \cdot g(d-2))^{d^C}\right)^{d^C} = (n^2 \cdot g(d-2))^{d^{2C}} = n^{2d^{2C}} \cdot g(d-2)^{d^{2C}}.$$

Hence,

$$g(d) \le n^{2d^C} \cdot n^{2d^{2C}} \cdot g(d-2)^{d^{2C}}.$$

By repeating the replacement of $g(i)$ with bounds including $g(i-1)$, we deduce that

$$g(d) \le n^{2d^C} \cdot n^{2d^{2C}} \cdot \ldots \cdot n^{2d^{dC}} \cdot (g(0))^{d^{dC}}.$$

Now, by Claim 11.6, we know that $g(0) \le (2n)^{d^C}$, so

$$\begin{aligned} g(d) \le n^{2d^C + 2d^{2C} + \ldots + 2d^{dC}} \cdot (g(0))^{d^{dC}} &\le n^{d \cdot 2d^{dC}} \cdot \left((2n)^{d^C}\right)^{d^{dC}} \\ &= n^{2d^{dC+1}} \cdot (2n)^{d^C \cdot d^{dC}} \\ &\le n^{2d^{dC+1}} \cdot n^{2d^{dC+C}} \\ &= n^{4d^{2dC+C+1}}. \end{aligned}$$

Altogether, this means that there is a self-covering run of length at most $n^{d^{\mathcal{O}(d)}}$.