

Infinite Automata 2025/26

Lecture Notes 5

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Definition 5.1. Universality of letter-labelled VASS (problem).

Input. A letter-labelled VASS V over the alphabet Σ .

Question. Is Σ^* the language that is recognised by V ?

Theorem 5.2. Universality of letter-labelled VASS is decidable.

Before we begin the proof of Theorem 5.2, we will introduce and work with downward closed sets of $\mathbb{N}^d$. Precisely, let $\mathbb{D} := \{X \subseteq \mathbb{N}^d : X = X \downarrow\}$. Here

$$X \downarrow := \{\mathbf{y} \in \mathbb{N}^d : \text{there exists } \mathbf{x} \in X \text{ such that } \mathbf{x} \geq \mathbf{y}\}.$$

In Exercise 5.3, we proved that $(\mathbb{D}, \subseteq)$ is a WQO. Fix $k \in \mathbb{N}$; $\mathbb{D}^k$ is the collection of k -tuples of sets in $\mathbb{D}$. Moreover, we defined the component-wise inclusion relation $\subseteq^k$ as follows. Let $X, Y \in \mathbb{D}^k$, $X \subseteq^k Y$ if and only if, for every $i \in \{1, \dots, k\}$, $X[i] \subseteq Y[i]$. The following claim follows from Exercise 5.3 and Dickson's Lemma (Lemma 4.5).

Claim 5.3. For every $k \in \mathbb{N}$, $(\mathbb{D}, \subseteq^k)$ is a WQO.

Proof of Theorem 5.2. Let $V = (\Sigma, Q, T, q_0, F)$ be a letter-labelled d -VASS. First observe that upon reading a word $w \in \Sigma^*$, it is only possible to reach a finite (exponential) number of different configurations. We need not maintain *all* possible configurations that can be reached by reading w , we only need to maintain the maximal configurations reached. For example, suppose that $(p, \mathbf{x})$ and $(p, \mathbf{y})$ can be reached after reading w_1 and $\mathbf{x} \leq \mathbf{y}$. Now suppose that from $(p, \mathbf{x})$ the word w_2 can be read. Since $\mathbf{x} \leq \mathbf{y}$, we know that w_2 can also be read starting from $(p, \mathbf{y})$ as well. Thus, when verifying whether the word $w_1 w_2$ is accepted by V or not, we only needed to consider the greater configuration $(p, \mathbf{y})$.

For a word $w \in \Sigma^*$, let $R_w \subseteq Q \times \mathbb{N}^d$ be the set of configurations that can be reached by reading w from the initial configuration $(q_0, \mathbf{0})$. We attribute, to w , the $|Q|$ -tuple of downward closed sets $X_w \in \mathbb{D}^{|Q|}$ such that, for every state $p \in Q$, $X_w[p] = \{\mathbf{v} : (p, \mathbf{v}) \in R_w\} \downarrow$. In other words, $X_w[p]$ is the downward closure of counter value vectors that can be reached by reading w and ending at state p . If it is not possible to reach the state p after reading w , then $X_w[p] = \emptyset$.

Since $X_w[p]$ is the downward closure of a finite set in $\mathbb{N}^d$, then $X_w[p]$ is also finite. Moreover $X_w[p]$ can be represented as the downward closure of finitely many maximal elements. It is also true that the representation of $X_w[p]$ as the downward closure of its maximal elements can be computed in finite time.

We say that X_w is an *accepting* tuple if there exists $q \in F$ such that $X_w[q] \neq \emptyset$. It immediately follows that w is accepted by V if and only if X_w is an accepting tuple.

Now, we will construct a tree $\mathcal{T}$ whose edges are labelled by letters in Σ and whose nodes are the tuples X_w . In a similar fashion to the proof of Theorem 4.6, consider generating this tree (in a BFS or DFS style). Since $(\mathbb{D}^{|Q|}, \subseteq^{|Q|})$ is a WQO (Claim 5.3), we know that in every path down the tree, there will eventually exist a node X_w which has an ancestor $X_{w'}$ (i.e. w' is a prefix of w) such that $X_{w'} \subseteq^{|Q|} X_w$. We shall terminate the exploration of a branch in the tree once we observe such a node X_w (that is greater than an ancestor). By König's Lemma (Lemma 4.4) and Claim 5.3, $\mathcal{T}$ is finite.

We will now argue that if there exists $w \in \Sigma^*$ such that w is *not* accepted by V , then there exists w' in $\mathcal{T}$ such that w' is not accepted by V . Suppose that there exists a word that is not accepted by V ; in fact let w be the shortest word that is not accepted by V . Assume, for sake

of contradiction, that w is not in $\mathcal{T}$. Consider the decomposition of w into subwords x , y , and z such that (i) $w = xyz$, (ii) X_{xy} is a leaf node of $\mathcal{T}$, and (iii) X_x is the ancestor tuple such that $X_x \subseteq^{|Q|} X_{xy}$. Note that $|y| > 0$ because X_x is an ancestor of X_{xy} . Since $X_x \subseteq^{|Q|} X_{xy}$, it is true that $X_{xz} \subseteq^{|Q|} X_{xyz}$. Moreover, since w is not accepted by V , we know that $X_{xyz} = X_w$ is not an accepting tuple. This means that for all $q \in F$, $X_w[q] = \emptyset$. Given that $X_{xz} \subseteq^{|Q|} X_{xyz}$, it follows that $X_{xyz}[q] = \emptyset$ implies that $X_{xz}[q] = \emptyset$. Thus X_{xz} is not an accepting tuple, and we can therefore conclude that $xz \in \Sigma^*$ is not accepted by V . Since $|y| > 0$, it is true that $|xz| < |xyz| = |w|$. This contradicts the minimality of the length of w .

Now, to conclude this proof, observe that in order to decide whether V accepts all words $w \in \Sigma^*$, it suffices to check that w is accepted by V for all words that have nodes in the tree $\mathcal{T}$. This algorithm therefore generates $\mathcal{T}$, at each step computes X_w and checks whether X_w is an accepting tuple. If at any point, a tuple is found that is not accepting, then the algorithm outputs “no” (the language of V is not universal). Otherwise, at the end of generating the tree, if all nodes are accepting, then the algorithm outputs “yes” (the language of V is universal). $\square$

The following definition differs only slightly from the original definition presented by Karp and Miller in 1969; they used VAS and we will use VASS. We define ω to be the symbol such that, for every $z \in \mathbb{Z}$, $\omega + z = \omega$ and $z < \omega$. We also define $\mathbb{N}_\omega := \mathbb{N} \cup \{\omega\}$.

Definition 5.4. Let $V = (Q, T)$ be a d -VASS and let $(p, \mathbf{u}) \in Q \times \mathbb{N}^d$ be a configuration. The *coverability tree* $\mathcal{T}$ of V rooted with $(p, \mathbf{u})$ is defined as follows. Nodes in the tree will be elements in T^* (i.e. runs in V). Each node is labelled by a “pseudo-configuration” $(q, \mathbf{v}) \in Q \times \mathbb{N}_\omega^d$. For convenience, we define the functions $\text{state}(\alpha)$ and $\text{vector}(\alpha)$ as one would expect: if α is labelled with $(q, \mathbf{v})$ then $\text{state}(\alpha) = q \in Q$ and $\text{vector}(\alpha) = \mathbf{v} \in \mathbb{N}_\omega^d$.

The root of $\mathcal{T}$ is $\varepsilon \in T^*$ and it is labelled with $\text{state}(\varepsilon) = p$ and $\text{vector}(\varepsilon) = \mathbf{u}$. Let β be an arbitrary node in $\mathcal{T}$. If there exists an ancestor α of β such that the labels are the same ($\text{state}(\alpha) = \text{state}(\beta)$ and $\text{vector}(\alpha) = \text{vector}(\beta)$) then β is a leaf node (it is the end of its branch of the tree and has no children). Assuming that is not the case (β is not a leaf), there will be one child γ of β for every transition such that $(\text{state}(\beta), \mathbf{x}, q) \in T$ such that $\text{vector}(\beta) + \mathbf{x} \geq 0$. The state-label of γ will be $\text{state}(\gamma) = q$ and the vector-label of γ will be defined as follows. First, we say that α is an *i-pumpable* ancestor of β if α is an ancestor of β , $\text{state}(\alpha) = \text{state}(\beta)$, $\text{vector}(\alpha) \leq \text{vector}(\beta)$, and $\text{vector}(\alpha)[i] < \text{vector}(\beta)[i]$. Now, for $i \in \{1, \dots, d\}$,

$$\text{vector}(\gamma)[i] = \begin{cases} \omega & \text{if } \beta \text{ has an } i\text{-pumpable ancestor,} \\ \text{vector}(\beta)[i] + \mathbf{x}[i] & \text{otherwise.} \end{cases}$$

Intuitively speaking there are two reasons that in which $\text{vector}(\gamma)[i]$ could be set to or equal to ω . The first is given by “*i-pumpable* ancestors”. This means that there is some ancestor which has the same state and which the vector-label of β is at least the vector-label of α . This means that the cycle $\beta - \alpha \in T^*$ can be repeated arbitrarily many times. Moreover, since we insist that $\text{vector}(\beta)[i] > \text{vector}(\alpha)[i]$, it is therefore possible to attain arbitrary high counter values on the i -th coordinate; hence we set $\text{vector}(\gamma)[i] = \omega$. The other way that $\text{vector}(\gamma)[i] = \omega$ is if $\text{vector}(\beta)[i] = \omega$ already, in this case, regardless of $\mathbf{x}$, it will be true that $\text{vector}(\gamma)[i] = \text{vector}(\beta)[i] + \mathbf{x}[i] = \omega$.

Lemma 5.5. Let V be a VASS and let $(p, \mathbf{u})$ be a configuration. The coverability tree $\mathcal{T}$ of V rooted with $(p, \mathbf{u})$ is finite.

Proof. Suppose, for sake of contradiction, that $\mathcal{T}$ is infinite. Since $\mathcal{T}$ is finitely branching, by König’s Lemma (Lemma 4.4), there must exist an infinite path down the tree. Since $(\mathbb{N}_\omega^d, \leq)$ is a WQO and by pigeonhole principle over the set of states Q , there exists an infinite sequence $(\alpha_i)_{i=1}^\infty$ of nodes in $\mathcal{T}$ such that $\text{state}(\alpha_1) = \text{state}(\alpha_2) = \dots$ and $\text{vector}(\alpha_1) \leq \text{vector}(\alpha_2) \leq \dots$. By definition of $\mathcal{T}$, we know that for every i , $\text{vector}(\alpha_i) \neq \text{vector}(\alpha_{i+1})$. Again, by definition of the tree, that means that $\text{vector}(\alpha_{i+1})$ must contain at least one more ω than $\text{vector}(\alpha_i)$. However, since there are only finitely many coordinates and the sequence of α_i is infinite, this cannot be the case. We therefore conclude that $\mathcal{T}$ must be finite. $\square$

Claim 5.6. Let V be a VASS and let $(p, \mathbf{u}), (q, \mathbf{v})$ be a pair of configurations. We define $R((p, \mathbf{u})) := \{(p', \mathbf{u}') : (p, \mathbf{u}) \xrightarrow{*}_V (p', \mathbf{u}')\}$. The following two statements are equivalent.

- (1) There exists $(q, \mathbf{v}') \in R((p, \mathbf{u}))$ such that $\mathbf{v}' \geq \mathbf{v}$.
- (2) There is a node α in $\mathcal{T}$ such that $\text{state}(\alpha) = q$ and $\text{vector}(\alpha) \geq \mathbf{v}$.

Proof sketch. For (1) $\implies$ (2): consider the run π from $(p, \mathbf{u})$ to $(q, \mathbf{v}')$ for some $\mathbf{v}' \geq \mathbf{v}$. Repeatedly apply the following procedure to π : (i) if there is a configuration $(r, \mathbf{y})$ that repeats in π , then delete the second occurrence of $(r, \mathbf{y})$ and all subsequent configurations; (ii) if there is a configuration $(r, \mathbf{y})$ such that there is a prior configuration $(r, \mathbf{x})$ such that $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x}[i] < \mathbf{y}[i]$, then replace the i -th entry of $\mathbf{y}$ with ω . This procedure must terminate because there are only finitely many (i) operations that are possible and between every (i) operation, there can only be finitely many (ii) operations. Let $(q, \mathbf{w}) \in Q \times \mathbb{N}_\omega^d$ be the final “pseudo-configuration” in the modified sequence. Observe that the final sequence obtained is exactly a route through $\mathcal{T}$ to $(q, \mathbf{w}) \geq (q, \mathbf{v}') \geq (q, \mathbf{v})$.

For (2) $\implies$ (1): consider the path down $\mathcal{T}$ from ε to α such that $\text{state}(\alpha) = q$ and $\text{vector}(\alpha) \geq \mathbf{v}$. Assume WLOG that the first $0 \leq h \leq d$ components of $\text{vector}(\alpha)$ are ω . Moreover, assume WLOG that the ω components are introduced in order via the path down the tree. Every time a new ω value is added to the tree, there must have been an i -pumpable ancestor. This can be used to create a cycle that is non-negative on all non- ω places and strictly positive on the ω place. One can obtain a cycle for each ω places and can create a valid run in the original VASS that follows the path down the tree, but whenever a new ω is introduced its corresponding i -pumpable cycle is taken a (very) large number of times — so much so that later cycles that may be negative on the ω components still do not bring the counters below zero. Intuitively speaking, it is possible, for every $N \in \mathbb{N}$ to find a configuration in $R((p, \mathbf{u}))$ with counter values $(N_1, \dots, N_h, v_{h+1}, \dots, v_d)$ for some $N_1, \dots, N_h \geq N$. $\square$