Infinite Automata 2025/26

Exercise Sheet 11

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Exercise 11.1. Prove that the minimal length runs witnessing reachability in unary-encoded 3-VASS can have exponential length.

Hint. Use the following lemma to perform repeated multiplication operations on the counter values.

Lemma. Controlling Counter Technique. Let Z be a d-VASS with zero test and $(s, \mathbf{x})$, $(t, \mathbf{y})$ be two configurations. Suppose Z has the property that on any run from $(s, \mathbf{x})$ to $(t, \mathbf{y})$, at most m zero tests are performed on each counter. Then there exist a $(d+1)$-VASS V and two configurations $(s', \mathbf{0})$, $(t', (\mathbf{y}, 0)$ such that

(1) $(s, \mathbf{x}) \xrightarrow{*}_Z (t, \mathbf{y})$ if and only if $(s', \mathbf{0}) \xrightarrow{*}_V (t', \mathbf{y}')$, and

(2) V can be constructed in $\mathcal{O}((\|Z\| + \|\mathbf{x}\|) \cdot m^d)$ time.

Exercise 11.2. Prove that there exists a $d \in \mathbb{N}$ such that reachability in unary-encoded d-VASS is NP-hard.

Hint 1. Consider the following encoding of an instance of 3-SAT with k variables $x_1, \ldots, x_k$ and ℓ clauses. Let $p_1, n_1, \ldots, p_k, n_k$ be the first $2k$ primes. An assignment of $x_1, \ldots, x_k$ will correspond to a natural number $N \in \mathbb{N}$ that is equal to the product of k primes. Specifically, $N = q_1 \cdot \ldots \cdot q_k$ where $q_i \in \{p_i, n_i\}$; here selecting $q_i = p_i$ corresponds to setting $x_i =$ `True` and selecting $q_i = n_i$ correspond to setting $x_i =$ `False`.

Hint 2. Create a VASS gadget (using zero tests) that starting with counters values $(v, 0)$ and can end with one of two possible counter value vectors: $(v \cdot p_i, 0)$ or $(v \cdot n_i, 0)$.

Hint 3. Consider a clause $(x_1 \vee \overline{x_2} \vee x_3)$. Checking whether $N = q_1 \cdot \ldots \cdot q_k$ satisfies this clause corresponds to checking whether $p_1 | N$ or $n_2 | N$ or $p_3 | N$.

Exercise 11.3. Prove that reachability in unary-encoded 3-dimensional linear path schemes is NP-hard.

Hint 1. Consider an alternative encoding of an instance of 3-SAT with k variables $x_1, \ldots, x_k$ and ℓ clauses. Let $p_1, \ldots, p_k$ be the first k primes. An assignment of $x_1, \ldots, x_k$ will correspond to a natural number $N \in \mathbb{N}$ where $N \equiv 0 \bmod p_i$ if $x_i =$ `False` and $N \equiv 1 \bmod p_1$ if $x_i =$ `True`.

Hint 2. Use non-divisibility tests to assert that (i) N is a valid assignment and (ii) N corresponds to a satisfying assignment.