Infinite Automata 2025/26

Exercise Sheet 8

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Exercise 8.1. Show that, from any initial configuration $(p, \mathbf{u})$, the reachability set of a given integer d-VASS ($\mathbb{Z}$-VASS) intersected with the positive quadrant $\mathbb{N}^d$ is semilinear.

Exercise 8.2. Show that, from any initial configuration $(p, \mathbf{u})$, the reachability set of a given bidirected VASS is semilinear.

Hint 1. Show that if there is a run from $(p, \mathbf{u})$ to $(q_1, \mathbf{v})$, and there is a run from $(p, \mathbf{u})$ to $(q_2, \mathbf{u}+\mathbf{x})$ then there is a run from $(p, \mathbf{u})$ to $(q_2, \mathbf{v}+\mathbf{x})$.

Hint 2. Show that if there is a run from $(p, \mathbf{u})$ to $(p, \mathbf{u}+\mathbf{r})$, then $\mathbf{r} = \mathbf{x_1} + \ldots + \mathbf{x_k}$ such that there is a run from $(p, \mathbf{u})$ to $(p, \mathbf{u}+\mathbf{x}_i)$ and $\mathbf{x}_i$ is minimal (i.e. there does not exist $\mathbf{y} \leq \mathbf{x}$ such that $(p, \mathbf{u}$ can reach $(p, \mathbf{u}+\mathbf{y})$).

Hint 3. Let B be the set of configurations $(q, \mathbf{v})$ that are reachable from $(p, \mathbf{u})$ and such that there does not exist a configuration $(q, \mathbf{w})$ that is also reachable from $(p, \mathbf{u})$ such that $\mathbf{w} \geq \mathbf{v}$. In other words B is the minimal set of reachable configurations from $(p, \mathbf{u})$. Let P be the set of vectors $\mathbf{x} \in \mathbb{N}^d$ in such that there is a run from $(p, \mathbf{u})$ to $(p, \mathbf{u}+\mathbf{x})$ and there does not exist a vector $\mathbf{y} \leq \mathbf{x}$ such that $\mathbf{y} \neq \mathbf{0}$ and $(p, \mathbf{u}+\mathbf{y})$ is reachable from $(p, \mathbf{u})$. Then, Reach$((p, \mathbf{u}))$ is equal to $\{(q, \mathbf{v}+\mathbf{x}_1+\ldots+\mathbf{x}_k) : (q, \mathbf{v}) \in B$ and $\mathbf{x}_i \in P\}$.

Exercise 8.3. Show the reachability problem for bidirected VASS is decidable.

Hint 1. Design two semi-procedures. One using runs and the other using semilinear separators.

Hint 2. If there is no run from $(p, \mathbf{u})$ to $(q, \mathbf{u})$, then we can guess the sets B and P and verify that Reach$((p, \mathbf{u})) = B + P^*$ (as per Hint 3 of Exercise 8.2) is an inductive invariant for reachability.

Exercise 8.4. Show that, if we have an exponential bound for the size (in the one-norm) of minimal solutions to a homogeneous systems of linear equations, then we also have an exponential bound for the size of minimal solutions of arbitrary (not necessarily homogeneous) systems of linear equations.