

Infinite Automata 2025/26

Exercise Sheet 7

Wojciech Czerwiński and Henry Sinclair-Banks

Exercise 7.1 *Generalisation of Exercise 6.6.* Show that the set of solutions of systems of integer linear inequalities (over $\mathbb{N}$) is semilinear.

Exercise 7.2 Show that the reachability relation of a linear path scheme (LPS) is semilinear.

Note. The reachability relation of a 2-VASS can be represented as a union of linear path schemes.

Exercise 7.3. Show that semilinear sets are closed under intersection.

Hint. Show that, for every linear set L , there exists a system of integer linear equations with a set distinguished variables such that the set of solutions to the system of integer linear equations projected onto the distinguished set of variables is exactly L .

Exercise 7.4. Show that each semilinear set can be expressed as a union of linear sets whose period vectors are linearly independent.

Hint 1. Show that if a set of periods P is linearly dependent then one can express $b + P^*$ as a finite union of linear sets with fewer periods.

Hint 2. Let P is a linearly dependent set of period vectors; choose a linearly independent subset $Q \subset P$. Every $\mathbf{p} \in P \setminus Q$ can be expressed as a linear combination of elements of Q . Let $\mathbf{p} = a_1 \mathbf{q}_1 + \dots + a_k \mathbf{q}_k$ for some $a_i \in \mathbb{Z}$ and suppose M is an upper bound on such that $|a_1|, \dots, |a_k| \leq M$. Every element of $b + P^*$ can be expressed in such a way that only of the follow is true: $\mathbf{p}$ is not used or there exists $i \in \{1, \dots, k\}$ such that $\mathbf{q}_i$ is used at most M times. Then one can reduce the number of periods by splitting into each of these $k + 1$ cases and removing the period that is used (zero or) a bounded number of times. In the case $\mathbf{q}_i$ is used at most M times, one needs to also split into $M + 1$ many cases for each value $x \in [0, M]$ for which $\mathbf{q}_i$ is used x many times.