Mathematics may not yet be ready for such problems

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The Collatz function is defined as follows. For a natural number n,

$$c(n) := \begin{cases} 3n+1 & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

For example, $c(5) = 16$ and $c(16) = 8$. For convenience, we shall use c^i to denote the i-th iterate of the Collatz function. For example $c^2(5) = 8$. One can iterate the Collatz function indefinitely, since for all n, $c(n) \geq 1$. For example, starting from $n = 5$, the Collatz sequence is $5, 16, 8, 4, 2, 1, 4, 2, 1, \ldots$ and this $(4, 2, 1)$-cycle can be repeated indefinitely. See Fig. 1 for some more examples.

						19	
	9					58	
	28					29	
	14			**15**		88	
7	7			46		44	
22	22			23		22	
11	11	**11**		70		11	
34	34	34		35		34	
17	17	17		106	**17**	17	
52	52	52		53	52	52	
26	26	26		160	26	26	
13	13	13	**13**	80	13	13	
40	40	40	40	40	40	40	
20	20	20	20	20	20	20	**21**
10	10	10	10	10	10	10	64
5	5	5	5	5	5	5	32
16	16	16	16	16	16	16	16
8	8	8	8	8	8	8	8
4	4	4	4	4	4	4	4
2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Fig. 1. The Collatz sequences for a selection of starting values.

It really seems that from any starting number, the $(4, 2, 1)$-cycle will be reached. This is exactly *Collatz's conjecture*. Formally, the conjecture asserts that for all n, there exists k such that $c^k(n) = 1$.

Proving the Collatz conjecture is one of the most famous open problems in mathematics. Lothar Collatz (1937) has been primarily attributed to this problem but there are several claims as to the original author of this conjecture and, accordingly, the conjecture goes by several names including: *the hailstone problem*, *the* $(3n+1)$*-conjecture*, *the Syracuse problem*, *Thwaites' conjecture*, *Ulam's conjecture*, and *Kakutani's problem*.

The *delay* of n is the number of Collatz iterations before n converges to 1, denoted $d(n)$. Formally, the $d(n)$ is the least k such that $c^k(n) = 1$. For example, $d(5) = 5$. We also know that $d(2n) = d(n) + 1$ because $2n$ is always even. Small starting values can yield immense delay values: starting from $n = 27$, its Collatz sequence peaks at 9232 and eventually reaches 1 after $d(27) = 111$ iterations. A number n is a *delay record* if its delay $d(n)$ is greater than the delay of any number less than n. For example, 7 is a delay record because the delays of 1, 2, 3, 4, 5, and 6 are all less than $\mathrm{d}(7) = 16$. See a table of delay records in Fig. 2.

n	$d(n)$	max
1	0	1
2	1	2
3	7	16
6	8	16
7	16	52
9	19	52
18	20	52
23	25	160
27	111	9232
54	112	9232
73	115	9232
97	118	9232
129	121	9232
⋮	⋮	⋮
871	178	190996
1161	181	190996
⋮	⋮	⋮
6171	261	975400
10971	267	975400
⋮	⋮	⋮
77031	350	21933016
106239	353	104674192
⋮	⋮	⋮
837799	524	2974984576

Fig. 2. Many of the delay records and the maximum values in their Collatz sequences with $n < 2^{20}$.

The surprising nature of the Collatz function, such as the similarities between the Collatz sequences in Fig. 1 and the seemingly random delay records in Fig. 2, indicate that approaching the Collatz conjecture is not going to be easy! Many well-known mathematicians have studied this problem; a lot is now known but a full proof has not (yet) been found. Jeffrey Lagarias, an expert on the Collatz conjecture, said *"This is an extraordinarily difficult problem, completely out of reach of present day mathematics"*. Perhaps more famously, Paul Erdős said *"Mathematics may not yet be ready for such problems"* when asked about the conjecture.

With any open conjecture, there are naturally two main lines of attack: prove it or disprove it. In the remainder of this article, we will focus on some theorems and arguments that support the idea that the conjecture is true. Before that, let's touch on the possibility of counterexample; there are two types of potential counterexamples. The first type is a cycle that is not the trivial $(4, 2, 1)$-cycle and the other type of counterexample is an infinitely diverging Collatz sequence. The Collatz conjecture has been extensively tested by computer searches. As of May 2025, it is known that for all $n \leq 2^{71}$, the Collatz sequence reaches 1. As for nontrivial cycles, if there is a cycle counterexample, then its length must be at least $355\,504\,839\,929 \approx 2^{38}$. The data suggests, quite possibly, that the Collatz conjecture is true!

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Before we continue, we shall introduce the shortcut version of the Collatz function:

$$s(n) := \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

This definition is equivalent to the original Collatz function; if n is odd, then $3n+1$ must be even, so after every an odd step, an even step $\frac{n}{2}$ proceeds.

Imagine for a moment that you are verifying the conjecture and you have checked that all $n \leq 1000$ converge to 1; now you would like to test whether $n = 1001$ converges. It is not necessary to exhaustively compute $s(1001) = 1502$, $s(1502) = 751$, $s(751) = 1127$, ... all the way to 1 because as soon as the sequence drops below 1001, you know it will converge as all numbers $n \leq 1000$ converge. Accordingly, we will define the *glide* of a starting number n to be the number of iterations before the shortcut Collatz sequence drops below its starting value, denoted $g(n)$. Formally, $g(n)$ is the least k such that $s^k(n) < n$. For example it takes just two iterations for the shortcut Collatz sequence starting at $n = 1001$ to reach 751, so $g(1001) = 2$. Fig. 3 demonstrates how the glide can be used to more efficiently test the Collatz conjecture.

```
n ← 0
while true:
  m ← n
  while m > 1:
    if m % 2:
      m ← m/2
    else:
      m ← (3m+1)/2
  print(m)
```

```
n ← 0
while true:
  m ← n
  while m >= n:
    if m % 2:
      m ← m/2
    else:
      m ← (3m+1)/2
  print(m)
```

Fig. 3. Glides are practically very useful. Suppose you have verified the Collatz conjecture up to $n-1$. As already discussed, if there exists k such that $s^k(n) \in \{1, \ldots, n-1\}$, then n also converges. Accordingly, the testing procedure on the right is much more efficient than the one on the left.

Even though the data suggests that the Collatz conjecture is true, the nearest-neighbouring *theorems* typically use the phrase "almost all". In 1976, Rino Terras proved that almost all starting numbers have finite glide. Let's unpack this statement by handling the "almost all". As N grows to infinity, the proportion Collatz sequences, starting from $n \leq N$, that drop below their starting value tends to 1. More precisely,

$$\lim_{N\to\infty} \frac{|\{n \text{ such that } n \leq N \text{ and } g(n) < \infty\}|}{N} = 1.$$

To see why Terras' theorem is important, consider removing the "almost" to obtain the stronger statement: "all starting numbers have finite glide". This statement is actually strong enough to allow us to prove the Collatz conjecture by induction. Suppose that all starting numbers $1, \ldots, n$ converge to 1. By the stronger version of Terras' theorem, we know that $n+1$ has finite glide. This means that there exists k such that $s^k(n+1) \leq n$. Now, since all numbers $1, \ldots, n$ converge, we therefore know that $n+1$ converges as well. This stengthening is therefore equivalent to the Collatz conjecture, so let's stick to Terras' original theorem! To be clear, Terras' theorem is not equivalent to the Collatz conjecture, see Fig. 4 for a brief explanation.

Fig. 4. It is important to note that Terras' theorem can be true at the same time as the Collatz conjecture being false. If there is an n_0 that does not converge, then there must exist infinitely many starting numbers that do not converge; clearly $2^k \cdot n_0$ and $c^k(n_0)$ do not converge for any $k \geq 1$. However, these non-convergent numbers are sparsely distributed and the proportion of numbers $n \leq N$ with finite glide still tends to 1 as N tends to infinity.

For the majority of the remainder of this article, we will sketch the proof of Terras' theorem. To begin, consider the first five elements of the shortcut Collatz sequence starting from 19: $(19, 29, 44, 22, 11)$. It's *parity sequence* is the sequence of '0's and '1's corresponding to when the Collatz sequence is even and odd: $\mathbf{p}_5 = (1, 1, 0, 0, 1)$. Here, the 5 refers to the length of the parity sequence. There are two nice facts about parity sequences. The first fact is that when b is a fixed value, all starting numbers $0 \cdot 2^k + b, 1 \cdot 2^k + b, 2 \cdot 2^k + b, 3 \cdot 2^k + b, \ldots$ will have the same parity sequence $\mathbf{p}_k$ up to length k. See Fig. 5 for an example. The second fact is that, for *any* parity sequence $\mathbf{p}_k$ of length k, there is a number n with the parity sequence $\mathbf{p}_k$. See Fig. 6 for two examples. Both facts can be proved by induction but we'll save the proofs for another day!

odd	**19**	**51**	**83**	**115**	**147**	**179**
odd	29	77	125	173	221	269
even	44	116	188	260	332	404
even	22	58	94	130	166	202
odd	11	29	47	65	83	101
???	17	44	71	98	125	152
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Fig. 5. Example of the first fact with $k = 5$ and $b = 19$. The following numbers have the same parity sequence for the first five steps (see the above table). $0 \cdot 2^5 + 19 = 19$, $1 \cdot 2^5 + 19 = 51$, $2 \cdot 2^5 + 19 = 83$, $3 \cdot 2^5 + 19 = 115$, $4 \cdot 2^5 + 19 = 147$, $5 \cdot 2^5 + 19 = 179$.

Fig. 6. The second fact tells us that all parity sequences are possible. For example, it is easy to see that (0,0,0,0,0) is the parity sequence of 32: (32, 16, 8, 4, 2). However, the parity sequence (1,1,1,1,1) is not so obvious; the first five numbers in the shortcut Collatz sequence of 31 are odd: (31, 47, 71, 107, 161).

By combining these facts, one can observe the following. Let n and n' be two starting numbers and let $\mathbf{p}_k$ and $\mathbf{p}'_k$ be the length-k parity sequences of n and n', respectively. It is true that $\mathbf{p}_k$ equals $\mathbf{p}'_k$ if and only if n and n' differ only by a multiple of 2^k. First, if n and n' differ only by a multiple of 2^k, then we can write $n = a2^k + b$ and $n' = a'2^k + b$ and notice that since they use the same value b, the first fact tells us that they have the same parity sequence $\mathbf{p}_k = \mathbf{p}'_k$. Now, to see the other direction of the if and only if, recall that, for every parity sequence, there is a starting number that has that parity sequence. Given that there are 2^k different parity sequences of length k and 2^k many different congruency classes modulo 2^k (i.e. choices for b), it must be true that there is a bijection between parity sequences of length k and congruency classes modulo 2^k. Thus, we conclude that if n and n' have the same parity sequences $\mathbf{p}_k = \mathbf{p}'_k$, then n and n' must be equal modulo 2^k and thus n and n' only differ by a multiple of 2^k.

Now, we will approximate the shortcut Collatz function. Let's look at the example parity sequence $(1, 1, 0, 0, 1)$. After the the first step, the value of $n = 19$ has increased to $\frac{3n+1}{2} = 29$ and after the second step it has increased again to $\frac{9n+5}{4} = 44$. From there it is halved twice to $\frac{9n+5}{8} = 22$ and $\frac{9n+5}{16} = 11$. Lastly, the value increases to $\frac{27n+31}{32} = 17$. We can under-approximate the shortcut Collatz function by ignoring the constant term in the numerators.

$$\begin{array}{ccccccccccc}
n & \nearrow & \frac{3n+1}{2} & \nearrow & \frac{3^2n+5}{2^2} & \searrow & \frac{3^2n+5}{2^3} & \searrow & \frac{3^2n+5}{2^4} & \nearrow & \frac{3^3n+31}{2^5} \\
\geq & & \geq & & \geq & & \geq & & \geq & & \geq \\
n & & \frac{3n}{2} & & \frac{3^2n}{2^2} & & \frac{3^2n}{2^3} & & \frac{3^2n}{2^4} & & \frac{3^3n}{2^5}
\end{array}$$

Fig. 7. To see why the parity sequence $(1, 1, 0, 0, 1)$ is convergent, consider the prefix $(1, 1, 0, 0)$ that has $d = 2$ odd steps out of $\ell = 4$ steps. The approximate value of $s^4(n)$ is $\frac{3^2 n}{2^4}$. Since $3^2 < 2^4$, the approximate value dropped below the starting value $\frac{3^2 n}{2^4} < n$. This also tells us, that any n which follows this parity sequence has approximate glide $\tilde{g}(n) = 4$.

Since we only removed constant terms, this approximation gets more and more accurate as n gets larger and larger. We shall call a parity sequence $\mathbf{p}_k$ *convergent* if its approximate shortcut Collatz function drops below its starting value. More precisely, if there exists a prefix of $\mathbf{p}_k$ of length ℓ with d odd steps such that $3^d < 2^\ell$, then $\mathbf{p}_k$ is convergent. For a number n with a convergent parity sequence $\mathbf{p}_k$, we say that the minimum $\ell \leq k$ such that $3^d < 2^\ell$ is the *approximate glide* of n, denoted $\widetilde{g}(n)$. See an example in Fig. 7.

The approximate glide turns out to be rather accurate. Let's fix our attention on a parity sequence $\mathbf{p}_k$ that converges at the k-th step. As this sequence converges at the final step, any number n with the parity sequence $\mathbf{p}_k$ has approximate glide $\widetilde{g}(n) = k$. We will prove that for large enough numbers n with parity sequence $\mathbf{p}_k$, the glide and approximate glide are equal $g(n) = \widetilde{g}(n) = k$.

Fig. 8. The approximate glide is *so* accurate that there is no known number n such that $g(n) \neq \tilde{g}(n)$. It is conjectured that, for every n, this is always the case!

First, we will argue that the glide $g(n)$ is at least the approximate glide $\widetilde{g}(n)$. Recall that the approximation of the shortcut Collatz function is never greater than the real value of the shortcut Collatz function. If $\widetilde{g}(n) = k$, then the approximation is above n for the first $k - 1$ steps, this means that the real value of the shortcut Collatz function is also above n for the first $k - 1$ steps. Accordingly, we know that the glide $g(n)$ is at least k.

Second, we will argue that the glide $g(n)$ is at most the approximate glide $\widetilde{g}(n)$. Since $\widetilde{g}(n) = k$, we know that the k-th step of the approximation $\frac{3^d n}{2^k}$ drops below its starting value n; this means that $\frac{3^d}{2^k} < 1$. It is true that there exists a constant C such that $s^k(n) = \frac{3^d n + C}{2^k}$ (see the earlier calculations for $n = 19$, for example). Notice that the difference between the the actual value of $s^k(n)$ and the approximate value is a constant, regardless of the choice of n:

$$s^k(n) - \frac{3^d n}{2^k} = \frac{3^d n + C}{2^k} - \frac{3^d n}{2^k} = \frac{C}{2^k}.$$

Now, look at the difference between the approximate shortcut Collatz function after k steps compared to the starting value: $n - \frac{3^d n}{2^k} = n(1 - \frac{3^d}{2^k})$. Since $\frac{3^d}{2^k} < 1$, it is true that $1 - \frac{3^d}{2^k} > 0$. It is therefore true that for any constant $C' \geq 0$, there exists n large enough such that $n - \frac{3^d n}{2^k} > C'$. Therefore, for sufficiently large n:

Fig. 9. We know that for any n with parity sequence $(1, 1, 0, 0, 1)$, $s^5(n) = \frac{3^3 n + 31}{2^5}$. We are interested in calculating when $s^5(n) < n$. It is true that $n > \frac{3^3 n + 31}{2^5}$ is equivalent to $n > \frac{31}{5} = 6.2$. So, for all $n > 6.2$, the approximate glide of n is equal to the glide of n. Since 19 follows this parity sequence and $19 > 6.2$, it is true that $g(19) = \tilde{g}(19)$.

$$s^k(n) = \frac{3^d n}{2^k} + \frac{C}{2^k} < n.$$

This implies that the glide $g(n)$ equals the approximate glide $\widetilde{g}(n)$. See an example for the parity sequence for $n = 19$ in Fig. 9.

Since there is a threshold above which the glide and approximate glide are equal, we therefore know only finitely many numbers n whose glide $g(n)$ is not equal to its approximate glide $\widetilde{g}(n)$ that follow a given parity sequence $\mathbf{p}_k$.

length k	#convergent	%convergent
1	1	50
2	3	75
3	6	75
4	13	81.25
5	28	87.5
6	56	87.5
7	115	89.84
8	237	92.58
9	474	92.58
10	960	93.75
⋮	⋮	⋮
20	1 021 248	97.39
⋮	⋮	⋮
30	1 060 970 550	98.81

Fig. 10. The number and proportion (rounded to 2 decimal places) of convergent parity sequences. Note that there are 2^k parity sequences of length k.

Now, we will argue that almost all parity sequences are convergent. Fig. 10 shows the proportion of convergent parity sequences for various lengths. Notice that if a parity sequence $\mathbf{p}_k$ contains an equal number of even steps and odd steps, then it is convergent. This is because an even step multiplies n by $\frac{1}{2}$ and an odd step multiplies n by $\frac{3}{2}$; if there are k even steps and k odd steps, the result will be $\frac{3^k}{2^{2k}}n = \frac{3^k}{4^k}n < n$. One can also view this as: a uniformly randomly selected parity sequence probably converges. In fact, for a parity sequence of length k to not be convergent, it must contain at least $d \geq \frac{k}{\log_2(3)} > 0.6k$ many odd steps. That is because after $k-d$ even steps and d odd steps, the approximate value will be $\frac{3^d n}{2^k}$ and for $\frac{3^d n}{2^k} \geq n$ to hold, $d \geq \frac{k}{\log_2(3)}$ many odd steps must be taken. We can use a Chernoff bound to argue that almost all parity sequences of length k contain less than $\frac{k}{\log_2(3)}$ many odd steps. We will use a simplified version of a Chernoff bounds that states: for any $0 < \alpha \leq \frac{1}{2}$, the probability that at least $(\frac{1}{2}+\alpha)k$ out of k flips of a fair coin turn are heads is at most $e^{-2\alpha^2 k}$. Since $\frac{1}{\log_2(3)} - \frac{1}{2} > 0.1$, we can simply take $\alpha = 0.1$ and conclude that the probability that a uniformly randomly selected parity sequence is convergent is at least $1 - e^{-2(0.1^2)k} = 1 - e^{-\frac{k}{50}}$. As k tends to infinity, this probability tends to 1. We conclude that almost all parity vectors are convergent.

Finally, we are now ready wrap up the proof of Terras' theorem. We just proved that the proportion of parity sequences $\mathbf{p}_k$ that do not converge tends to zero as the length k tends to infinity. Remember that if $\mathbf{p}_k$ is convergent, then all numbers n that follow this parity sequence have finite approximate glide $\widetilde{g}(n) \leq k < \infty$. Furthermore, for a given parity sequence, there are only finitely many numbers n such that $g(n) \neq \widetilde{g}(n)$. Now, by combining these facts, we deduce that almost all numbers n satisfy $g(n) = \widetilde{g}(n) < \infty$. In other words, almost all starting numbers have finite glide.

It turns out there are some results that are stronger than Terras' theorem. There are two results which argue that the minimum value of the Collatz sequence $n, c(n), c^2(n), \ldots$ is almost always at most n^θ. Note that if $\theta < 1$, then $n^\theta < n$ and so the glide of n is finite. The first result of this kind was by Allouche (1979) with $\theta \approx 0.8691$. This was subsequently improved by Korec (1994) with $\theta \approx 0.7924$. More recently, in 2019, Terence Tao proved something similar: for any function $f : \mathbb{N} \to \mathbb{R}_+$ such that $f(n)$ tends to infinity as n tends to infinity, the minimum value of the Collatz sequence starting at n is almost always* less than $f(n)$. So pick your favourite function that grows to infinity —perhaps $\log(\log(\log(n)))$— and observe that almost all numbers n will reach a value less than $\log(\log(\log(n)))$, remarkable!

*The only downside to Tao's result is that here "almost always" is used with a logarithmic density measure, not the typical natural density measure.

Although we are "almost there", bridging the gap from Terras', Allouche's, Korec's, or Tao's theorems to prove that the Collatz conjecture is true requires substantial new approaches, insights, and ideas. It is not known whether any of aforementioned theorems can be used to prove that almost all starting numbers have finite delay. Recall that the delay is the number of steps before reaching 1 (compared to the glide which is the number of steps before reaching a number less than the starting number). Even if one could prove that almost all starting numbers have finite delay, then it would still not be clear whether all starting numbers have finite delay. In short, even if we positively prove the almost all version of the Collatz conjecture, there is still plenty of work to proving the Collatz conjecture in full. It could very much be the case that the Collatz conjecture is false and that the minimum counterexample is just about to be discovered!