CapablePtrs: Securely Compiling Partial Programs using the Pointers-as-Capabilities Principle (Technical Report)

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1 The target language (CHERIExpress)

Our target language models a platform that supports memory and object capabilities, and is strongly inspired by the CHERI system [1, 2], a MIPS-based capability-machine architecture. CHERI offers fine-grained memory capabilities through hardware support, and it offers object capabilities through a combination of hardware support, kernel support and a user-space library (libcheri).

Accordingly, we model in this section a low-level target language, which we call CHERIExpress. This language includes abstractions that mimic the interfaces offered by libcheri as well as CHERI’s capabilities. Our model of capabilities draws heavily from a prior model of a capability machine [3].

1.1 Values, expressions, and commands

Values in CHERIExpress are denoted by \( V = \mathbb{Z} \cup \text{Cap} \) and range over integers \( \mathbb{Z} \) and memory capabilities \( \text{Cap} = \{ \kappa, \delta \} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \). Memory capabilities are code or data capabilities, denoted by \( \kappa \) and \( \delta \) respectively, where the \( \kappa \)-labeled elements describe a range of the code memory \( M_c \) and an offset within this range, and the \( \delta \)-labeled elements describe the same for the data memory \( M_d \).

We separate capabilities from integers to model unforgeability of capabilities, which is a key design feature in CHERI [1, 2]. Formal arguments of how this unforgeability is guaranteed by the CHERI architecture are beyond the scope of this paper, but can be found in [3].

Definition 1 (Unforged code/data capability).

We use the judgment \( \vdash_x (y, s, e, \text{off}) \) to mean that \( y = x \) and that \( (y, s, e, \text{off}) \in \{ y \} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) which means that \( (y, s, e, \text{off}) \) is an unforged capability value of type \( x \).

Definition 2 (Valid code/data capability).

We use the judgment \( \models_x (y, s, e, \text{off}) \) to mean that \( \vdash_x (y, s, e, \text{off}) \) and that \( s + \text{off} \in [s, e) \) which is the condition necessary for valid access using this capability.

Validity of a code/data capability \((\sigma, s, e, \text{off})\) ensures that it is of the intended capability type \( x \), and that its offset lies within the legal range that it prescribes.

Definition 3 (Subset relation and disjoint capabilities).

We use the judgment \((x, s_1, e_1, ..., \_ ) \subseteq (x, s_2, e_2, ..., \_ )\) to mean \([s_1, e_1] \subseteq [s_2, e_2]\) and similarly \((x, s_1, e_1, \_ ) \cap (x, s_2, e_2, \_) = \_\) to mean that \([s_1, e_1] \cap [s_2, e_2] = \_\).

Lemma 1 (The subset and disjointness relations are offset oblivious).

\[
\forall x, \sigma_1, \epsilon_1, \sigma_2, \epsilon_2, \text{off}_1, \text{off}_2, \text{off}_1', \text{off}_2'.
\]

\[
((x, \sigma_1, \epsilon_1, \text{off}_1) \subseteq (x, \sigma_2, \epsilon_2, \text{off}_2) \implies (x, \sigma_1, \epsilon_1, \text{off}_1') \subseteq (x, \sigma_2, \epsilon_2, \text{off}_2') \)
\]

Proof.

Immediate by Definition 3.

Definition 4 (Comparing a capability to a set of addresses).

We overload the notation \( \subseteq \) to represent a relation over \( \text{Cap} \times 2^\mathbb{Z} \) between a capability and a set of integers where \((\_, s, e, \_) \subseteq X\) means that the interval \([s, e)\) of integers is a subset of \(X\) \((s, e) \subseteq X\).

Definition 5 (Membership of a capability’s address in a set of addresses).

We similarly use the set membership notation \( \in \) to mean with \((\_, s, e, \text{off}) \in X\) that the address \(s + \text{off}\) is a member in the set \(X\) of natural numbers \((i.e., s + \text{off} \in X)\).

Definition 6 (Equal-bounds capabilities).

We use the judgment \((x, \sigma_1, \epsilon_1, \_) \equiv (x, \sigma_2, \epsilon_2, \_) \) to mean that the bounds of two capabilities are the same \((i.e., the two capabilities give authority over the same range of memory addresses)\). Notice that \(a \equiv b\) is equivalent to \(a \subseteq b \cap b \subseteq a\) for any two capabilities \(a\) and \(b\).

And we define the function \( \text{inc}: \text{Cap} \times \mathbb{Z} \rightarrow \text{Cap} \) as \( \text{inc}((x, s, e, \text{off}), z) \equiv (x, s, e, \text{off} + z) \) which increments the offset of a capability by \(z\).
Memory notation

Code and data memories \((\mathcal{M}_c : \mathbb{N} \xrightarrow{\text{fin}} \text{Cmd} \text{ and } \mathcal{M}_d : \mathbb{Z} \xrightarrow{\text{fin}} \mathcal{V})\) are finite maps from addresses – that are natural numbers – to commands and values respectively. Memory values have been described above. Below we describe expressions and commands. But we first fix some notation regarding code and data memories:

- We refer to the type \(\mathbb{N} \xrightarrow{\text{fin}} \text{Cmd}\) as \CodeMemory\ and to the type \(\mathbb{Z} \xrightarrow{\text{fin}} \mathcal{V}\) as \DataMemory\.
- The operator \(\sqcup\) is used to refer to the disjoint union of sets or functions. For functions \(f\) and \(g\) with \(\text{dom}(f) \cap \text{dom}(g) = \emptyset\), the function \((f \sqcup g)\) has domain \(\text{dom}(f) \cup \text{dom}(g)\) and is defined as \((f \sqcup g)(x) \overset{\text{def}}{=} f(x)\) if \(x \in \text{dom}(f)\), and \(g(x)\) otherwise. We use the notation \(\mathcal{M}_c = \bigsqcup_i \mathcal{M}_{c_i}\) to mean the linking of several code memories \(\mathcal{M}_{c_i}\) with disjoint mapped addresses into one code memory \(\mathcal{M}_c\), and similarly for other constructs that are maps or functions.

Commands in CHERIExpress

Figure 1 shows the semantics of CHERIExpress commands. The semantics is given by the reduction relation \(\rightarrow \subseteq \text{TargetState} \times \text{TargetState}\). The reduction relation is additionally parameterized by \(\nabla \in \mathbb{Z}\) which prescribes the total amount of memory available for dynamic allocation. We omit it from the symbol \(\rightarrow\), and always write just \(\rightarrow\) for convenience. Every statement that mentions the reduction relation \(\rightarrow\) should be understood to be in the scope of one outermost universal quantification over \(\nabla\) unless otherwise is explicitly mentioned. The type \TargetState\ is defined in the section below. An auxiliary relation \(\succsim\) is used to describe the behavior of the \Invoke\ command in the case when there is enough stack space. This is useful for re-factoring and proof purposes. Commands \text{Cmd} in CHERIExpress are the following:

- **Assign** \(\mathcal{E}_L \mathcal{E}_R\) which evaluates the expression \(\mathcal{E}_R\) to a value \(v \in \mathcal{V}\), evaluates the expression \(\mathcal{E}_L\) to a data capability value \(c \in \{\delta\} \times \mathbb{Z} \times \mathbb{Z}\), and stores in the data memory \(\mathcal{M}_d\) the value \(v\) at the address indicated by \(c\) (the address \((s + o)\) for \(c = (\delta, s, e, o)\)).

- **Alloc** \(\mathcal{E}_L \mathcal{E}_{\text{size}}\) which allocates new memory and stores a data capability giving authority over the newly-allocated memory. The parameter \(\nabla\) is the first unavailable address indicating the limit of memory usage. \Alloc\ fails (i.e., execution gets stuck) if this limit is reached.

- **JumpIfZero** \(\mathcal{E}_{\text{cond}} \mathcal{E}_{\text{off}}\) is a conditional jump which evaluates the expression \(\mathcal{E}_{\text{cond}}\) to a value \(v \in \mathbb{N}\), and if \(v \neq 0\), then it evaluates the expression \(\mathcal{E}_{\text{off}}\) to an offset that is added to \(\text{pcc}\). Otherwise \((v = 0)\), nothing is done.

- **Invoke** \(\mid id fid \pi\) \(^1\), which is used to invoke an object capability. Our target platform is configured in the \imp\ component of the initial machine state, see below) with a fixed number of object capabilities identified by module identifiers \(mid \in \text{ModID}\), and each object capability supports invocation of a fixed number of functions specified by function identifiers \(fid \in \text{FunID}\). Each secure call to a function \(fid\) gets access via \(\text{stc}\) to a new data stack frame of size \(\phi(fid)\) for local use. Argument values are also written by the \Invoke\ command in this region. This latter design choice is a simpler alternative to modeling a register file.

- **CReturn**, which is used to return from a call that has been performed using \Invoke\ . The rules \invoke\ and \creturn\ in fig. 1 specify the exact operations performed to push and pop the necessary capabilities to/from the trusted stack.

\(^1\)We use the notation \(\pi\) to denote that \(\pi\) has a list type. And we also use the same notation for types (i.e., as a type constructor). For instance, we write \(\mathbb{N}\) to denote the type of lists of natural numbers.
Figure 1: Evaluation of commands \textit{Cmd} in \textit{CHERIExpress}. The reduction relation is parameterized by $\nabla$. We omit it from the symbol $\rightarrow$ for convenience.

\[
\begin{align*}
\text{(assign)} & \quad \vdash_{\kappa} \text{pcc} \quad \text{pcc'} = \text{inc}(\text{pcc}, 1) \\
\mathcal{M}_c(\text{pcc}) & = \text{Assign} \mathcal{E}_L \mathcal{E}_R \\
\mathcal{E}_L, \mathcal{M}_d, \text{ddc}, \text{stc}, \text{pcc} \downarrow v & \quad \vdash_{\delta} c \quad \vdash_{\delta} v \quad \implies (v \cap \text{stc} = 0 \lor c \subseteq \text{stc}) \\
\mathcal{M}'_d & = \mathcal{M}_d[c \mapsto v] \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{M}_c(\text{pcc}) & = \text{Alloc} \mathcal{E}_L \mathcal{E}_{\text{size}} \\
\mathcal{E}_{\text{size}}, \mathcal{M}_d, \text{ddc}, \text{stc}, \text{pcc} \downarrow v & \quad \vdash_{\delta} c \quad \mathcal{M}'_d = \mathcal{M}_d[c \mapsto \{\delta, \text{nalloc} - v, \text{nalloc}, 0\}, \forall i \mapsto \forall i \in [\text{nalloc} - v, \text{nalloc}]] \\
\text{nalloc}' & = \text{nalloc} - v \quad \text{nalloc}' > \nabla \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{M}_c(\text{pcc}) & = \text{JumpIfZero} \mathcal{E}_{\text{cond}} \mathcal{E}_{\text{off}} \\
\mathcal{E}_{\text{off}}, \mathcal{M}_d, \text{ddc}, \text{stc}, \text{pcc} \downarrow \text{off} & \quad \text{off} \in \mathbb{Z} \\
\mathcal{E}_c & = \text{inc}(\text{pcc}, \text{off}) \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{M}_c(\text{pcc}) & = \text{Cinvoke} \quad \text{mid}, \text{fid} \quad \text{stc'} = \text{push}(\text{stk}, (\text{ddc}, \text{pcc}, \text{mid}, \text{fid})) \\
\phi(\text{mid}, \text{fid}) & = (\text{nArgs}, \text{nLocal}) \\
\mathcal{M}_d & = \text{stk}[s + \text{off} + i \mapsto v_i \forall i \in [0, \text{nArgs})] (s + \text{off} + \text{nArgs} + i \mapsto \forall i \in [0, \text{nLocal}]) \\
\text{mstc}' & = \text{mstc}[\text{mid} \mapsto \text{stc}'] \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{M}_c(\text{pcc}) & = \text{Cret} \quad \text{stk}', (\text{ddc'}, \text{pcc'}, \text{mid}, \text{fid}) = \text{pop}(\text{stk}) \\
\phi(\text{mid}, \text{fid}) & = (\text{nArgs}, \text{nLocal}) \\
\text{mstc} & = \text{mstc}[\text{mid} \mapsto (\delta, s, e, \text{off'})] \\
\exists \text{mid}'. \quad \text{pcc}' & = \text{inc}(\text{mid}'. \text{pcc} \land \text{stc} = \text{mstc}[\text{mid}']) \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{M}_c(\text{pcc}) & = \text{Exit} \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{M}_c(\text{pcc}) & = \text{Exit} \\
\end{align*}
\]
CHERIExpress program state

A state \( \langle M_c, M_d, stk, imp, \phi, ddc, stc, pcc, mstc, nalloc \rangle \) of a program in CHERIExpress consists of:

- code and data memories, \( M_c \) and \( M_d \) as defined earlier (We define \( M_d(\delta, s, e, o) \) \( \overset{\text{def}}{=} M_d(s + o) \), and similarly for update expressions and for \( M_c \) with \( \kappa \)-labeled values. We also (ab)use the set membership notation \( (_-, s, _, \text{off}) \in X \) for \( X \subseteq \mathbb{N} \) to mean \( s + \text{off} \in X \). We use it to say that the capability points to an address within a certain range of addresses, say \( pcc \in \text{dom}(M_c) \)),

- a trusted call stack \( stk : \text{Cap} \times \text{Cap} \times \text{ModID} \times \text{FunID} \), which is a list of 4-tuples; each tuple consists of two capabilities, a module ID, and a function ID. The trusted call stack stores the history of the values of \( ddc, pcc \) at the call locations. It also stores the identifier of the function (and module) that is being called. The storing of the function identifier allows us to build into the target language an assumption that it implements safe management of the data part of the stack frames.

- a map of imports \( \text{imp} : \text{ModID} \rightarrow \text{CapObj} \) that for each module identifier, keeps an object capability \( \text{CapObj} = (\{\kappa\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}) \times (\{\delta\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}) \times (\text{FunID} \rightarrow \mathbb{N}) \). An object capability consists of
  
  - a code capability that grants access to the module’s code region in \( M_c \),
  - a data capability that grants access to the module’s data region in \( M_d \),
  - an offsets map, that for each function identifier in the module, specifies the offset within the module’s code memory at which the function’s code starts (i.e., this map of offsets describes the legitimate entry points to the module).

- a map of call frame sizes \( \phi : (\text{ModID} \times \text{FunID}) \rightarrow (\mathbb{N} \times \mathbb{N}) \) that for each function (given by the module identifier and the function identifier) gives the number of arguments and the number of local variables that this function allocates.

- three capability registers/variables:
  
  - \( ddc : \{\delta\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z} \), the data capability (which specifies the region in the data memory \( M_d \) that is private to the active module),
  - \( stc : \{\delta\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z} \), the stack-data capability (which specifies the region in the data memory \( M_d \) that corresponds to the current activation record),
  - and \( pcc : \{\kappa\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z} \), the program counter capability (which specifies the region in the code memory \( M_c \) in which the currently-executing module is defined),

- a map \( \text{mstc} : \text{ModID} \rightarrow \text{Cap} \) that for each module identifier keeps the most recent value of its stack capability. This value is managed by the trusted Cinvoke and Creturn commands. The map records the most recent update to the \( stc \) capability. Updates to \( \text{mstc} \) made done by only the two commands Cinvoke and Creturn.

- a marker \( \text{nalloc} : \mathbb{Z} \) that holds the first non-allocated address in \( M_d \) in the direction of growth of the heap (i.e., the dynamically-allocated segment of \( M_d \)).

The type of CHERIExpress program states is denoted by \( \text{TargetState} = \text{CodeMemory} \times \text{DataMemory} \times (\text{Cap} \times \text{Cap} \times \text{Cap}) \times (\text{ModID} \rightarrow \text{CapObj}) \times ((\text{ModID} \times \text{FunID}) \rightarrow (\mathbb{N} \times \mathbb{N})) \times (\{\delta\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}) \times (\{\delta\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}) \times (\{\kappa\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}) \times (\text{ModID} \rightarrow \text{Cap}) \times \mathbb{Z} \).

It is worth noting that the map of imports \( \text{imp} \), and the code memory \( M_c \) are fixed at load time, and their contents are not modified by any instruction.

**Lemma 2** (Reduction does not change call frame sizes, imports map or code memory).
\[
\forall s, s'. s \rightarrow^* s' \implies (s.\phi = s'.\phi \land s.\text{imp} = s'.\text{imp} \land s.M_c = s'.M_c)
\]
Proof. By induction on the reduction steps and inspecting the rules of Figure 1.

Lemma 3 (A reduction is enabled only on a valid program counter).

∀s. (∃s’. s → s’ ∨ s ≈ s’) \implies \vdash_{\kappa} s.pcc

Proof. By inversion of s → s’ (resp. s ≈ s’).

Definition 7 (Code region of an imports map).

\text{code\_region} : (\text{ModID} \to \text{CapObj}) \to 2^{\mathbb{Z}}

\text{code\_region}(imp) \overset{\text{def}}{=} \bigcup_{\text{mid} \in \text{dom}(imp)} \text{imp(mid).pcc.} \sigma, \text{imp(mid).pcc.e}

The syntax of the language enables the use of capabilities that are expressible in terms of two distinguished names, “ddc”, and “stc” denoting data capability, and stack capability respectively. Notice that the program counter capability register is not addressable. Instead, the jump instruction can only increment the offset of the capability value in that register. Effectively, there is no way for code capabilities to live in memory. This is proved in Lemma 52.

Expressions in CHERIExpress are denoted by the grammar

\[ \mathcal{E} ::= \]
\[ Z \]
\[ | \text{ddc} \]
\[ | \text{stc} \]
\[ | \text{inc}(\mathcal{E}, \mathcal{E}) \]
\[ | \text{deref}(\mathcal{E}) \]
\[ | \text{lim}(\mathcal{E}, \mathcal{E}, \mathcal{E}) \]
\[ | \text{capType}(\mathcal{E}) \]
\[ | \text{capStart}(\mathcal{E}) \]
\[ | \text{capEnd}(\mathcal{E}) \]
\[ | \text{capOff}(\mathcal{E}) \]
\[ | \mathcal{E} \oplus \mathcal{E} \]

where \( \oplus ::= + | - | * \), and \( \mathbb{Z} \) is the set of integers. The forms ddc and stc are the distinguished names for the corresponding capabilities. An expression \( \text{inc}(\mathcal{E}, Z) \) increments the offset of a capability value. An expression \( \text{deref}(\mathcal{E}) \) evaluates to the value at the memory address pointed to by a capability only if it is a valid capability according to Definition 2. The expression \( \text{lim}(\mathcal{E}, \mathcal{E}, \mathcal{E}) \) evaluates to a shrunk copy of the capability given by its first argument. The second and third arguments determine the new range of memory prescribed by the shrunk copy. The expressions \( \text{capType}(\mathcal{E}) \), \( \text{capStart}(\mathcal{E}) \), \( \text{capEnd}(\mathcal{E}) \), and \( \text{capOff}(\mathcal{E}) \) select the corresponding fields of the capability value given by evaluating their argument expression. The evaluation of expressions \( \mathcal{E} \) to values \( V \) is given by rules of the form \( \mathcal{E}, \mathcal{M}_d, \text{ddc}, \text{stc}, \text{pcc} \Downarrow V \) listed in fig. 2.

Lemma 4 (Expression evaluation cannot forge code capabilities).

\[ \forall a, s, \mathcal{E}. \]
\[ s.\text{ddc} \notin \{\kappa\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \land s.\text{stc} \notin \{\kappa\} \times \mathbb{Z} \times \mathbb{Z} \]
\[ \land (s.\mathcal{M}_d(a) \neq (\kappa, \sigma_{a}, e_{a}, _{a}) \land \mathcal{E}, s.\mathcal{M}_d, s.\text{ddc}, s.\text{stc}, s.\text{pcc} \Downarrow v \]
\[ \implies v \neq (\kappa, _{a}, _{a}, _{a}) \]
Figure 2: Evaluation of expressions $\mathcal{E}$ in CHERIExpress

<table>
<thead>
<tr>
<th>(evalconst)</th>
<th>(evalddc)</th>
<th>(evalstc)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \in \mathbb{Z}$, $\mathcal{M}_d$, ddc, stc, pcc $\downarrow n$</td>
<td>ddc, $\mathcal{M}_d$, ddc, stc, pcc $\downarrow$ ddc</td>
<td>stc, $\mathcal{M}_d$, ddc, stc, pcc $\downarrow$ stc</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(evalCapType)</th>
<th>(evalCapStart)</th>
<th>(evalCapEnd)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{E}, \mathcal{M}_d, \text{ddc, stc, pcc} \downarrow v$</td>
<td>$\mathcal{E}, \mathcal{M}_d, \text{ddc, stc, pcc} \downarrow v$</td>
<td>$\mathcal{E}, \mathcal{M}_d, \text{ddc, stc, pcc} \downarrow v$</td>
</tr>
<tr>
<td>$v \in {\kappa} \times \mathbb{Z} \times \mathbb{Z} \Rightarrow v' = 0$</td>
<td>$v \in {\delta} \times \mathbb{Z} \times \mathbb{Z} \Rightarrow v' = 1$</td>
<td>$v \in {\delta} \times \mathbb{Z} \times \mathbb{Z} \Rightarrow v' = 2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(evalCapOff)</th>
<th>(evalBinOp)</th>
<th>(evalLim)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{E}, \mathcal{M}_d, \text{ddc, stc, pcc} \downarrow v$</td>
<td>$\mathcal{E}_1, \mathcal{M}_d, \text{ddc, stc, pcc} \downarrow v_1$</td>
<td>$\mathcal{E}, \mathcal{M}_d, \text{ddc, stc, pcc} \downarrow v$</td>
</tr>
<tr>
<td>$v = (x, s, e, _)$</td>
<td>$v = (x, s, e, _)$</td>
<td>$v = (x, s, e, _)$</td>
</tr>
<tr>
<td>$v' = s$</td>
<td>$v' = v_1 \oplus v_2$</td>
<td>$v' = M_d(s + \text{off})$</td>
</tr>
<tr>
<td>$\mathcal{E}_1 \oplus \mathcal{E}_2, \mathcal{M}_d, \text{ddc, stc, pcc} \downarrow v'$</td>
<td>$\mathcal{E}_2 \oplus \mathcal{M}_d, \text{ddc, stc, pcc} \downarrow v_2$</td>
<td>$\mathcal{E}, \mathcal{M}_d, \text{ddc, stc, pcc} \downarrow v$</td>
</tr>
<tr>
<td>$v_2 \in \mathbb{Z}$</td>
<td></td>
<td>$v = (x, s, e, _)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$v' = v_2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(evalDeref)</th>
<th>(evalLim)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{E}, \mathcal{M}_d, \text{ddc, stc, pcc} \downarrow v$</td>
<td>$\mathcal{E}_s, \mathcal{M}_d, \text{ddc, stc, pcc} \downarrow s'$, $\mathcal{E}_e, \mathcal{M}_d, \text{ddc, stc, pcc} \downarrow e'$</td>
</tr>
<tr>
<td></td>
<td>$s', e' \in \mathbb{Z}$</td>
</tr>
<tr>
<td></td>
<td>$v = (x, s, e, _)$</td>
</tr>
<tr>
<td></td>
<td>$v' = (x, s', e', s)$</td>
</tr>
<tr>
<td></td>
<td>$v' = (x, s', e', 0)$</td>
</tr>
</tbody>
</table>
Proof.
Easy by induction on the evaluation $E, s.M_d, s.ddc, s.stc, s.pcc \Downarrow v$.

1.2 Target setup, and initial and terminal states

Having defined the program state, we now define a target setup

$$ TargetSetup \equiv CodeMemory \times DataMemory \times (ModID \to CapObj) \times (ModID \to Cap) \times ((ModID \times FunID) \to (N \times N)) $$

as a tuple of code memory, data memory, imports map, stack capabilities map, and call-frame-sizes map.

Definition 8 (Disjoint object capabilities).
For $c, c' \in CapObj$, $c \cap c' = \emptyset \equiv c.1 \cap c'.1 = \emptyset \land c.2 \cap c'.2 = \emptyset$ where disjointness of capabilities is as in Definition 3.

We hence define the linking

$\times : TargetSetup \to TargetSetup \to \text{Option}(TargetSetup)$ of two target setups $t_1$ and $t_2 \in TargetSetup$ as follows:

Definition 9 (Valid Linking). Valid linking of $t_1, t_2 \in TargetSetup$ is the component-wise disjoint union of code memories $t_1.M_c, t_2.M_c$, data memories $t_1.M_d, t_2.M_d$, imports maps $t_1.imp, t_2.imp$, and call-frame-sizes maps $t_1.\phi, t_2.\phi$ under the well-formedness conditions given by the rule valid-linking in Figure 3.

Design choices for linking

The disjointness conditions on the address ranges and on the capability ranges in rule valid-linking are not surprising. But notice the non-commutativity of the valid linking operator $\times$. The linking operator is designed to be aware of the context. All the context (i.e., untrusted) modules should be put on the left-hand side of $\times$. The right-hand side operand should include all and only the trusted modules (if any). In case only untrusted modules are being linked, the order does not really matter.

There are two noteworthy design choices here that cause the linking operator $\times$ to be non-commutative. They are expressed by the two conditions $\max(\text{dom}(M_{c1})) < \min(\text{dom}(M_{c2}))$ and $\min(\text{dom}(M_{d1})) > \max(\text{dom}(M_{d2}))$ of the rule valid-linking. The first of these conditions is a necessary security measure, while the second condition is required only as an artifact of our security proof techniques. The first condition ensures that the code memory segment of the context is always placed before the code memory segment of the trusted/compiled program. This ensures hiding (away from the context) information about the size of the code segment of the trusted program. The second condition ensures a reverse order on the data segments of the context and the program. This is a restriction that is required only as a result of our proof technique. In particular, we want to avoid reasoning about the scenario where the data layout of the program is shifted by a fixed amount of memory. The reason is that this places an unnecessary restriction on the way we have to construct a distinguishing context for two programs that we know are distinguishable.

An initial state of a CHERIExpress program is one where the trusted stack is empty, the free memory marker captures the correct amount of dynamically-allocated memory (i.e., zero memory consumption), and the main function is about to start execution (the local stack of the main module contains the corresponding frame). We refer to a state $s$ that is initial for setup $t$ as $t \vdash_i s$.

Definition 10 (Initial state). A state $s$ is initial for a target setup $t$ (written $t \vdash_i s$) iff the preconditions described by rule initial-state in Figure 3 hold.
Figure 3: Valid linking of two TargetSetup’s – Initial state of a TargetSetup – Execution state invariant

(valid-program)

\[ t = (\mathcal{M}_c, \mathcal{M}_d, \text{imp}, \text{mstc}_t, \phi) \]
\[ \text{modIDs} = \text{dom}(\text{imp}) = \text{dom}(\text{mstc}_t) \]
\[ \forall \text{mid} \in \text{modIDs}. \text{valid} = \text{imp}(	ext{mid}) \& \text{ddc} \& \text{mstc} \]
\[ \text{dom}(\mathcal{M}_c) = \bigcup_{\text{mid} \in \text{modIDs}} [\text{imp}(	ext{mid}).\text{ddc}, \sigma, \text{imp}(	ext{mid}).\text{ddc}.e] \]
\[ \text{dom}(\mathcal{M}_d) = \bigcup_{\text{mid} \in \text{modIDs}} [\text{imp}(	ext{mid}).\text{ddc}, \sigma, \text{imp}(	ext{mid}).\text{ddc}.e] \cup [\text{mstc}_t(mid), \sigma, \text{mstc}_t(mid).e] \]
\[ \forall \text{funIDs} = [\text{fid} | \text{fid} \in \text{dom}(\text{imp}(	ext{mid}).\text{offs}) \& \text{mid} \in \text{modIDs}] \]
\[ \text{all}_\text{distinct}(\text{funIDs}) \]
\[ \text{dom}(\phi) = \{(\text{mid}, \text{fid}) | \text{fid} \in \text{dom}(\text{imp}(	ext{mid}).\text{offs}) \& \text{mid} \in \text{modIDs} \} \]
\[ \vdash_{\text{valid}} t \]

(valid-linking)

\[ \forall i \in \{1, 2\}, \text{i}_i = (\mathcal{M}_{c_i}, \mathcal{M}_{d_i}, \text{imp}_i, \text{mstc}_i, \phi_i) \& \vdash_{\text{valid}} \text{i}_i \]
\[ t = (\mathcal{M}_c \cup \mathcal{M}_{d_1} \cup \mathcal{M}_{d_2}, \text{imp}_1 \cup \text{imp}_2, \text{mstc}_1 \cup \text{mstc}_2, \phi_1 \cup \phi_2) \]
\[ \text{min(dom}(\mathcal{M}_{d_1})) > \text{max(dom}(\mathcal{M}_{d_2})) \]
\[ \vdash_{\text{valid}} t \]

(initial-state)

\[ t_1 \bowtie t_2 = [t] \]
\[ \vdash_{\text{valid}} t \]
\[ s = (\mathcal{M}_c, \mathcal{M}_d, \text{stk}, \text{imp}, \phi, \text{ddc}, \text{stk}, \text{pcc}, \text{mstc}, \text{nalloc}) \]
\[ \text{pcc} = (\kappa, p, \sigma, p, \text{offs}(\text{main})) \]
\[ \text{ddc} = d \]
\[ \text{mainMod} = (\text{main}, \phi) \]
\[ n\text{Alloc} = -1 \]
\[ \vdash_{\text{valid}} t \]

(exec-state)

\[ t = (\mathcal{M}_c, \mathcal{M}_d, \text{imp}, \text{mstc}_t, \phi) \]
\[ s = (\mathcal{M}_c, \mathcal{M}_d, \text{stk}, \text{imp}, \phi, \text{ddc}, \text{stk}, \text{pcc}, \text{mstc}, \text{nalloc}) \]
\[ \vdash_{\text{valid}} t \]
\[ \vdash_{\text{valid}} \text{pcc} \]
\[ \vdash_{\text{valid}} \text{ddc} \]
\[ \text{modIDs} = \text{dom}(\text{imp}) = \text{dom}(\text{mstc}) = \text{dom}(\text{mstc}_t) \]
\[ \forall \text{mid} \in \text{modIDs}. \text{mstc} \]
\[ \forall \text{mid} \in \text{modIDs}. \text{mstc} \]
\[ \forall \text{mid} \in \text{modIDs}. \text{ddc} \]
\[ \text{reachable-addresses}(\bigcup_{\text{mid} \in \text{modIDs}} [\text{imp}(	ext{mid}).\text{ddc}, \sigma, \text{imp}(	ext{mid}).\text{ddc}.e] \cup [\text{mstc}(\text{mid}), \sigma, \text{mstc}(\text{mid}).e]) \cup [\text{nalloc}, -1] \]
\[ \forall \text{mid}, a. a \in \text{reachable-addresses}(\{\text{mstc}(\text{mid}), \text{imp}(	ext{mid}).\text{ddc}\}, \mathcal{M}_d) \implies \]
\[ a \notin \bigcup_{\text{mid} \in \text{modIDs} \setminus \{\text{mid}\}} [\text{mstc}(\text{mid}).\sigma, \text{mstc}(\text{mid}).e] \]
\[ \forall a, \text{mid} \in \text{modIDs}. \text{M}_d(a) = (\delta, \sigma, e, _) \& \sigma, e \leq \text{mstc} \implies a \in [\text{mstc}(\text{mid}).\sigma, \text{mstc}(\text{mid}).e] \]
\[ \forall a. \text{M}_d(a) \neq (\kappa, \sigma, e, _) \implies [\sigma, e] \subseteq \text{dom}(\mathcal{M}_d) \]
\[ \text{stk} \neq \text{nil} \implies \text{pcc} = \text{imp}(\text{top}(\text{stk}), \text{mid}).\text{pcc} \]
\[ \forall i \in [1, \text{length}(	ext{stk}) - 1]. \text{stk}(i).\text{pcc} = \text{imp}(\text{stk}(i - 1), \text{mid}).\text{pcc} \]
\[ \vdash_{\text{exec}} s \]
Definition 11 (Initial state function).

\[ \text{initial\_state}(t, \text{mainMod}) \overset{\text{def}}{=} \{ \]
\[ t.M_c, \]
\[ \{ a \mapsto 0 \mid a \in \text{dom}(t.M_d) \}, \]
\[ \text{nil}, \]
\[ t.\text{imp}, \]
\[ t.\phi, \]
\[ t.\text{imp}(\text{mainMod}).\text{ddc}, \]
\[ (\delta, t.\text{mstc}(\text{mainMod}).\sigma, t.\text{mstc}(\text{mainMod}).e, t.\phi(\text{mainMod}, \text{main}).nAr g + t.\phi(\text{mainMod}, \text{main}).nLocal), \]
\[ t.\text{imp}(\text{mainMod}).\text{pcc}, \]
\[ \{ \text{mid} \mapsto (\delta, t.\text{mstc}(\text{mid}).\sigma, t.\text{mstc}(\text{mid}).e, 0) \mid \text{mid} \in \text{dom}(t.\text{mstc}) \setminus \{ \text{mainMod} \} \} \] \]
\[ \{ \text{mainMod} \mapsto (\delta, t.\text{mstc}(\text{mainMod}).\sigma, t.\text{mstc}(\text{mainMod}).e, t.\phi(\text{mainMod}, \text{main}).nAr g + t.\phi(\text{mainMod}, \text{main}).nLocal) \}, \]
\[ -1 \} \}
\]

Definition 12 (Main module).

\[ \text{main\_module}(t) = \text{mid} \iff \text{main} \in \text{dom}(t.\text{imp}(\text{mid}).\text{offs}) \]

Claim 1 (The function \text{initial\_state} and the judgment \( \vdash \) are compatible).

\[ \forall t, s, \text{mainMod}. \]
\[ \text{initial\_state}(t, \text{mainMod}) = s \land \]
\[ \vdash \text{valid} t \land \]
\[ \text{main} \in \text{dom}(t.\text{imp}(\text{mainMod}).\text{offs}) \]
\[ \implies \]
\[ t \vdash s \]

Proof. Follows easily after unfolding the assumptions using Definition 11, and inversion of the goal using rule initial-state. \qed

Definition 13 (Terminal state).

A program state \( s = (M_c, M_d, stk, imp, \phi, ddc, stc, pcc, mstc, nalloc) \) is \text{terminal}, written \( \vdash_t s \) iff \( M_c(\text{pcc}) = \text{Exit} \).

Definition 14 (Addition of an offset \( \omega \) to the data memory).

\[ M_d + \omega \overset{\text{def}}{=} \{ a + \omega \mapsto M_d(a) \mid a \in \text{dom}(M_d) \} \]

Definition 15 (Addition of an offset \( \omega \) to the \text{imp} map).

\[ \overset{\text{def}}{=} \{ \text{mid} \mapsto (\text{pcc}, (\delta, ddc.\sigma + \omega, ddc.e + \omega, ddc.offs), offs) \mid (\text{mid} \mapsto (\text{pcc}, ddc, offs)) \in \text{imp} \} \]

Definition 16 (Addition of an offset \( \omega \) to a program \( t \)).

\[ t + \omega \overset{\text{def}}{=} (t.M_c, t.M_d + \omega, t.\text{imp} + \omega, t.mstc, t.\phi) \]

Given two target setups \( t_1, t_2 \in \text{TargetSetup} \), we write \( t_1[t_2] \downarrow \) (convergence) to mean that \( t_1 \sqcap t_2 \) is defined, that there is at least one valid initial state, and that for all possible initial states, there is a reduction to a terminal state.
Definition 17 (Linkability, loadability, and convergence of execution in the target language).

\[
\nabla \vdash C[t_1] \Downarrow \overset{\text{def}}{=} \exists t', C \times t_1 = [t'] \land \\
\exists s_t, \text{initial\_state}(t', \text{main\_module}(t')) \rightarrow s_t \land \vdash s_t
\]

Definition 18 (Target contextual equivalence).

\[
t_1 \simeq \nabla t_2 \overset{\text{def}}{=} \forall C, \nabla \vdash C[t_1] \Downarrow \iff \nabla \vdash C[t_2] \Downarrow
\]

Definition 19 (Valid execution state). A state \(s\) is a valid execution state for a target setup \(t\) (written \(t \vdash_{\text{exec}} s\)) iff the preconditions described by rule \(\text{exec\_state}\) in Figure 3 hold.

Lemma 5 (Initial states are valid execution states). \(\forall t, s. t \vdash i s \Rightarrow t \vdash_{\text{exec}} s\)

We skip the details here. By inversion of our goal using \(\text{exec\_state}\), all subgoals follow easily from preconditions of the rule \(\text{initial\_state}\).

1.3 Memory Reachability

Definition 20 (Accessible addresses).

\[
\text{access}_{M_d} : \mathbb{2}^\mathbb{Z} \rightarrow \mathbb{2}^\mathbb{Z}
\]

\[
\text{access}_{M_d} A \overset{\text{def}}{=} A \cup \bigcup_{a \in A, M_d(a) = (\delta, s, e, _)} [s, e)
\]

Definition 21 (k-accessible addresses).

\[
\text{access}_{0, M_d} A \overset{\text{def}}{=} A
\]

\[
\text{access}_{k+1, M_d} \overset{\text{def}}{=} \text{access}_{M_d}(\text{access}_{k, M_d} A)
\]

Definition 22 (Reachable addresses).

\[
\text{reachable\_addresses} : (2^{\delta} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z} \times \text{DataMemory}) \rightarrow \mathbb{2}^\mathbb{Z}
\]

\[
\text{reachable\_addresses}(C, M_d) \overset{\text{def}}{=} \bigcup_{k \in [0, |M_d|]} \text{access}_{k, M_d}(\bigcup_{c \in C} [c, s, c, e])
\]

\[
\text{reachable\_addresses\_closure} : (\mathbb{2}^\mathbb{Z} \times \text{DataMemory}) \rightarrow \mathbb{2}^\mathbb{Z}
\]

\[
\text{reachable\_addresses\_closure}(A, M_d) \overset{\text{def}}{=} \bigcup_{k \in [0, |M_d|]} \text{access}_{k, M_d} A
\]

Lemma 6 (Reachability is not affected by offsets, only bounds).

\[
\forall c, M_d, c'. c \equiv c' \implies \text{reachable\_addresses}(\{c\}, M_d) = \text{reachable\_addresses}(\{c'\}, M_d)
\]

Proof. Immediate by Definitions 6 and 22. \(\Box\)

Lemma 7 (\(\text{access}_{M_d}\) is expansive).

\[
\forall A, M_d. \text{access}_{M_d} A \supseteq A
\]

Proof. Immediate by Definition 20 and the reflexivity of \(\supseteq\). \(\Box\)

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Lemma 8 (access_{n,M_d} is expansive).

$$\forall n, A, M_d. \text{access}_{n,M_d} A \supseteq A$$

Proof. We prove it by induction on $n$.

- **Base case $n = 0$:**
  Immediate by Definition 21; access_{0,M_d} A = A \supseteq A.

- **Inductive case:**
  Assuming for an arbitrary $k$ that $\forall A. \text{access}_{k,M_d} A \supseteq A$, we show for an arbitrary $B$ that access_{k+1,M_d} B \supseteq B.
  By Definition 21, our goal becomes access_{M_d}(access_{k,M_d} B) \supseteq B.
  But by assumption (the induction hypothesis), we have by universal instantiation that access_{k,M_d} B \supseteq B.
  And by Lemma 7, we have access_{M_d}(access_{k,M_d} B)) \supseteq access_{k,M_d} (B).
  So, by transitivity of $\supseteq$, we have our goal.

Lemma 9 (Fixed points lead to convergence of access_{k,M_d}).

$$\forall k, M_d, A. k > 0 \implies (\text{access}_{k,M_d} A = A \implies \text{access}_{k+1,M_d} A = A)$$

Proof.

- We fix arbitrary $k, A, M_d$ and assume both antecedents.
- By the assumptions and Definition 21, we have (*):
  $A = \text{access}_{M_d}(\text{access}_{k-1,M_d} A)$.
- Then by expansiveness of access_{M_d} (Lemma 7), we obtain:
  $A = \text{access}_{M_d}(\text{access}_{k-1,M_d} A) \supseteq \text{access}_{k-1,M_d} A$.
- We also have by expansiveness of access_{k-1,M_d} (Lemma 8) that:
  $A = \text{access}_{M_d}(\text{access}_{k-1,M_d} A) \supseteq access_{k-1,M_d} A \supseteq A$.
- Thus, we conclude:
  access_{k-1,M_d} A = A.
- We substitute this equality in (*) to get (**):
  access_{M_d} A = A.
- Our goal is to show the consequent of the lemma statement: access_{k+1,M_d} A = A.
- By Definition 21, our goal becomes access_{M_d}(access_{k,M_d} A) = A.
- And by the assumption access_{k,M_d} A = A, our goal becomes access_{M_d} A = A.
- But this goal is exactly statement (**) that we already obtained above.

Lemma 10 (In an empty memory, only the starting addresses are reachable).

$$\forall C, M_d. \left( (\forall v \in \text{range}(M_d) \implies v \neq (\delta, \_,-,\_)) \implies \text{reachable_addresses}(C, M_d) = \bigcup_{c \in C} [c.\sigma, c.e] \right)$$
Proof. Immediate by Definitions 20 to 22.

Lemma 11 (k-accessibility either adds a new memory address or a fixed point has been reached).

\[ \forall k, A, M_d. \ k > 0 \implies \ \text{access}_{k+1, M_d} A \supseteq \text{access}_{k, M_d} A \implies \ (\exists a. \ a \in \text{dom}(M_d) \land a \in \text{access}_{k, M_d} A \setminus \text{access}_{k-1, M_d} A) \]

Proof. We fix arbitrary \( k, A \) and \( M_d \), and we assume both antecedents.

- By Definitions 20 and 21, we have from the assumption that:
  \[ \text{access}_{k, M_d} A \cup \bigcup_{a \in \text{access}_{k, M_d} A} [s, e) \supseteq \text{access}_{k, M_d} A \]

- So the set
  \[ \bigcup_{a \in \text{access}_{k, M_d} A} [s, e) \neq \emptyset, \text{ and in particular:} \]

\[ (\exists a, a'. \ a \in \text{access}_{k, M_d} A \land M_d(a) = (\delta, s, e, \_)) \land a' \in [s, e) \land a' \notin \text{access}_{k, M_d} A. \]

- Suppose for the sake of contradiction that \( a \in \text{access}_{k-1, M_d} A. \)
  
  - By Definitions 20 and 21, we know that
    \[ (**) \text{access}_{k, M_d} A = \text{access}_{k-1, M_d} A \cup \bigcup_{a \in \text{access}_{k-1, M_d} A} [s, e) \]

  - From (**), we know that our obtained \( a \) satisfies \( M_d(a) = (\delta, s, e, \_)) \) and that our \( a' \) satisfies \( a' \in [s, e) \).

  - Thus, we conclude that \( a' \in \bigcup_{a \in \text{access}_{k-1, M_d} A} [s, e) \).

  - Thus by (**), \( a' \in \text{access}_{k, M_d} A. \) But this contradicts conjunct \( a' \notin \text{access}_{k, M_d} A \) of (**).

- Thus, necessarily \( a \notin \text{access}_{k-1, M_d} A. \)

- Thus, the obtained \( a \) from (**) satisfies our goal:
  \[ a \in \text{dom}(M_d) \land a \in \text{access}_{k, M_d} A \setminus \text{access}_{k-1, M_d} A. \]

Lemma 12 (k-accessibility set contains at least k mapped addresses).

\[ \forall k, A, M_d. \ \text{access}_{k+1, M_d} A \supseteq \text{access}_{k, M_d} A \implies \ |\{a | a \in \text{access}_{k, M_d} A \land a \in \text{dom}(M_d)\}| > k \]

Proof. We fix arbitrary \( A \) and \( M_d. \)

We prove it by induction on \( k. \)

- Base case \((k = 0):\)
  Our goal is: \(|\{a | a \in \text{access}_{0, M_d} A \land a \in \text{dom}(M_d)\}| > 0.\)

  We have by assuming the antecedent that \( \text{access}_{1, M_d} A \supseteq \text{access}_{0, M_d} A. \)

  By Definitions 20 and 21, this simplifies to \( A \cup \bigcup_{a \in A, M_d(a) = (\delta, s, e, \_)} [s, e) \supseteq A. \)

  Thus, \( \exists a, a'. \ a \in A \land M_d(a) = (\delta, s, e, \_) \land a' \in [s, e). \)

  Thus, the set \( \{a | a \in A \land a \in \text{dom}(M_d)\} \neq \emptyset. \)

  By Definition 21, we substitute \( A \) by \( \text{access}_{0, M_d} A \) to get our goal:
  \( |\{a | a \in \text{access}_{0, M_d} A \land a \in \text{dom}(M_d)\}| > 0, \text{i.e.,} \)
  \[ |\{a | a \in \text{access}_{0, M_d} A \land a \in \text{dom}(M_d)\}| > 0. \]
• **Inductive case** ($k > 0$):

Here, we have by the inductive hypothesis:

(*) $\text{access}_{k,M_d}A \supseteq \text{access}_{k-1,M_d}A \implies |\{a \mid a \in \text{access}_{k-1,M_d}A \land a \in \text{dom}(M_d)\}| > k - 1$

We have by assuming the antecedent that $\text{access}_{k+1,M_d}A \supseteq \text{access}_{k,M_d}A$.

Thus by Lemma 11, we have that:

(**) $\exists a^*. a^* \in \text{dom}(M_d) \land a^* \in \text{access}_{k,M_d}A \setminus \text{access}_{k-1,M_d}A$.

The latter gives us by the definition of $\supseteq$ that $\text{access}_{k,M_d}A \supseteq \text{access}_{k-1,M_d}A$.

Thus, by instantiating the induction hypothesis (*), we get:

(***) $|\{a \mid a \in \text{access}_{k-1,M_d}A \land a \in \text{dom}(M_d)\}| > k - 1$.

We rewrite it as: (***) $|\{a \mid a \in \text{access}_{k-1,M_d}A \land a \in \text{dom}(M_d)\}| \geq k$

But by (**), we already also obtained $a^*$ with:

$a^* \in \text{dom}(M_d) \land a^* \in \text{access}_{k,M_d}A \setminus \text{access}_{k-1,M_d}A$.

Thus, we can conclude that:

$\{a \mid a \in \text{access}_{k,M_d}A \land a \in \text{dom}(M_d)\} \supseteq \{a \mid a \in \text{access}_{k-1,M_d}A \land a \in \text{dom}(M_d)\} \cup \{a^*\}$. (The left operand of $\cup$ is contained by expansiveness (Lemma 8)).

Thus, we have:

$|\{a \mid a \in \text{access}_{k,M_d}A \land a \in \text{dom}(M_d)\}| \geq |\{a \mid a \in \text{access}_{k-1,M_d}A \land a \in \text{dom}(M_d)\}| + |\{a^*\}|$.

Thus, by (***) and simplification:

$|\{a \mid a \in \text{access}_{k,M_d}A \land a \in \text{dom}(M_d)\}| \geq k + 1 > k$

\[\square\]

**Lemma 13** ($|M_d|$-accessibility suffices).

$\forall A, M_d, k. k \geq 0 \implies \text{access}_{|M_d|+k,M_d}A = \text{access}_{|M_d|,M_d}A$

**Proof.** We fix arbitrary $A$ and $M_d$, and prove it by induction on $k$.

• **Base case** ($k = 0$):

  Holds by reflexivity.

• **Inductive case** ($k > 0$):

  We assume $\text{access}_{|M_d|,M_d}A = \text{access}_{|M_d|+k,M_d}A$.

  Suppose for the sake of contradiction that $\text{access}_{|M_d|+k,M_d}A \supseteq \text{access}_{|M_d|+k-1,M_d}A$.

  Then, we know by Lemma 12 that necessarily

  $|\{a \mid a \in \text{access}_{|M_d|+k-1,M_d}A \land a \in \text{dom}(M_d)\}| > |M_d| + k - 1$.

  But $k > 0$. Thus, $k - 1 \geq 0$.

  So, our statement says

  $|\{a \mid a \in \text{access}_{|M_d|+k-1,M_d}A \land a \in \text{dom}(M_d)\}| > |M_d|$.

  But this is immediately a contradiction because

  $|\{a \mid a \in \text{dom}(M_d)\}| = |M_d|$,

  and

  $\{a \mid a \in \text{dom}(M_d)\} \supseteq \{a \mid a \in \text{access}_{|M_d|+k-1,M_d}A \land a \in \text{dom}(M_d)\}$.

  Thus, necessarily by our contradictory assumption and Lemma 8:

  $\text{access}_{|M_d|+k,M_d}A = \text{access}_{|M_d|+k-1,M_d}A$.

  So, by substitution from our inductive hypothesis, we get our goal:

  $\text{access}_{|M_d|+k,M_d}A = \text{access}_{|M_d|,M_d}A$

\[\square\]
Lemma 14 (Invariance to non-δ-capability values).

\[ \forall C, M_d, a, v. \\
\text{\(v \neq (\delta, \_, \_, \_) \land M_d(a) = v\)} \implies \text{reachable_addresses}(C, M_d) = \text{reachable_addresses}(C, M_d[a \mapsto 0]) \]

Proof.

- We fix arbitrary \(C, M_d, a\), and \(v\). We assume the antecedents \(v \neq (\delta, \_, \_, \_) \land M_d(a) = v\).
- Our goal is \(\text{reachable_addresses}(C, M_d) = \text{reachable_addresses}(C, M_d[a \mapsto 0])\).
- By Definition 22, it suffices to show that:
  \(\forall n. \text{access}_{n, M_d} A = \text{access}_{n, M_d[a \mapsto 0]} A\).
- We prove it by induction on \(n\).
  - Base case \((n = 0)\):
    By Definition 21, \(\text{access}_{0, M_d} A = \text{access}_{0, M_d[a \mapsto 0]} A = A\).
  - Inductive case \((n > 0)\):
    By the induction hypothesis, we have:
    \(\text{access}_{n-1, M_d} A = \text{access}_{n-1, M_d[a \mapsto 0]} A = s_{\text{ind}}\).
    By unfolding Definition 21, our goal becomes (after substitution):
    \(\text{access}_{M_d s_{\text{ind}}} = \text{access}_{M_d[a \mapsto 0] s_{\text{ind}}}\).
    By Definition 20, our goal is:
    \(s_{\text{ind}} \cup \bigcup_{a' \in s_{\text{ind}}} \left[ s, e \right] = s_{\text{ind}} \cup \bigcup_{a' \in s_{\text{ind}}} \left[ s, e \right] = (\delta, s, e, \_)
    
    Thus, it suffices to show that:
    \(\forall a', s, e. \in s_{\text{ind}}, M_d(a') = (\delta, s, e, \_) \iff M_d(a \mapsto 0)(a') = (\delta, s, e, \_).
    
    We prove it for an arbitrary \(a', s, e\) by distinguishing the following cases:
    * Case \(a' \neq a\):
      In this case, by the definition (stability) of the function update operator, we have:
      \(M_d(a') = M_d(a \mapsto 0)(a')\), which implies our goal:
      \(M_d(a') = (\delta, s, e, \_) \iff M_d(a \mapsto 0)(a') = (\delta, s, e, \_).
    * Case \(a' = a\):
      
      “\(\implies\)”: In this case, suppose \(M_d(a) = (\delta, s, e, \_)\). Then, we get a contradiction to our assumption that \(v \neq (\delta, \_, \_, \_)\). So, any goal is provable.
      
      “\(\iff\)”: In this case, suppose \(M_d(a \mapsto 0)(a) = (\delta, s, e, \_).\) This is immediately a contradiction by the disjointness of \(\mathcal{Z}\) and \(\{\delta\} \times \mathcal{Z} \times \mathcal{Z}\). So, any goal is provable.

Lemma 15 (Overwriting a non-δ-capability value does not shrink the accessibility set).

\[ \forall k, M_d, A, a, v. \ M_d(a) \neq (\delta, \_, \_, \_) \implies \text{access}_{k, M_d} A \subseteq \text{access}_{k, M_d[a \mapsto v]} A \]

Proof. We fix arbitrary \(M_d, A\), and \(v\), and assume the antecedent. We prove it by induction on \(k\).

- Base case \((k = 0)\):
  In this case, our goal is to show that
  \(\text{access}_{0, M_d} A \subseteq \text{access}_{0, M_d[a \mapsto v]} A\). By Definition 21, we have:
  \(\text{access}_{0, M_d} A = \text{access}_{0, M_d[a \mapsto v]} A = A\) which satisfies our goal.

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• **Inductive case** \((k > 0)\):

Here, the I.H. gives us \(\text{access}_{k-1,M_d}A \subseteq \text{access}_{k-1,M_d[a\rightarrow v]}A\).

We pick an arbitrary \(a' \in \text{access}_{k,M_d}A\).

By Definitions 20 and 21, we distinguish two cases:

- **Case** \(a' \in \text{access}_{k-1,M_d}A\):

  In this case, by the I.H., we know \(a' \in \text{access}_{k-1,M_d[a\rightarrow v]}A\).

  So by expansiveness (Lemma 8), we have our goal.

- **Case** \(a' \in \text{access}_{k-1,M_d}A\):

  We pick an arbitrary \(k > 0\) and \(a'' \in \text{access}_{k-1,M_d}A\).

  By Definitions 20 and 21, we distinguish two cases:

  - **Case** \(a'' = a\):

    This case is impossible because by assumption we know \(M_d(a) \neq (\delta, \_1, \_2, \_3)\).

  - **Case** \(a'' \neq a\):

    In this case, we know that \(M_d(a \rightarrow v)(a'') = M_d(a''')(a'') = (\delta, s, e, \_).\)

    Thus, we have that

    \[
    a' \in \bigcup_{a'' \in \text{access}_{k-1,M_d}A, M_d(a'') = (\delta, s, e, \_)} [s, e] \bigcup_{a'' \in \text{access}_{k-1,M_d}A, M_d(a'') = (\delta, s, e, \_)} [s, e].
    \]

    But by the I.H., this gives us:

    \[
    a' \in \bigcup_{a'' \in \text{access}_{k-1,M_d}A, M_d(a'') = (\delta, s, e, \_)} [s, e].
    \]

    By Definition 21 of \(a' \in \text{access}_{k,M_d[a\rightarrow v]}A\), our goal is satisfied.

\[\quad\]

**Lemma 16** (Additivity of \(\text{access}_{M_d}\)).

\[
\forall A_1, A_2, M_d. \, \text{access}_{M_d}(A_1 \cup A_2) = \text{access}_{M_d}A_1 \cup \text{access}_{M_d}A_2
\]

**Proof.**

• By Definition 20, our goal becomes:

\[
A_1 \cup A_2 \cup \bigcup_{a \in A_1 \cup A_2, M_d(a) = (\delta, s, e, \_)} [s, e] = A_1 \cup \bigcup_{a \in A_1, M_d(a) = (\delta, s, e, \_)} [s, e] \cup A_2 \cup \bigcup_{a \in A_2, M_d(a) = (\delta, s, e, \_)} [s, e].
\]

• Then, it suffices to show that:

\[
\bigcup_{a \in A_1 \cup A_2, M_d(a) = (\delta, s, e, \_)} [s, e] = \bigcup_{a \in A_1, M_d(a) = (\delta, s, e, \_)} [s, e] \cup \bigcup_{a \in A_2, M_d(a) = (\delta, s, e, \_)} [s, e].
\]

• The above goal can be shown as follows:

  - Pick an arbitrary \(a' \in \bigcup_{a \in A_1 \cup A_2, M_d(a) = (\delta, s, e, \_)} [s, e]\).

  - Notice that by the definition of \(\cup\), this is equivalent to:

    \[
    \exists a. \, a \in A_1 \cup A_2 \land M_d(a) = (\delta, s, e, \_ \_) \land a' \in [s, e]
    \]

  - By the definition of \(\cup\), this is equivalent to:

    \[
    \exists a. \, (a \in A_1 \lor a \in A_2) \land M_d(a) = (\delta, s, e, \_ \_ \_ \_ ) \land a' \in [s, e]
    \]

  - By distributivity, this is equivalent to:

    \[
    \exists a. \, (a \in A_1 \land M_d(a) = (\delta, s, e, \_ \_ \_ ) \land a' \in [s, e]) \lor (a \in A_2 \land M_d(a) = (\delta, s, e, \_ \_ \_ \_ ) \land a' \in [s, e])
    \]

  - By folding back the definition of \(\cup\), this is equivalent to:

    \[
    a' \in \bigcup_{a \in A_1, M_d(a) = (\delta, s, e, \_ \_ \_ )} [s, e] \cup \bigcup_{a \in A_2, M_d(a) = (\delta, s, e, \_ \_ \_ \_ )} [s, e]
    \]
This concludes the proof of our sufficient goal.

\[\forall k, A_1, A_2, \mathcal{M}_d. \text{access}_{k, \mathcal{M}_d} A_1 \cup A_2 = \text{access}_{k, \mathcal{M}_d} A_1 \cup \text{access}_{k, \mathcal{M}_d} A_2\]

**Proof.** We fix arbitrary \(A_1, A_2,\) and \(\mathcal{M}_d,\) and prove it by induction on \(k.\)

- **Base case (\(k = 0\):)**
  Our goal is to show that \(\text{access}_{0, \mathcal{M}_d} A_1 \cup A_2 = \text{access}_{0, \mathcal{M}_d} A_1 \cup \text{access}_{0, \mathcal{M}_d} A_2.\)
  By unfolding Definition 21, it becomes \(A_1 \cup A_2 = A_1 \cup A_2\) which holds by reflexivity.

- **Inductive case (\(k > 0\):)**
  By the induction hypothesis, we have:
  \[\text{access}_{k-1, \mathcal{M}_d} A_1 \cup A_2 = \text{access}_{k-1, \mathcal{M}_d} A_1 \cup \text{access}_{k-1, \mathcal{M}_d} A_2.\]
  By Definition 21, our goal is to show that:
  \[\text{access}_{\mathcal{M}_d} (\text{access}_{k-1, \mathcal{M}_d} A_1 \cup A_2) = \text{access}_{\mathcal{M}_d} (\text{access}_{k-1, \mathcal{M}_d} A_1) \cup \text{access}_{\mathcal{M}_d} (\text{access}_{k-1, \mathcal{M}_d} A_2)\]
  By substitution using the induction hypothesis, our goal becomes:
  \[\text{access}_{\mathcal{M}_d} (\text{access}_{k-1, \mathcal{M}_d} A_1 \cup \text{access}_{k-1, \mathcal{M}_d} A_2) = \text{access}_{\mathcal{M}_d} (\text{access}_{k-1, \mathcal{M}_d} A_1) \cup \text{access}_{\mathcal{M}_d} (\text{access}_{k-1, \mathcal{M}_d} A_2)\]
  This goal can be directly satisfied by Lemma 16.

**Lemma 18** (Additivity of \(\text{reachable\_addresses}\) in the first argument).

\[\forall C_1, C_2, \mathcal{M}_d. \text{reachable\_addresses}(C_1 \cup C_2, \mathcal{M}_d) = \text{reachable\_addresses}(C_1, \mathcal{M}_d) \cup \text{reachable\_addresses}(C_2, \mathcal{M}_d)\]

**Proof.**

- We fix arbitrary \(C_1, C_2,\) and \(\mathcal{M}_d.\)
- By Definition 22, our goal becomes
  \[\bigcup_{n \in [0, |\mathcal{M}_d|]} \text{access}_{n, \mathcal{M}_d}(\text{addr}(C_1 \cup C_2)) = \bigcup_{n \in [0, |\mathcal{M}_d|]} \text{access}_{n, \mathcal{M}_d}(\text{addr}(C_1)) \cup \bigcup_{n \in [0, |\mathcal{M}_d|]} \text{access}_{n, \mathcal{M}_d}(\text{addr}(C_2))\]
  where \(\text{addr}(C) \overset{\text{def}}{=} \bigcup_{c \in C} [c.s, c.e].\)
- **Claim** (**\(\text{addr\ is\ additive}\)**): \(\text{addr}(C_1 \cup C_2) = \text{addr}(C_1) \cup \text{addr}(C_2).\)
- It suffices for our goal to show that:
  \[\forall n. \text{access}_{n, \mathcal{M}_d}(\text{addr}(C_1 \cup C_2)) = \text{access}_{n, \mathcal{M}_d}(\text{addr}(C_1)) \cup \text{access}_{n, \mathcal{M}_d}(\text{addr}(C_2)).\]
- By the claimed additivity of \(\text{addr},\) it suffices to show that:
  \[\forall n. \text{access}_{n, \mathcal{M}_d}(\text{addr}(C_1) \cup \text{addr}(C_2)) = \text{access}_{n, \mathcal{M}_d}(\text{addr}(C_1)) \cup \text{access}_{n, \mathcal{M}_d}(\text{addr}(C_2)).\]
- The latter directly follows by Lemma 17.

□
**Lemma 19** (Additivity of `reachable_addresses` in the first argument using `addr`).

\[
\forall C, C_1, C_2, M_d. \\
\text{addr}(C) = \text{addr}(C_1) \cup \text{addr}(C_2) \\
\implies \text{reachable_addresses}(C, M_d) = \text{reachable_addresses}(C_1, M_d) \cup \text{reachable_addresses}(C_2, M_d)
\]

**Proof.** Similar to the proof of Lemma 18.

**Lemma 20** (Invariance to capability’s location so long as it is reachable).

\[
\forall C, M_d, a, c. \\
M_d(a) \neq (\delta, \_ , \_ , \_ ) \land c = (\delta, \_ , \_ , \_ ) \land \\
a \in \text{reachable_addresses}(C, M_d) \\
\implies \text{reachable_addresses}(C \cup \{c\}, M_d) = \text{reachable_addresses}(C, M_d[a \mapsto c])
\]

**Proof.**

- We fix arbitrary \(a, c, C,\) and \(M_d,\) and assume the antecedent:
  \(M_d(a) \neq (\delta, \_ , \_ , \_ ) \land c = (\delta, \_ , \_ , \_ ) \land a \in \text{reachable_addresses}(C, M_d).\)
- We let \(A = \text{addr}(C)\) where \(\text{addr}(C) \overset{\text{def}}{=} \bigcup_{c \in C} [c, s, c, e].\)
- From the antecedent, and by Definition 22 and the definition of \(\cup,\) we thus have:
  \((\ast) \exists k_a. a \in \text{access}_{k_a, M_d} A\)
- By Lemma 18, our goal can be rewritten as:
  \(\text{reachable_addresses}(C, M_d) \cup \text{reachable_addresses}(\{c\}, M_d) = \text{reachable_addresses}(C, M_d[a \mapsto c]).\)
- By Definition 22, it is equivalent to show that:
  \(\forall b. b \in \bigcup_{n \in [0, |M_d|]} \text{access}_{n, M_d} A \cup \bigcup_{n \in [0, |M_d|]} \text{access}_{n, M_d} (\text{addr}(\{c\})) \) \iff \(b \in \bigcup_{n \in [0, |M_d[a \mapsto c]|]} \text{access}_{n, M_d[a \mapsto c]} A\)

We have two proof obligations:

- **Goal “\(\implies\)”:**
  - Here, we assume for an arbitrary \(b\) that:
    \(b \in \bigcup_{n \in [0, |M_d|]} \text{access}_{n, M_d} A \cup \bigcup_{n \in [0, |M_d|]} \text{access}_{n, M_d} (\text{addr}(\{c\})) \)
  - Our goal is to show that:
    \(b \in \bigcup_{n \in [0, |M_d[a \mapsto c]|]} \text{access}_{n, M_d[a \mapsto c]} A\)
  - We consider the two possible cases from our assumption:
    1. **Case** \(b \in \bigcup_{n \in [0, |M_d|]} \text{access}_{n, M_d} A:\)
      - By the definition of \(\cup,\) we have:
        \((\ast\ast) \exists k_b, k_b \in [0, |M_d|] \land b \in \text{access}_{k_b, M_d} A.\)
      - Under our lemma’s antecedents, we show the following:
        \(\forall k, b', b' \in \text{access}_{k, M_d} A \implies b' \in \text{access}_{k, M_d[a \mapsto c]} A\)
Case \( b' = a \): 
In this case, our goal already follows by Lemma 23 which states that an update to a location (in this case, \( a \)) does not affect its own accessibility.

Case \( b' \neq a \): 
Here, we prove our statement by induction on \( k \):
(a) Base case \( k = 0 \):
We assume \( b' \in \text{access}_{a,M_d} A \), i.e., by Definition 21, that \( b' \in A \).
Our goal is to show that \( b' \in \text{access}_{a,M_d[a \mapsto c]} A \), which by Definition 21 is \( b' \in A \).

(b) Inductive case \( k > 0 \):
By the induction hypothesis, we have:
\[
\forall b', b \in \text{access}_{k-1,M_d} A \implies b' \in \text{access}_{k-1,M_d[a \mapsto c]} A.
\]
We assume \( b' \in \text{access}_{k,M_d} A \), and our goal is to show that \( b' \in \text{access}_{k,M_d[a \mapsto c]} A \).

Case \( b' \in \text{access}_{k-1,M_d} A \):
In this case, by instantiating the induction hypothesis, we conclude:
\( b' \in \text{access}_{k-1,M_d[a \mapsto c]} A \).
By Definition 21, and expansiveness (Lemma 7) of \( \text{access}_{M_d[a \mapsto c]} \), we obtain our goal:
\( b' \in \text{access}_{k,M_d[a \mapsto c]} A \).

Case \( b' \notin \text{access}_{k-1,M_d} A \):
By the definition of \( \cup \), we have:
\( \exists a'. a' \in \text{access}_{k-1,M_d} A \land M_d(a') = (\delta, s, e, _) \land b' \in [s, e) \).
By the induction hypothesis, we have:
\( a' \in \text{access}_{k-1,M_d[a \mapsto c]} A \).
So, we distinguish two cases:
A. Case \( a' \neq a \):
Here, by the definition/stability of the function update operator, we have that:
\( M_d(a) = M_d(a') = (\delta, s, e, _) \).
So our goal is satisfied by seeing that we have the judgment:
\( \exists a'. a' \in \text{access}_{k-1,M_d[a \mapsto c]} A \land M_d(a) = (\delta, s, e, _) \land b' \in [s, e) \).
So, by folding back the definition of \( \cup \) and Definition 21 of \( \text{access}_{k,M_d[a \mapsto c]} A \), we see that indeed \( b' \in \text{access}_{k,M_d[a \mapsto c]} A \).

B. Case \( a' = a \):
Here, conjunct \( M_d(a') = (\delta, s, e, _) \) contradicts our antecedent \( M_d(a) \neq (\delta, _, _, _) \).
So any goal is provable.

Having shown our boxed statement, we now instantiate it with \( b \) and \( k_b \) from (***) to obtain:
\( b \in \text{access}_{k_b,M_d[a \mapsto c]} A \).
Thus, by \( k_b \in [0, |M_d|] \) of (***), and the definition of \( \cup \), we have our goal:
\( b \in \bigcup_{n \in [0, |M_d|]} \text{access}_{n,M_d[a \mapsto c]} A \).

2. Case \( b \in \bigcup_{n \in [0, |M_d|]} \text{access}_{n,M_d}(\text{addr}(\{c\})) \):
By the definition of \( \cup \), we have:
\( \exists k_b, k_b \in [0, |M_d|] \land b \in \text{access}_{k_b,M_d}(\text{addr}(\{c\})) \).
From (**), we know \( k_a \).
Under our lemma’s antecedents, we show the following:
\[
\forall k_a', k_a', b', b' \in \text{access}_{k_a,M_d[a \mapsto c]} A \land a \in \text{access}_{k_a,M_d} A \implies b' \in \text{access}_{k_a'+k_a'+1,M_d[a \mapsto c]} A
\]
We consider two cases:
(a) Case \( b' = a \):
In this case, by Lemma 23, we know \( b' \in \text{access}_{k_a',M_d[a \mapsto c]} A \). Thus by Lemma 8, we know \( b' \in \text{access}_{k_a'+k_a'+1,M_d[a \mapsto c]} A \).
*Case $b' \neq a$:
In this case, we prove it by induction on $k'_a$.

(a) **Base case $k'_a = 0$:***
In this case, we know by the antecedent and unfolding Definition 21 that $a \in A$. We prove our goal by induction on $k'_b$.

i. **Base case ($k'_b = 0$):***
In this case, we know by the antecedent and Definition 21 that $b' \in \text{addr}(\{c\})$, and our goal is to show that:

$$\forall k'_b, b' \in \text{access}_{k'_b-1, M_d} \text{addr}(\{c\}) \land a \in A \implies b' \in \text{access}_{k'_b, M_d[a \mapsto c]} A$$

By assumption, we have $a \in A$ and $b' \in \text{access}_{k'_b, M_d} \text{addr}(\{c\})$, and our goal is to show that $b' \in \text{access}_{k'_b+1, M_d[a \mapsto c]} A$.

From the assumption $b' \in \text{access}_{k'_b, M_d} \text{addr}(\{c\})$, we know by Definition 21 that there are two possible cases:

- **Case $b' \in \text{access}_{k'_b-1, M_d} \text{addr}(\{c\})$:***
  In this case, we instantiate the induction hypothesis and obtain:
  $$b' \in \text{access}_{k'_b, M_d[a \mapsto c]} A.$$  
  Thus, our goal is satisfied by expansiveness (Lemma 8).

- **Case $b' \in \bigcup_{a \in A} [s, e) \cup \{k'\} * \text{add}(\{c\})$, $M_d(a^*) = (\delta, s, e, \_)$:**
  In this case we obtain $a^*$ by the definition of $\bigcup$, and we distinguish the following two cases:
  - **Case $a^* = a$:***
    This case is impossible because the $\bigcup$-condition $M_d(a^*) = (\delta, \_1, \_1, \_1)$ contradicts our lemma’s assumed antecedent.
  - **Case $a^* \neq a$:***
    In this case, we conclude from $a^* \in \text{access}_{k'_b-1, M_d} \text{addr}(\{c\})$ and the induction hypothesis that $a^* \in \text{access}_{k'_b, M_d[a \mapsto c]} A$.
    Thus, given that $b' \in [s, e)$ where $M_d(a \mapsto c)(a^*) = (\delta, s, e, \_)$, we conclude by folding Definition 21 that $b' \in \text{access}_{k'_b+1, M_d[a \mapsto c]} A$ by membership in the right operand of $\bigcup$ in Definition 21.
    This last conclusion is our goal.

(b) **Inductive case $k'_a > 0$:***
By the induction hypothesis, we have (IHka):

$$\forall k'_b, b', b' \in \text{access}_{k'_b, M_d} \text{addr}(\{c\}) \land a \in \text{access}_{k'_b-1, M_d} A \implies b' \in \text{access}_{k'_b, M_d[a \mapsto c]} A$$

And, our goal is to show that:

$$\forall k'_b, b', b' \in \text{access}_{k'_b, M_d} \text{addr}(\{c\}) \land a \in \text{access}_{k'_b, M_d} A \implies b' \in \text{access}_{k'_a, k'_b+1, M_d[a \mapsto c]} A$$

Again, we prove our goal by induction on $k'_b$.

i. **Base case ($k'_b = 0$):***
In this case, we know $b' \in \text{addr}(\{c\})$, and $a \in \text{access}_{k'_b, M_d} A$, and our goal is to show that $b' \in \text{access}_{k'_b+1, M_d[a \mapsto c]} A$.
By Lemma 15, we know that \( a \in \text{access}_{k_a, \mathcal{M}_d[a \rightarrow c]} A \).
For our goal, it suffices to show that:
\[
b' \in \bigcup \{s, e\}
\]
\[
a^* \in \text{access}_{k_a', \mathcal{M}_d[a \rightarrow c]} A, \mathcal{M}_d[a \rightarrow c](a^*) = (\delta, s, e, c, \_)
\]
We pick \( a^* := a \), so we know from just above that \( a \in \text{access}_{k_a', \mathcal{M}_d[a \rightarrow c]} A \) holds, and then it suffices to show that \( b' \in [s, e] \) where \( c = (\delta, s, e, c, \_). \)

The latter follows by our assumption \( b' \in \text{addr}(\{c\}) \) by unfolding our definition of \( \text{addr} \) given in the beginning.

ii. **Inductive case** \( (k_b > 0): \)

In this case, we know by the I.H. that \( (\text{IIhb}) \):
\[
\forall b', b' \in \text{access}_{k_a', \mathcal{M}_d[a \rightarrow c]} \text{addr}(\{c\}) \land a \in \text{access}_{k_a', \mathcal{M}_d[a \rightarrow c]} A \implies b' \in \text{access}_{k_a' + k_b', \mathcal{M}_d[a \rightarrow c]} A
\]
We assume the antecedents of our goal for arbitrary \( b' \):
\[
b' \in \text{access}_{k_a', \mathcal{M}_d[a \rightarrow c]} \text{addr}(\{c\}) \land a \in \text{access}_{k_a', \mathcal{M}_d[a \rightarrow c]} A.
\]
By Definition 21, we distinguish the following three cases:

- **Case** \( a \in \text{access}_{k_a', \mathcal{M}_d[a \rightarrow c]} A \):
  In this case, we obtain by \( (\text{IIha}) \) that \( b' \in \text{access}_{k_a' + k_b', \mathcal{M}_d[a \rightarrow c]} A \).
  By Lemma 8, we have our goal.

- **Case** \( b' \in \text{access}_{k_a', \mathcal{M}_d[a \rightarrow c]} \text{addr}(\{c\}) \):
  In this case, we obtain by \( (\text{IIhb}) \) that \( b' \in \text{access}_{k_a' + k_b', \mathcal{M}_d[a \rightarrow c]} A \).
  By Lemma 8, we have our goal.

- **Case** \( a \notin \text{access}_{k_a', \mathcal{M}_d[a \rightarrow c]} A \land b' \notin \text{access}_{k_a', \mathcal{M}_d[a \rightarrow c]} \text{addr}(\{c\}) \):
  Equivalently (from the unfolding of Definition 21 in both of our antecedents), we know in this case that:
  \[
a \in \bigcup_{a^* \in \text{access}_{k_a', \mathcal{M}_d[a \rightarrow c]} A, \mathcal{M}_d(a^*) = (\delta, s_a, e_a, c, \_)} \land b' \in \bigcup_{b^* \in \text{access}_{k_b', \mathcal{M}_d[a \rightarrow c]} \text{addr}(\{c\}) \land \mathcal{M}_d(b^*) = (\delta, s_b, e_b, c, \_)} \bigcup [s_b, e_b)
  \]
  From the right conjunct, we obtain \( b^* \) satisfying
  \[
b^* \in \text{access}_{k_b', \mathcal{M}_d[a \rightarrow c]} \text{addr}(\{c\}) \land \mathcal{M}_d(b^*) = (\delta, s_b, e_b, c, \_), \land b' \in [s_b, e_b).
  \]
  By Definitions 20 and 21 and the definition of \( \cup \), it suffices for our goal
  \[
b' \in \text{access}_{k_a' + k_b', \mathcal{M}_d[a \rightarrow c]} A \text{ to show that } b' \in \bigcup [s_b, e_b)
  \]
  We satisfy the latter by picking the \( b^* \) we obtained above noticing that it satisfies \( b^* \in \text{access}_{k_b', \mathcal{M}_d[a \rightarrow c]} A \) by Lemma 15, and that it satisfies \( \mathcal{M}_d(a \rightarrow c)(b^*) = (\delta, s_b, e_b, c, \_) \) because \( b^* \neq a \) must hold (otherwise, we contradict our antecedent \( \mathcal{M}_d(a) 
eq (\delta, \_, \_, \_, \_). \))
  This concludes our case.

This concludes the proof of our boxed statement; we instantiate it by \((**2)\) and \((*)\) to obtain \((**2*)\):
\[
b \in \text{access}_{k_a + k_b, \mathcal{M}_d[a \rightarrow c]} A.
\]
Recall that our goal is to show that \( \exists n, n \in [0, |\mathcal{M}_d(a \rightarrow c)|] \land b \in \text{access}_{n, \mathcal{M}_d[a \rightarrow c]} A \).
We distinguish two cases for \( k_a + k_b + 1 \):

- **Case** \( k_a + k_b + 1 \leq |\mathcal{M}_d(a \rightarrow c)| \):
  In this case, our goal follows directly from \((**2*)\).

- **Case** \( k_a + k_b + 1 > |\mathcal{M}_d(a \rightarrow c)| \):
  In this case, we know by Lemma 13 that:
  \[
  \text{access}_{k_a + k_b + 1, \mathcal{M}_d[a \rightarrow c]} A = \text{access}_{|\mathcal{M}_d(a \rightarrow c)|, \mathcal{M}_d[a \rightarrow c]} A.
  \]
  So, we pick \( n := |\mathcal{M}_d(a \rightarrow c)| \) satisfying our goal.

This concludes **Goal “\( \implies \)”.**

- **Goal “\( \impliedby \)”:**
Lemma 21

We prove the general statement:

\[ b \in \bigcup_{n \in [0, |M_d[a \mapsto c]|]} \text{access}_{n, M_d[a \mapsto c]} A \]

Our goal is to show that:

\[ b \in ( \bigcup_{n \in [0, |M_d|]} \text{access}_{n, M_d} A \cup \bigcup_{n \in [0, |M_d|]} \text{access}_{n, M_d} \{ \text{addr}(\{c\}) \} ). \]

By assumption, we know (#):

\[ \exists n. n \in [0, |M_d[a \mapsto c]|] \land b \in \text{access}_{n, M_d[a \mapsto c]} A. \]

We prove the general statement:

\[ \forall n, b'. b' \in \text{access}_{n, M_d[a \mapsto c]} A \implies b' \in \text{access}_{n, M_d} A \lor b' \in \text{access}_{n, M_d} \{ \text{addr}(\{c\}) \} \]

We prove our goal by induction on \( n \).

* **Base case** \( (n = 0) \):

In this case, we know \( b' \in \text{access}_{0, M_d[a \mapsto c]} A = A = \text{access}_{0, M_d} A \).

So, our goal is satisfied by satisfying the left disjunct.

* **Inductive case** \( (n > 0) \):

The induction hypothesis gives us:

\[ \forall b'. b' \in \text{access}_{n-1, M_d[a \mapsto c]} A \implies b' \in \text{access}_{n-1, M_d} A \lor b' \in \text{access}_{n-1, M_d} \{ \text{addr}(\{c\}) \} \]

By assumption and Definitions 20 and 21, we distinguish two cases:

* **Case** \( b' \in \text{access}_{n-1, M_d[a \mapsto c]} A \):

In this case, we have by the induction hypothesis that:

\[ b' \in \text{access}_{n-1, M_d} A \lor b' \in \text{access}_{n-1, M_d} \{ \text{addr}(\{c\}) \}. \]

So, in either case (left disjunct or right disjunct holds), we have our goal by unfolding Definition 21 in our goal and applying Lemma 7.

* **Case** \( \exists b''. b'' \in \text{access}_{n-1, M_d[a \mapsto c]} A \land M_d[a \mapsto c](b'') = (\delta, s, e, _) \land b' \in [s, e] \):

By the induction hypothesis, we know:

\[ b'' \in \text{access}_{n-1, M_d} A \lor b'' \in \text{access}_{n-1, M_d} \{ \text{addr}(\{c\}) \}. \]

We distinguish two cases:

- **Case** \( b'' \neq a \):

  In this case, we know that \( M_d[a \mapsto c](b'') = M_d(b'') = (\delta, s, e, _) \).

  So, by Definition 21, we can conclude:

  \[ b' \in \text{access}_{n, M_d} A \text{ in case } b'' \in \text{access}_{n-1, M_d} A, \text{ and } \]

  \[ b' \in \text{access}_{n, M_d} \{ \text{addr}(\{c\}) \} \text{ in case } b'' \in \text{access}_{n-1, M_d} \{ \text{addr}(\{c\}) \}. \]

- **Case** \( b'' = a \):

  In this case, we know that \( c = M_d[a \mapsto c](b'') = (\delta, s, e, _) \land b' \in [s, e] \).

  So, in particular, we know \( b' \in \text{addr}(\{c\}) \).

  So, by Definition 21, we know \( b' \in \text{access}_{0, M_d} \{ \text{addr}(\{c\}) \} \).

  So, by \( n > 0 \), and by expansiveness (Lemmas 7 and 8), we conclude:

  \[ b' \in \text{access}_{n, M_d} \{ \text{addr}(\{c\}) \}, \text{ which satisfies the right disjunct of our goal.} \]

This concludes the proof of our boxed statement.

Instantiating it with (#) gives us by Lemma 13 an \( n \) satisfying our goal.

This concludes **Goal \( ^* \iff \) \**, which concludes the proof of Lemma 20.

\[ \square \]

**Lemma 21** (Invariance to unreachable memory updates).

\[ \forall C, M_d, a, v. a \notin \text{reachable_addresses}(C, M_d) \implies \text{reachable_addresses}(C, M_d) = \text{reachable_addresses}(C, M_d[a \mapsto v]) \]
Proof.

- We fix arbitrary $C, M_d, a$, and $v$. We assume the antecedent.
- By unfolding Definition 22, and the definition of $\cup$, our antecedent can be re-written as $(†)$:
  \[ \forall n \in [0, |M_d|], a \notin access_{n, M_d} addr(C), \]
  where $addr(C) \overset{\text{def}}{=} \bigcup_{c \in C} [c, s, c, e]$.
- Thus, by Lemma 22, we conclude that $(‡)$:
  \[ \forall n \in [0, |M_d|], access_{n, M_d} addr(C) = access_{n, M_d[a \rightarrow v]} addr(C). \]
- Thus, by identities of $\cup$, we have that $(*)$:
  \[ \bigcup_{n \in [0, |M_d|]} access_{n, M_d} addr(C) = \bigcup_{n \in [0, |M_d|]} access_{n, M_d[a \rightarrow v]} addr(C) \]
- (Intuition) By looking at the right-hand side, we notice that the set union could be missing one extra step for the expression to satisfy reachable_addresses($C, M_d[a \rightarrow v]$). The intuition is $|M_d[a \rightarrow v]| \in [|M_d|, |M_d| + 1]$.

In particular, we distinguish the two possible cases:

- **Case** $a \in dom(M_d)$:
  
  This case, $|M_d| = |M_d[a \rightarrow v]|$. So, we assume for the sake of contradiction that:
  
  ($)$ $access_{|M_d| + 1, M_d[a \rightarrow v]}(addr(C)) \supseteq access_{|M_d|, M_d[a \rightarrow v]}(addr(C))$.
  
  (Notice that by Lemma 8, necessarily $access_{|M_d| + 1, M_d[a \rightarrow v]}(addr(C)) \supseteq access_{|M_d|, M_d[a \rightarrow v]}(addr(C))$).
  
  * In this case, we know by unfolding Definitions 20 and 21 that $(† † †)$:
    
    $\exists a', a''$. $a' \notin access_{|M_d|, M_d[a \rightarrow v]}(addr(C)) \land a'' \in access_{|M_d|, M_d[a \rightarrow v]}(addr(C)) \land M_d[a \rightarrow v](a'' \in (δ, s, e, _) \land a' \in [s, e])$.
  
  * We distinguish two cases for $a''$:
  
    - **Case** $a'' = a$:
      
      In this case, we know by $(† † † †)$ that $a \in access_{|M_d|, M_d[a \rightarrow v]}(addr(C))$.
      
      But by $(† † †)$, this means that $a \in access_{|M_d|, M_d}(addr(C))$. But this contradicts $(†)$. So, any goal is provable.
    
    - **Case** $a'' \neq a$:
      
      Again, we know by $(† † † †)$ that $a'' \notin access_{|M_d|, M_d[a \rightarrow v]}(addr(C))$.
      
      And again by $(† † †)$, this means that $a'' \notin access_{|M_d|, M_d}(addr(C))$.
      
      And by conjunct $M_d[a \rightarrow v](a' \in (δ, s, e, _) \land a' \in [s, e])$, we know that $a'' \in dom(M_d)$.
      
      Thus, we have by $(† † † † †)$ that the following expression holds:
      
      $a'' \in access_{|M_d|, M_d}(addr(C)) \land M_d(a' \in (δ, s, e, _) \land a' \in [s, e])$.
      
      This gives us by folding Definition 21 that:
      
      $a' \in access_{|M_d| + 1, M_d}(addr(C))$.
      
      But we know from $(† † † †)$ that $a' \notin access_{|M_d|, M_d[a \rightarrow v]}(addr(C))$, which by $(† † †)$ gives us $a' \notin access_{|M_d|, M_d}(addr(C))$.
      
      This means that $a' \in access_{|M_d| + 1, M_d}(addr(C)) \setminus access_{|M_d|, M_d}(addr(C))$, i.e., access_{|M_d| + 1, M_d}(addr(C)) \supseteq access_{|M_d|, M_d}(addr(C))$.
By Lemma 12, we, hence, conclude:

\(\forall a, k, M_d, A, v. \ a \notin \text{access}_{k, M_d}A \implies \text{access}_{k, M_d}A = \text{access}_{k, M_d[a \mapsto v]}A\)

**Proof.** We prove it by induction on

- **Base case** \((k = 0)\):
  Fix arbitrary \(a, A, v, \) and \(M_d\).
  By Definition 21, we have that \(\text{access}_{0, A} = A = \text{access}_{0, M_d}A = \text{access}_{0, M_d[a \mapsto v]}A\).

- **Inductive case** \((k > 0)\):
  The induction hypothesis gives us \((*)\):
  \(\forall a, M_d, A, v. \ a \notin \text{access}_{k-1, M_d}A \implies \text{access}_{k-1, M_d}A = \text{access}_{k-1, M_d[a \mapsto v]}A\)
  We fix arbitrary \(a, M_d, A, \) and \(v, \) and we assume \(a \notin \text{access}_{k, M_d}A\).

Now, by Definitions 20 and 21, we have:

\[
\text{access}_{k, M_d}A = \text{access}_{k-1, M_d}A \cup \bigcup_{a' \in \text{access}_{k-1, M_d}A} \{s, e\} \cup_{M_d[a']=(\delta, s, v)}
\]

Thus, by our assumption together with the definition of \(\cup, \) we conclude:

- (**1) \(a \notin \text{access}_{k-1, M_d}A, \) and
- (**2) \(a \notin \bigcup_{a' \in \text{access}_{k-1, M_d}A} \{s, e\} \cup_{M_d[a']=(\delta, s, v)}\)

By (**1) and (*), we have (**3):

\[
\text{access}_{k-1, M_d}A = \text{access}_{k-1, M_d[a \mapsto v]}A.
\]

Now, in order to show our goal \((\text{access}_{k, M_d}A = \text{access}_{k, M_d[a \mapsto v]}A)\), it suffices by Definitions 20 and 21 to show that both:

- (g1) \(\text{access}_{k-1, M_d}A = \text{access}_{k-1, M_d[a \mapsto v]}A, \) and
- (g2) \(\bigcup_{a' \in \text{access}_{k-1, M_d[a \mapsto v]}A, M_d[a']=(\delta, s, v)} \{s, e\} \cup_{M_d[a']=(\delta, s, v)} \)

We already have (g1) by (**3). By substitution using (**3), our goal (g2) becomes:

- Case \(a' \neq a:\)
  By the definition of function update, we have our goal:
  \(M_d[a'] = M_d[a \mapsto v](a')\)
- Case \(a' = a:\)
  Impossible because by substituting using (**3) in (**1), we get a contradiction to \(a' \in \text{access}_{k-1, M_d[a \mapsto v]}A\).

This concludes the inductive case, which concludes the proof of Lemma 22.

\(\square\)
Lemma 23 (Updating a location does not affect its own k-accessibility).

\[ \forall a, A, k_a, M_d, v. \ a \in \text{access}_{k_a,M_d}A \implies a \in \text{access}_{k_a,M_d[a \rightarrow v]}A \]

Proof. We fix arbitrary \( a, A, k_a, M_d \), and \( v \). We assume the antecedent \( a \in \text{access}_{k_a,M_d}A \).

Our goal is to show that \( a \in \text{access}_{k_a,M_d[a \rightarrow v]}A \).

Assume for the sake of contradiction the contrary of our goal: \( (a \notin \text{access}_{k_a,M_d[a \rightarrow v]}A) \).

Then:

- By Lemma 22, we conclude that:
  \[ \text{access}_{k_a,M_d[a \rightarrow v]}A = \text{access}_{k_a,M_d[a \rightarrow v]}[a \rightarrow M_d(a)]A, \]
  which simplifies to:
  \[ \text{access}_{k_a,M_d[a \rightarrow v]}A = \text{access}_{k_a,M_d}A. \]

- Substituting using this equality into our latest assumption, we get:
  \[ a \notin \text{access}_{k_a,M_d}A. \]

This contradicts our antecedent, so our latest assumption must be false.

This concludes the proof of Lemma 23.

Lemma 24 (Updating a location does not affect its own reachability).

\[ \forall C, a, v, M_d. \ a \in \text{reachable_addresses}(C, M_d) \implies a \in \text{reachable_addresses}(C, M_d[a \rightarrow v]) \]

Proof.

- We fix arbitrary \( C, a, v, M_d \), and assume the antecedent.
- By assumption and unfolding Definition 22, we have \( a \in \bigcup_{k \in [0,|M_d|]} \text{access}_{k,M_d}\text{addr}(C), \)
  where \( \text{addr}(C) \stackrel{\text{def}}{=} \bigcup_{c \in C} \{c,s,c.e\}. \)
- Thus, by the definition of \( \bigcup \), we have (*) : \( \exists k_a \in [0,|M_d|]. \ a \in \text{access}_{k_a,M_d}\text{addr}(C). \)
- And then by Lemma 23, we conclude that (**) : \( a \in \text{access}_{k_a,M_d[a \rightarrow v]}\text{addr}(C). \)
- And by the definition of the function update operator, we notice that \( k_a \in [0,|M_d|] \implies k_a \in [0,|M_d[a \rightarrow v]|] \) which gives us \( k_a \in [0,|M_d[a \rightarrow v]|] \) by (*)
- Thus, by definition of \( \bigcup \), we have from (**) that : \( a \in \bigcup_{k \in [0,|M_d[a \rightarrow v]|]} \text{access}_{k,M_d[a \rightarrow v]}\text{addr}(C). \)
- Thus, by folding using Definition 22, we get our goal: \( a \in \text{reachable_addresses}(C, M_d[a \rightarrow v]). \)

Lemma 25 (Completeness of \( \text{reachable_addresses} \)).

\[ \forall \mathcal{E}, M_d, ddc, stc, pcc. \]
\[ ddc = (\delta, \_ , \_ , \_ ) \land stc = (\delta, \_ , \_ , \_ ) \land \]
\[ \mathcal{E}, M_d, ddc, stc, pcc \downarrow (\delta, s, e, off) \implies (s, e) \subseteq \text{reachable_addresses}(\{stc, ddc\}, M_d) \]

Proof. We prove it by induction on the evaluation \( \mathcal{E}, M_d, ddc, stc, pcc \downarrow (\delta, s, e, off) \) of the expression \( \mathcal{E} \):

- Case \( \text{evalconst} \).
• Case evalCapType,
• Case evalCapStart,
• Case evalCapEnd,
• Case evalCapOff, and
• Case evalBinOp:
  These are all vacuous cases because of disjointness of the integer values and the data capability values.
• Case evalddc, and
  • Case evalDeref:
    ∃ Therefore, we can re-write this conclusion as:
    we conclude that
    Hence, by Definitions 20 and 21 of
    and by assuming
    and by Definition 22, we have:
    and
    Thus, we can re-write this conclusion as:
    Hence, by Definitions 20 and 21 of
    and by assuming
    and by the inductive hypothesis
    we conclude that
    (the antecedent of our goal),
    we conclude that
    Thus, we can re-write this conclusion as:
    Hence, by Definition 21, our goal is satisfied.
  • Case evalIncCap:
    Here, the goal follows directly from the inductive hypothesis.
    We obtain the preconditions
    together with the inductive hypothesis that
    And our goal is to show that:
    Re-writing our goal by Definition 22, it is required to show that:
    We observe that
    And by the induction hypothesis
    and
    Hence, by Definition 22, we have:
    Hence, by Definitions 20 and 21 of
    and by assuming
    and by the inductive hypothesis
    we conclude that
    And our goal is to show that:
    These two cases are similar. We show the proof for evalddc.
    Let
    By evalddc, our goal is to show that
    These two cases are similar. We show the proof for evalddc.
    Let
    By evalddc, our goal is to show that
    These two cases are similar. We show the proof for evalddc.
    Let
    By evalddc, our goal is to show that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
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    And our goal is to show that:
    By Definition 22, our goal is to show that:
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    By Definition 22, our goal is to show that:
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    By Definition 22, our goal is to show that:
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    By Definition 22, our goal is to show that:
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    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
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    By Definition 22, our goal is to show that:
    We obtain the preconditions
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    By Definition 22, our goal is to show that:
    We obtain the preconditions
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    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
    By Definition 22, our goal is to show that:
    We obtain the preconditions
    Here, the goal follows directly from the inductive hypothesis.
    We observe that
    And our goal is to show that:
But by Lemma 13 about sufficiency of $|M_d|$-accessibility, our conclusion is equivalent to:
\[ \exists k. \ k \in [0, |M_d|] \land [s', e') \subseteq access_{k, M_d} \bigcup_{c \in \{ddc, stc\}} [c, s, e), \]
which satisfies our goal.

- **Case evalLim:**
  Here, we obtain the preconditions $E', M_d, ddc, stc, pcc \downarrow v', v' = (x, \sigma', e', _) \in Cap$, with the inductive hypothesis that $x = \delta \implies [\sigma', e') \subseteq reachable_addresses\{ddc, stc\}, M_d)$. We also obtain the preconditions $[\sigma, e) \subseteq [\sigma', e')$, and $E, M_d, ddc, stc, pcc \downarrow (x, \sigma, e, _) and our goal is to show given $x = \delta$ that $[\sigma, e) \subseteq reachable_addresses\{ddc, stc\}, M_d)$. So, our goal follows immediately by transitivity of $\subseteq$.

**Lemma 26** (Expression evaluation cannot forge data capabilities).
\[
\forall s, E, \sigma, e .
\models s.ddc \land
\models s.stc \land
E, s.M_d, s.ddc, s.stc, s.pcc \downarrow (\delta, \sigma, e, _)
\implies
((\delta, \sigma, e, _) \subseteq s.ddc \lor
(\delta, \sigma, e, _) \subseteq s.stc \lor
\exists a. (\delta, \sigma, e, _) \subseteq s.M_d(a) \land a \in reachable_addresses\{ddc, stc\}, s.M_d))
\]

**Proof.**

- We assume the antecedents
- And we prove our goal by induction on the evaluation of expression $E$
  1. **Case evalconst,**
  2. **Case evalBinOp,**
  3. **Case evalCapType,**
  4. **Case evalCapStart,**
  5. **Case evalCapEnd,**
  6. **Case evalCapOff:**
     In all of these cases, we notice that $E, _, _, _, _ \downarrow z$ with $z \in \mathbb{Z}$.
     This contradicts our assumed antecedent $E, s.M_d, s.ddc, s.stc, s.pcc \downarrow (\delta, \sigma, e, _)$ because $(\delta, _, _, _) \notin \mathbb{Z}$. So these cases are impossible.
  7. **Case evalddc:**
     In this case, we choose the leftmost disjunct, so our goal becomes $s.ddc \subseteq s.ddc$ which by the reflexivity of $\subseteq$ (Definition 3) is immediate.
  8. **Case evalstc:**
     In this case, we choose the middle disjunct, so our goal becomes $s.stc \subseteq s.stc$ which by the reflexivity of $\subseteq$ (Definition 3) is immediate.

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9. Case evalDeref:
Here, we obtain the preconditions:
\[ E, s.M_d, s.ddc, s.stc, s.pcc \downarrow (\delta, \sigma, e, \text{off}), \vdash_{\delta} v, \text{ and } v' = s.M_d(\sigma + \text{off}). \]
By instantiating Lemma 25 using the preconditions \[ E, s.M_d, s.ddc, s.stc, s.pcc \downarrow (\delta, \sigma, e, \text{off}), \vdash_{\delta} v, \text{ and our lemma assumptions, we conclude (\ast):} \]
\[ \sigma + \text{off} \in \text{reachable_addresses}\{s.ddc, s.stc\}, s.M_d \]
Now, we choose the rightmost disjunct of our goal.
We thus have two subgoals to prove.
The left subgoal (after the choice of \[ a = \sigma + \text{off} \]) is immediate by the preconditions obtained above.
The right conjunct is exactly (\ast) that we proved above.

10. Case evalIncCap:
Here, by Lemma 1 about the obliviousness of \( \subseteq \) to the capability offset, our goal is immediate from the induction hypothesis.

11. Case evalLim:
Here, our goal follows by the transitivity of \( \subseteq \) from the induction hypothesis, and assumptions.

This concludes the proof of Lemma 26. \( \square \)

**Definition 23** (Derivable capability). A capability \( c^* = (x, s, e, \_ ) \) is derivable from a set of capabilities \( C : 2^\text{Cap} \) on memory \( M_d \), written \( C, M_d \vdash c^* \) iff \( \forall a \in [s, e]. a \in \text{reachable_addresses}(C, M_d) \).

**Lemma 27** (Upward closure of derivability).
\[ \forall c, C, C', M_d. C, M_d \vdash c \wedge C \subseteq C' \implies C', M_d \vdash c \]

**Proof.**
- Take \( C'' \) such that \( C' = C \cup C'' \).
- By Definition 23, our goal is to show that:
  \( \forall a \in [c.s, c.e]. a \in \text{reachable_addresses}(C \cup C'', M_d) \)
- By additivity (Lemma 18), it is equivalent to show that:
  \( \forall a \in [c.s, c.e]. a \in \text{reachable_addresses}(C, M_d) \cup \text{reachable_addresses}(C'', M_d) \)
- The assumption \( C, M_d \vdash c \) gives us:
  \( \forall a \in [c.s, c.e]. a \in \text{reachable_addresses}(C, M_d) \) (by Definition 23) which suffices for our goal. \( \square \)

**Lemma 28** (Reachability traverses all derivable capabilities).
\[ \forall C, M_d, c. C, M_d \vdash c \implies \text{reachable_addresses}(C, M_d) \supseteq \text{reachable_addresses}\{c\}, M_d \]

**Proof.**
- We fix arbitrary \( C, M_d \), and \( c \), and assume the antecedent \( C, M_d \vdash c \).
- By Definition 23, we thus have:
  \( \forall a \in [c.s, c.e]. a \in \text{reachable_addresses}(C, M_d) \).
Lemma 29

We prove our statement by induction on $k$.

- **Base case** ($k = 0$):
  
  We fix an arbitrary $a$.

  In this case, by Definition 21, we have from our antecedent that:
  
  $a \in \{c.s, c.e\}$.

  In this case, by universal instantiation of (*), we get:
  
  $\exists k. k' \in [0, |M_d|] \land a \in \text{access}_{k', M_d} \bigcup_{c \in C} [c'.s, c'.e)$, which is our goal.

- **Inductive case** ($k > 0$):

  Here, by the induction hypothesis, we have:
  
  $\forall a. a \in \text{access}_{k-1, M_d} [c.s, c.e] \implies \exists k'. k' \in [0, |M_d|] \land a \in \text{access}_{k', M_d} \bigcup_{c \in C} [c'.s, c'.e)$

  We fix an arbitrary $a$, and we assume the antecedent:
  
  $a \in \text{access}_{k, M_d} [c.s, c.e]$

  We distinguish two cases by Definitions 20 and 21:

  * **Case** $a \in \text{access}_{k-1, M_d} [c.s, c.e]$:

    In this case, the induction hypothesis gives us that:
    
    $\exists k'. k' \in [0, |M_d|] \land a \in \text{access}_{k', M_d} \bigcup_{c \in C} [c'.s, c'.e)$, which is our goal.

  * **Case** $a' \in \text{access}_{k-1, M_d} [c.s, c.e] \land M_d(a') = (\delta, s, e, _) \land a \in [s, e]$:

    In this case, the induction hypothesis gives us that:
    
    $\exists k'. k' \in [0, |M_d|] \land a' \in \text{access}_{k', M_d} \bigcup_{c \in C} [c'.s, c'.e)$

    Thus, by Definition 21 of $a \in \text{access}_{k+1, M_d} \bigcup_{c \in C} [c'.s, c'.e)$, and by the case conditions
    
    $M_d(a') = (\delta, s, e, _) \land a \in [s, e)$, we obtain:
    
    $\exists k''. k'' \in [1, |M_d| + 1] \land a \in \text{access}_{k'', M_d} \bigcup_{c \in C} [c'.s, c'.e)$.

    By Lemma 13, we know we have:
    
    $\exists k'''. k''' \in [1, |M_d|] \land a \in \text{access}_{k''', M_d} \bigcup_{c \in C} [c'.s, c'.e)$, which suffices for our goal.

$\Box$

**Lemma 29** (Preservation of reachability equivalence under safe memory updates).

$\forall C, M_{d1}, M_{d2}, r_1, r_2, a, v.$

$r_1 = \text{reachable_addresses}(C, M_{d1}) \land r_2 = \text{reachable_addresses}(C, M_{d2}) \land$

$r_1 = r_2 \land M_{d1}[ r_1 = M_{d2}[ r_2 \land (C, M_{d1} \models v \lor v \notin \{\delta\} \times Z \times Z)]$

$\implies \text{reachable_addresses}(C, M_{d1}[a \mapsto v]) = \text{reachable_addresses}(C, M_{d2}[a \mapsto v])$
Proof.

- We fix arbitrary \( C, M_{d_1}, M_{d_2}, r_1, r_2, \hat{a}, v \).
- We assume the antecedents \( r_1 = \text{reachable_addresses}(C, M_{d_1}), r_2 = \text{reachable_addresses}(C, M_{d_2}) \), \( r_1 = r_2 \), \( M_{d_1} | r_1 = M_{d_2} | r_2 \), and \( (C, M_{d_1} \models (\hat{a} \rightarrow v)) \cup \{\delta\} \times Z \times Z \), which by \( r_1 = r_2 \) and by Definition 23 gives us also that \( (C, M_{d_2} \models (\hat{a} \rightarrow v)) \cup \{\delta\} \times Z \times Z \).

We now distinguish two cases:

- Case \( \hat{a} \in r_1 \):
  
  In this case, we know from the assumptions \( r_1 = r_2 \) and \( M_{d_1} | r_1 = M_{d_2} | r_2 \) that \( M_{d_1}(\hat{a}) = M_{d_2}(\hat{a}) \).
  
  We distinguish four different cases:

  - Case \( M_{d_1}(\hat{a}) \neq (\delta, \_ , \_ , \_ ) \land v \neq (\delta, \_ , \_ , \_ ) \):
    
    * In this case, we know by Lemma 14 about irrelevance of non-\( \delta \)-capability values that \( r_1 = \text{reachable_addresses}(C, M_{d_1}[\hat{a} \rightarrow 0]) = \text{reachable_addresses}(C, M_{d_1}[\hat{a} \rightarrow v]) \).
    
    And because \( M_{d_2}(\hat{a}) = M_{d_1}(\hat{a}) \neq (\delta, \_ , \_ , \_ ) \), we analogously then have by Lemma 14 that \( r_2 = \text{reachable_addresses}(C, M_{d_1}[\hat{a} \rightarrow 0]) = \text{reachable_addresses}(C, M_{d_2}[\hat{a} \rightarrow v]) \).
    
    * So by substitution in the assumption \( r_1 = r_2 \), we get our goal \( \text{reachable_addresses}(C, M_{d_1}[\hat{a} \rightarrow v]) = \text{reachable_addresses}(C, M_{d_2}[\hat{a} \rightarrow v]) \).

  - Case \( M_{d_1}(\hat{a}) \neq (\delta, \_ , \_ , \_ ) \land v = (\delta, s, e, \_ ) \):
    
    * By Lemma 20 about invariance to the location of \( v \), we have: \( \text{reachable_addresses}(C, M_{d_1}[\hat{a} \rightarrow v]) = \text{reachable_addresses}(C \cup \{ v \}, M_{d_1}) \).
    
    * So, by Lemma 18 about “additivity in the first argument”, we get: \( \text{reachable_addresses}(C, M_{d_1}[\hat{a} \rightarrow v]) = \text{reachable_addresses}(C, M_{d_1}) \cup \text{reachable_addresses}(\{ v \}, M_{d_1}) \).
    
    * By the assumption \( C, M_{d_1} \models (\hat{a} \rightarrow v) \cup \{\delta\} \times Z \times Z \), we have in this case that \( C, M_{d_1} \models v \), resp. \( C, M_{d_2} \models v \).
    
    * So, by Lemma 28, we have that: \( \text{reachable_addresses}(\{ v \}, M_{d_1}) \subseteq \text{reachable_addresses}(C, M_{d_1}) \).
    
    * Thus, we obtain: \( \text{reachable_addresses}(C, M_{d_1}[\hat{a} \rightarrow v]) = \text{reachable_addresses}(C, M_{d_1}) = r_1 \).
    
    * By an argument analogous to the above, we have that: \( \text{reachable_addresses}(C, M_{d_2}[\hat{a} \rightarrow v]) = \text{reachable_addresses}(C, M_{d_2}) = r_2 \).
    
    * So by substitution in the assumption \( r_1 = r_2 \), we get our goal: \( \text{reachable_addresses}(C, M_{d_2}[\hat{a} \rightarrow v]) = \text{reachable_addresses}(C, M_{d_2}[\hat{a} \rightarrow v]) \).

  - Case \( M_{d_1}(\hat{a}) = (\delta, s_a, e_a, \_ ) \land v = (\delta, s, e, \_ ) \):
    
    In this case, we break down the memory update operation into two memory updates, namely, the update \( \lambda x. x[\hat{a} \rightarrow 0] \) followed by \( \lambda x. x[\hat{a} \rightarrow v] \).
    
    * So, we notice that \( M_{d_1}[\hat{a} \rightarrow v] = M_{d_1}[\hat{a} \rightarrow 0][\hat{a} \rightarrow v] \).
    
    * Thus, by Lemma 20 about invariance to a capability’s location, we get: \( \text{reachable_addresses}(C, M_{d_1}[\hat{a} \rightarrow v]) = \text{reachable_addresses}(C \cup \{ v \}, M_{d_1}[\hat{a} \rightarrow 0]) \).
    
    * Thus, by additivity (Lemma 18), we get: \( \text{reachable_addresses}(C, M_{d_1}[\hat{a} \rightarrow v]) = \text{reachable_addresses}(C, M_{d_1}[\hat{a} \rightarrow 0]) \cup \text{reachable_addresses}(\{ v \}, M_{d_1}[\hat{a} \rightarrow 0]) \).
    
    * Now recall that by assumption we know \( C, M_{d_1} \models v \), so we can use Lemma 28 to get: \( (\uparrow \uparrow 1) \text{reachable_addresses}(C, M_{d_1}[\hat{a} \rightarrow v]) = \text{reachable_addresses}(C, M_{d_1}[\hat{a} \rightarrow 0]) \).
    
    * By a similar argument, we also have for \( M_{d_2} \) that: \( (\uparrow \uparrow 2) \text{reachable_addresses}(C, M_{d_2}[\hat{a} \rightarrow v]) = \text{reachable_addresses}(C, M_{d_2}[\hat{a} \rightarrow 0]) \).

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* Next we work out the right-hand side of the $M_1$ equality to reach the right-hand side of the $M_2$ equality, thus satisfying our goal.

* First, we notice that by $\hat{a} \in r_1$, and by Lemma 24, we have that:
  \[ \hat{a} \in \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto 0]). \]

* Thus, we can now use Lemma 20 with the instantiation $M_d := M_{d1}[\hat{a} \mapsto 0], c := M_{d1}(\hat{a})$ to get:
  \[ \text{reachable_addresses}(C \cup \{M_{d1}(\hat{a})\}, M_{d1}[\hat{a} \mapsto 0]) = \text{reachable_addresses}(C, M_{d1}) = r_1. \]

* So, by additivity (Lemma 18), we conclude that:
  \[ \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto 0]) \subseteq r_1 \]

* Thus, we pick an arbitrary $a' \notin r_1$, and we know that:
  it also satisfies $a' \notin \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto 0]).$

Thus, we know by Lemma 21 about invariance to unreachable memory updates that:
\[ \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto 0]) = \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto 0][a' \mapsto M_{d2}(a')]). \]

* Now by applying Lemma 21 inductively on the list of successive updates to $M_{d1}$ at addresses from $\{a' | a' \in \text{dom}(M_{d1}) \cup \text{dom}(M_{d2}) \setminus r_1\}$, and by the assumption $M_{d1}|r_1 = M_{d2}|r_1$, we get the desired transformation:
  \[ \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto 0]) = \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto 0][a' \mapsto M_{d2}(a')]). \]

* By substituting the above equality in (‡ 1), we get our goal by (‡ 2):
  \[ \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto v]) = \text{reachable_addresses}(C, M_{d2}[\hat{a} \mapsto v]). \]

- Case $M_{d1}(\hat{a}) = (\delta, s, c_a, _) \land v \neq (\delta, _, _, _)$:
  This case is very similar to the case above (unsurprisingly strictly shorter).

  * First, we notice that by $\hat{a} \in r_1$, and by Lemma 24, we have that:
    \[ \hat{a} \in \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto v]). \]

  * Thus, we can now use Lemma 20 with the instantiation $M_d := M_{d1}[\hat{a} \mapsto v], c := M_{d1}(\hat{a})$ to get:
    \[ \text{reachable_addresses}(C \cup \{M_{d1}(\hat{a})\}, M_{d1}[\hat{a} \mapsto v]) = \text{reachable_addresses}(C, M_{d1}) = r_1. \]

  * So, by additivity (Lemma 18), we conclude that:
    \[ \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto v]) \subseteq r_1 \]

  * Thus, we pick an arbitrary $a' \notin r_1$, and we know that:
    it also satisfies $a' \notin \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto v]).$

Thus, we know by Lemma 21 about invariance to unreachable memory updates that:
\[ \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto v]) = \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto v][a' \mapsto M_{d2}(a')]). \]

* Now by applying Lemma 21 inductively on the list of successive updates to $M_{d1}$ at addresses from $\{a' | a' \in \text{dom}(M_{d1}) \cup \text{dom}(M_{d2}) \setminus r_1\}$, and by the assumption $M_{d1}|r_1 = M_{d2}|r_1$, we get our goal:
  \[ \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto v]) = \text{reachable_addresses}(C, M_{d2}[\hat{a} \mapsto v]). \]

- Case $\hat{a} \notin r_1$:
  By assumption $r_1 = r_2$, we also have that $\hat{a} \notin r_2$.

  Thus, by Lemma 21, we have that
  \[ \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto v]) = r_1, \text{ and } \text{reachable_addresses}(C, M_{d2}[\hat{a} \mapsto v]) = r_2. \]

  By substitution these two claims in the assumption $r_1 = r_2$, our goal
  \[ \text{reachable_addresses}(C, M_{d1}[\hat{a} \mapsto v]) = \text{reachable_addresses}(C, M_{d2}[\hat{a} \mapsto v]) \]

\[ \Box \]

**Definition 24** (Shrunken access: Access set without using the capability at location $a$).
\[ \chi(A, M_d, a) \overset{\text{def}}{=} A \cup \{a^* | a^* \in [\sigma, e] \land M_d(a') = (\delta, \sigma, e, _) \land a' \in A \setminus \{a\} \} \]

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**Definition 25** (Shrunk $k$-th access: $K$-th access set without using the capability at location $a$).

\[
\chi_0(A,M_d,a) \overset{\text{def}}{=} \chi(A,M_d,a)
\]

\[
\chi_k(A,M_d,a) \overset{\text{def}}{=} \chi(\chi_{k-1}(A,M_d,a),M_d,a)
\]

**Lemma 30** (Additivity of $\chi_k$).

\[
\forall k, A_1, A_2, M_d, a. \chi_k(A_1 \cup A_2, M_d, a) = \chi_k(A_1, M_d, a) \cup \chi_k(A_2, M_d, a)
\]

**Proof.** By induction on $k$. Similar to Lemma 17.

**Lemma 31** ($\chi_k$ is upper-bounded by $k$-accessibility).

\[
\forall k, M_d, A, a. \chi_k(A,M_d,a) \subseteq \text{access}_{k,M_d}A
\]

**Proof.** Immediate by Definitions 21 and 25.

**Lemma 32** (One capability is potentially lost from accessible addresses as a result of a non-capability update).

\[
\forall A, a, M_d, v. v \neq (\delta, s, e, \_) \implies \text{access}_{M_d[a \mapsto v]}A = \chi(A,M_d,a)
\]

**Proof.** Follows from Definitions 20 and 24 by observing that $M_d[a \mapsto v](a) \neq (\delta, s, e, \_)$ and that $M_d[a \mapsto v](a') = M_d(a')$ for $a' \neq a$.

**Lemma 33** ($\chi_k$ captures $k$-accessibility after potential deletion of a capability).

\[
\forall A, a, M_d, v. v \neq (\delta, s, e, \_) \implies \text{access}_{k,M_d[a \mapsto v]}A = \chi_k(A,M_d,a)
\]

**Proof.** Follows by induction on $k$ from Definitions 21 and 25 using Lemma 32.

**Lemma 34** (Reachability is captured by union over $\chi_k$ after potential deletion of a capability).

\[
\forall C, M_d, a, v. v \neq (\delta, s, e, \_) \implies \text{reachable_addresses}(C,M_d[a \mapsto v]) = \bigcup_k (\chi_k(\bigcup_{c \in C} [c,\sigma, c.e],M_d,a))
\]

**Proof.** Immediate by Definition 22 and lemma 33.

**Lemma 35** (Accessible addresses shrink by non-$\delta$-capability updates).

\[
\forall A, a, M_d, v. v \neq (\delta, s, e, \_) \implies \text{access}_{M_d[a \mapsto v]}A \subseteq \text{access}_{M_d}A
\]

**Proof.** Immediate by Definition 20 and Lemma 32. Here is an alternative proof:

- By Definition 20, our goal is to show that:
  \[
  A \cup \bigcup_{a' \in A, M_d[a \mapsto v](a')=(\delta,s,e,\_)} [s,e] \subseteq A \cup \bigcup_{a' \in A, M_d(a')=(\delta,s,e,\_)} [s,e]
  \]

- Thus, it suffices to show that:
  \[
  \bigcup_{a' \in A, M_d[a \mapsto v](a')=(\delta,s,e,\_)} [s,e] \subseteq \bigcup_{a' \in A, M_d(a')=(\delta,s,e,\_)} [s,e]
  \]

- We consider an arbitrary $a' \in A$, and distinguish the following two cases:
We prove it by induction on $k$.

**Proof.**

**Lemma 36** (k-accessible addresses shrink by non-$\delta$-capability updates).

$$\forall k,A,a,M_d,v.\ v \neq (\delta,\_\_\_,\_\_) \implies \text{access}_{k,M_d[a \mapsto v]}A \subseteq \text{access}_{k,M_d}A$$

**Proof.**

We prove it by induction on $k$:

- **Base case ($k = 0$):**
  
  Trivial by $A \subseteq A$.

- **Inductive case ($k > 0$):**
  
  By the inductive hypothesis, we know $\text{access}_{k-1,M_d[a \mapsto v]}A \subseteq \text{access}_{k-1,M_d}A$.

  By Definition 21, our goal is to show that:
  $$\text{access}_{M_d[a \mapsto v]}(\text{access}_{k-1,M_d[a \mapsto v]} A) \subseteq \text{access}_{M_d}(\text{access}_{k-1,M_d} A)$$

  We rewrite the inductive hypothesis as: $\exists B. \text{access}_{k-1,M_d} A = B \cup \text{access}_{k-1,M_d[a \mapsto v]} A$.

  Thus, by substitution, our goal becomes:
  $$\text{access}_{M_d[a \mapsto v]}(\text{access}_{k-1,M_d[a \mapsto v]} A) \subseteq \text{access}_{M_d}(B \cup \text{access}_{k-1,M_d[a \mapsto v]} A)$$

  By additivity of $\text{access}_{M_d}$ (Lemma 16), it is equivalent to show:
  $$\text{access}_{M_d[a \mapsto v]}(\text{access}_{k-1,M_d[a \mapsto v]} A) \subseteq \text{access}_{M_d}(\text{access}_{k-1,M_d[a \mapsto v]} A)$$

  By transitivity of $\subseteq$ (set identities), it suffices to show that:
  $$\text{access}_{M_d[a \mapsto v]}(\text{access}_{k-1,M_d[a \mapsto v]} A) \subseteq \text{access}_{M_d}(\text{access}_{k-1,M_d[a \mapsto v]} A)$$

  The latter follows immediately by Lemma 35, which proves our goal.

**Lemma 37** (Reachability shrinks by non-$\delta$-capability updates).

$$\forall C,M_d,a,v.\ v \neq (\delta,\_\_\_,\_\_ \_\_ \_\_ \_\_ \_\_\_) \implies \text{reachable_addresses}(C,M_d[a \mapsto v]) \subseteq \text{reachable_addresses}(C,M_d)$$

**Proof.**

- By Definition 22, it is equivalent to show that:
  $$\bigcup_{k \in [0,|M_d[a \mapsto v]|]} \text{access}_{k,M_d[a \mapsto v]}(\bigcup_{c \in C} [c,s,c,e]) \subseteq \bigcup_{k \in [0,|M_d|]} \text{access}_{k,M_d}(\bigcup_{c \in C} [c,s,c,e])$$

- By preservation of $\subseteq$ under $\cup$ (set identities), it suffices to show that:
  $$\forall k \in [0,|M_d[a \mapsto v]|]. \text{access}_{k,M_d[a \mapsto v]}(\bigcup_{c \in C} [c,s,c,e]) \subseteq \text{access}_{k,M_d}(\bigcup_{c \in C} [c,s,c,e])$$

- But for an arbitrary $k$, the assertion $\text{access}_{k,M_d[a \mapsto v]}(\bigcup_{c \in C} [c,s,c,e]) \subseteq \text{access}_{k,M_d}(\bigcup_{c \in C} [c,s,c,e])$ follows immediately by Lemma 36. This concludes the proof.
Lemma 38 (Safe memory updates only shrink reachability).

\[
\forall C, M_d, \hat{a}, v.
\]
\[
\hat{a} \in \text{reachable_addresses}(C, M_d) \land
(C, M_d \models v \lor v \notin \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})
\implies \text{reachable_addresses}(C, M_d[\hat{a} \mapsto v]) \subseteq \text{reachable_addresses}(C, M_d)
\]

Proof. Similarly to the proof of Lemma 29, we distinguish the following four cases:

- **Case** \( v \neq (\delta, \_, \_, \_ \hat{\_}) \land M_d(\hat{a}) \neq (\delta, \_, \_, \_ \hat{\_}) \), and
- **Case** \( v \neq (\delta, \_, \_, \_ \hat{\_}) \land M_d(\hat{a}) = (\delta, \sigma, e, \_ \hat{\_}) \):
  In these two cases, our goal follows immediately by Lemma 37.
- **Case** \( C, M_d = v \land M_d(\hat{a}) \neq (\delta, \_, \_, \_ \hat{\_}) \):
  By Definition 23, we know \( v = (\delta, \sigma_v, e_v, \_ \hat{\_}) \).
  Thus, by Lemma 20, we know that:
  \[
  \text{reachable_addresses}(C \cup \{v\}, M_d) = \text{reachable_addresses}(C, M_d[\hat{a} \mapsto v])
  \]
  Thus, by additivity – Lemma 18, we have (*):
  \[
  \text{reachable_addresses}(C, M_d) \cup \text{reachable_addresses}(\{v\}, M_d) = \text{reachable_addresses}(C, M_d[\hat{a} \mapsto v])
  \]
  But by Lemma 28, we know:
  \[
  \text{reachable_addresses}(\{v\}, M_d) \subseteq \text{reachable_addresses}(C, M_d)
  \]
  Thus, we can rewrite (*) as:
  \[
  \text{reachable_addresses}(C, M_d) = \text{reachable_addresses}(C, M_d[\hat{a} \mapsto v])
  \]
  which suffices for our goal.
- **Case** \( C, M_d = v \land M_d(\hat{a}) = (\delta, \sigma, e, \_ \hat{\_}) \):
  By Definition 23, we know \( v = (\delta, \sigma_v, e_v, \_ \hat{\_}) \).
  Thus, by Lemma 20, we know that:
  \[
  \text{reachable_addresses}(C \cup \{v\}, M_d[\hat{a} \mapsto 0]) = \text{reachable_addresses}(C, M_d[\hat{a} \mapsto v])
  \]
  Thus, by additivity – Lemma 18, we have (**):
  \[
  \text{reachable_addresses}(C, M_d[\hat{a} \mapsto 0]) \cup \text{reachable_addresses}(\{v\}, M_d[\hat{a} \mapsto 0]) = \text{reachable_addresses}(C, M_d[\hat{a} \mapsto v])
  \]
  We consider an arbitrary address \( a \in \text{reachable_addresses}(C, M_d[\hat{a} \mapsto v]) \). We distinguish the two possible cases that arise from (**):
  - **Case** \( a \in \text{reachable_addresses}(C, M_d[\hat{a} \mapsto 0]) \):
    In this case, we know by Lemma 37, and the definition of \( \subseteq \) that \( a \in \text{reachable_addresses}(C, M_d) \), which by definition of \( \subseteq \) gives us our goal.
  - **Case** \( a \in \text{reachable_addresses}(\{v\}, M_d[\hat{a} \mapsto 0]) \):
    Analogously, here, we know by Lemma 37, and the definition of \( \subseteq \) that:
    \( a \in \text{reachable_addresses}(\{v\}, M_d) \).
    But by Lemma 28, and the definition of \( \subseteq \), we know that \( a \in \text{reachable_addresses}(C, M_d) \), which by the definition of \( \subseteq \) gives our goal.

\[\square\]
Lemma 39 (Safe allocation adds only allocated addresses to k-accessibility).

\[
\forall A, \mathcal{M}_d, \hat{a}, a_a, \sigma, e, k.
\forall a \in [\sigma, e). \mathcal{M}_d[\hat{a} \mapsto (\delta, \sigma, e, _)](a) = v \implies v \notin \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \land
a_a \in \text{access}_k, \mathcal{M}_d[\hat{a} \mapsto (\delta, \sigma, e, _)]A
\implies a_a \in \text{access}_k, \mathcal{M}_d A \lor a_a \in [\sigma, e)
\]

Proof.

- We fix arbitrary \( A, \mathcal{M}_d, \hat{a}, \sigma, e, \) and we assume the antecedents.
- We prove our goal by induction on \( k \).

- **Base case \((k = 0)\):**
  - We fix arbitrary \( a_a \).
  - By Definition 21, we unfold \( a_a \in \text{access}_{0, \mathcal{M}_d[\hat{a} \mapsto (\delta, \sigma, e, _)]}A \) to get \( a_a \in A \).
  - By Definition 21, we thus conclude \( a_a \in \text{access}_{0, \mathcal{M}_d} A \) satisfying our goal (the left disjunct).

- **Inductive case \((k > 0)\):**
  - By the inductive hypothesis, we have:
    \( \forall a. \ a \in \text{access}_{k-1, \mathcal{M}_d[\hat{a} \mapsto (\delta, \sigma, e, _)]}A \implies a \in \text{access}_{k-1, \mathcal{M}_d} A \lor a \in [\sigma, e) \).
  - We fix arbitrary \( a_a \).
  - By Definition 21, we unfold \( a_a \in \text{access}_{k, \mathcal{M}_d[\hat{a} \mapsto (\delta, \sigma, e, _)]} \) to get:
    \( a_a \in \text{access}_{k, \mathcal{M}_d}[\hat{a} \mapsto (\delta, \sigma, e, _)](\text{access}_{k-1, \mathcal{M}_d[\hat{a} \mapsto (\delta, \sigma, e, _)]}A) \).
  - By Definition 20, we distinguish two cases:
    - **Case \( a_a \in \text{access}_{k-1, \mathcal{M}_d[\hat{a} \mapsto (\delta, \sigma, e, _)]}A \)**:
      - By the inductive hypothesis, we thus have:
        \( a_a \in \text{access}_{k-1, \mathcal{M}_d} A \lor a_a \in [\sigma, e) \).
        - Two cases are possible:
          - **Case \( a_a \in \text{access}_{k-1, \mathcal{M}_d} A \)**:
            - By Lemma 8, we immediately obtain our goal (the left disjunct).
          - **Case \( a_a \in [\sigma, e) \)**:
            - This is immediately the right disjunct of our goal.
    - **Case \( 3a^*, \sigma^*, e^*, a_a \in [\sigma^*, e^*) \land \mathcal{M}_d[\hat{a} \mapsto (\delta, \sigma, e, _)](a^*) = (\delta, \sigma^*, e^*, _) \land a^* \in \text{access}_{k-1, \mathcal{M}_d[\hat{a} \mapsto (\delta, \sigma, e, _)]}A \)**:
        - By instantiating the inductive hypothesis with \( a^* \in \text{access}_{k-1, \mathcal{M}_d[\hat{a} \mapsto (\delta, \sigma, e, _)]} \), we obtain:
          \( a^* \in \text{access}_{k-1, \mathcal{M}_d} A \lor a^* \in [\sigma, e) \).
        - So, we consider the two possible cases:
          - **Case \( a^* \in [\sigma, e) \)**:
            - In this case, we instantiate this assumed antecedent of our lemma:
              \( \forall a \in [\sigma, e). \mathcal{M}_d[\hat{a} \mapsto (\delta, \sigma, e, _)](a) = v \implies v \notin \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) and get a contradiction to the conjunct \( \mathcal{M}_d[\hat{a} \mapsto (\delta, \sigma, e, _)](a^*) = (\delta, \sigma^*, e^*, _) \).
            - So, this case is impossible.
          - **Case \( a^* \in \text{access}_{k-1, \mathcal{M}_d} A \)**:
            - Here, we further distinguish two cases:
              **Case \( a^* = \hat{a} \)**:
              - In this case, \( [\sigma^*, e^*) = [\sigma, e) \). Thus, by substitution, we immediately obtain
                \( a_a \in [\sigma, e) \) which satisfies our goal (the right disjunct).
            - **Case \( a^* \neq \hat{a} \)**:
              - In this case, we know \( a^* \in \text{dom}(\mathcal{M}_d) \) and \( \mathcal{M}_d(a^*) = (\delta, \sigma^*, e^*, _) \).
And already we know \( a_a \in [\sigma^*, e^*) \) and \( a^* \in \text{access}_{k-1, M_d} A \).

So, by Definitions 20 and 21, we have:

\[ a_a \in \text{access}_{k, M_d} A \]

which satisfies our goal (the left disjunct).

This concludes the two cases arising from the instantiated inductive hypothesis.

This concludes the two cases arising from Definition 20, and thus concludes the inductive case of our lemma.

- This concludes the proof of Lemma 39.

**Lemma 40** (Safe allocation adds only allocated addresses to reachability).

\[
\forall C, M_d, \hat{a}, a_a, \sigma, e.
\forall a \in [\sigma, e). \ M_d[\hat{a} \mapsto (\delta, \sigma, e, _)](a) = v \implies v \notin \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \land
\]

\[
\exists a_a \in \text{reachable_addresses}(C, M_d[\hat{a} \mapsto (\delta, \sigma, e, _)])
\implies a_a \in \text{reachable_addresses}(C, M_d) \lor a_a \in [\sigma, e)
\]

**Proof.**

- We fix arbitrary \( C, M_d, \hat{a}, a_a, \sigma \) and \( e \), and assume the antecedents.

- From the antecedent \( a_a \in \text{reachable_addresses}(C, M_d[\hat{a} \mapsto (\delta, \sigma, e, _)]) \) and by Definition 22, we have:

\[
\exists k. k \in [0, |M_d[\hat{a} \mapsto _]|] \land a_a \in \text{access}_{k, M_d[\hat{a} \mapsto (\delta, \sigma, e, _)]}(\bigcup_{c \in C} [c.\sigma, c.e])
\]

- Thus, by Lemma 39, we have:

\[
a_a \in \text{access}_{k, M_d}(\bigcup_{c \in C} [c.\sigma, c.e]) \lor a_a \in [\sigma, e)
\]

- We distinguish the following two cases:

  - **Case** \( a_a \in \text{access}_{k, M_d}(\bigcup_{c \in C} [c.\sigma, c.e]) \):

    In this case, we would like to show the left disjunct of our goal.

    By Definition 22, we would like to show that:

    \[
    \exists k. k \in [0, |M_d|] \land a_a \in \text{access}_{k, M_d}(\bigcup_{c \in C} [c.\sigma, c.e])
    \]

    Since we know our obtained \( k \) from above satisfies \( k \geq |M_d| \), then Lemma 13 suffices for the above re-statement of our goal.

  - **Case** \( a_a \in [\sigma, e) \):

    Here, immediately our goal holds (its right disjunct).

\[\Box\]

**Lemma 41** (Safe allocation causes reduction of \( k \)-accessibility to \( \chi_k \) and addition of exactly the allocated addresses).

\[
\forall A, M_d, \hat{a}, a_a, \sigma, e, k.
\forall a \in [\sigma, e). \ M_d[\hat{a} \mapsto (\delta, \sigma, e, _)](a) = v \implies v \notin \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \land
\]

\[
\hat{a} \in \text{access}_{k, M_d} A
\]

\[
\implies \text{access}_{k, M_d[\hat{a} \mapsto (\delta, \sigma, e, _)]} A = \chi_k(A, M_d, \hat{a}) \cup [\sigma, e)
\]

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Proof. The proof should follow by induction on $k$, and should be similar to the proof of Lemma 39.

Lemma 42 (Effect of assigning a derivable capability).
\[
\forall C, M_d, a, c.
C, M_d \vDash c \land a \in \text{reachable_addresses}(C, M_d)
\implies \text{reachable_addresses}(C, M_d[a \mapsto c]) = \bigcup_k \chi_k \left( \bigcup_{c' \in C} [e', \sigma, c' e] \cup [c, \sigma, c e), M_d, a] \right)
\]

Proof. Follows from Lemmas 17, 18, 20, 30 and 34.

Lemma 43 (Assigning a derivable capability does not enlarge reachability).
\[
\forall C, M_d, a, c.
C, M_d \vDash c \land a \in \text{reachable_addresses}(C, M_d)
\implies \text{reachable_addresses}(C, M_d[a \mapsto c]) \subseteq \text{reachable_addresses}(C, M_d)
\]

Proof. After substitution using Lemma 42, we apply Lemma 30 to get two subgoals that are provable using Lemma 31 and Lemma 28 respectively.

Definition 26 (Sub-capability-closed predicate). For a predicate $P : V \to \mathbb{B}$, sub-capability closure is defined as follows:
\[
\text{subcap_closed}(P) \overset{\text{def}}{=} \forall x, \sigma, e, \text{off}, \sigma', e'. P(x, \sigma, e, \text{off}) \land [\sigma', e'] \subseteq [\sigma, e] \implies P(x, \sigma', e', \text{off})
\]

Definition 27 (Z-trivial predicate). For a predicate $P : V \to \mathbb{B}$, Z-triviality is defined as follows:
\[
\text{z_trivial}(P) \overset{\text{def}}{=} \forall z \in \mathbb{Z}. P z
\]

Definition 28 (Offset-oblivious predicate). For a predicate $P : V \to \mathbb{B}$, offset obliviousness is defined as follows:
\[
\text{offset_oblivious}(P) \overset{\text{def}}{=} \forall x, \sigma, e, \text{off}_1, \text{off}_2. P(x, \sigma, e, \text{off}_1) \implies P(x, \sigma, e, \text{off}_2)
\]

Definition 29 (Allocation-compatible predicate). For a predicate $P : V \to \mathbb{B}$, and an allocation bound $\nabla$, allocation compatibility is defined as follows:
\[
\text{allocation_compatible}(P, \nabla) \overset{\text{def}}{=} \forall \sigma, e. [\sigma, e] \subseteq (\nabla, -1) \implies P(\delta, \sigma, e, 0)
\]

Definition 30 (State-universal predicate). A predicate $P : V \to \mathbb{B}$ holds universally for all values of a program state $s$ when:
\[
\text{state_universal}(P, s) \overset{\text{def}}{=} \forall a. P(s, M_d(a)) \land P(s, \text{ddc}) \land P(s, \text{stc}) \land P(s, \text{pcc}) \land P(s, \text{imp}(\text{mid}).\text{pcc}) \land P(s, \text{imp}(\text{mid}).\text{dcc}) \land P(s, \text{mstc}(\text{mid})) \land P((cc, dc, \_\_, \_\_) \in \text{stk}. P(cc) \land P(dc)
\]
Lemma 44 (Predicates that are guaranteed to hold on the result of expression evaluation).

\[ \forall \mathcal{E}, s, v. \]
\[ \mathcal{E}, s, M_d, s, ddc, s, stc, s, pcc \downarrow v \land \]
\[ \text{state\_universal}(P, s) \land \]
\[ \text{offset\_oblivious}(P) \land \]
\[ \text{z\_trivial}(P) \land \]
\[ \text{subcap\_closed}(P) \]
\[ \implies P(v) \]

Proof.
We assume the antecedents, and prove it by induction on expression evaluation.

1. Case evalconst,
2. Case evalBinOp,
3. Case evalCapType,
4. Case evalCapStart,
5. Case evalCapEnd, and
6. Case evalCapOff:
   All of these subgoals follow immediately by assumption \text{z\_trivial}(P) (unfolding Definition 27).
7. Case evalIncCap:
   Follows from the induction hypothesis, and by assumption \text{offset\_oblivious}(P) (unfolding Definition 28).
8. Case evalDeref:
   Follows from the assumption \text{state\_universal}(P, s) (unfolding Definition 30).
9. Case evalLim:
   Follows from the induction hypothesis, and by assumption \text{subcap\_closed}(P) (unfolding Definition 26).
10. Case evalddc, and
11. Case evalstc:
   Follow from assumption \text{state\_universal}(P, s) (unfolding Definition 30).
Lemma 45 (Preservation of state universality of predicates).

\[ \forall P, s, s'. \\
\text{s.aloc} < 0 \land \\
\text{state\_universal}(P, s) \land \\
\text{allocation\_compatible}(P, s'.\text{alloc} - 1) \land \\
\text{offset\_oblivious}(P) \land \\
\text{z\_trivial}(P) \land \\
\text{subcap\_closed}(P) \land \\
\text{s} \rightarrow^* \text{s}' \\
\implies \\
\text{state\_universal}(P, s') \land \text{s'.alloc} < 0 \]

We prove state\_universal\((P, s')\) by induction on \(\text{s} \rightarrow^* \text{s}'\):

- **Base case:**
  Immediate by assumption.

- **Inductive case:**
  Here, we have \(s''\) with state\_universal\((P, s'')\), \(s''.\text{alloc} < 0\), and \(s'' \rightarrow s'\). Our goal state\_universal\((P, s')\) consists of the following subgoals (by unfolding Definition 30):

  1. \(\forall a. P(s'.\text{Md}(a))\)
  2. \(P(s'.\text{ddc})\)
  3. \(P(s'.\text{stc})\)
  4. \(P(s'.\text{pcc})\)
  5. \(\forall \text{mid}. P(s'.\text{imp(mid)}).\text{pcc} \land P(s'.\text{imp(mid)}).\text{dcc} \land P(s'.\text{mstc(mid)})\)
  6. \(\forall (cc, dc, _, _) \in s'.\text{stk}. P(cc) \land P(dc)\)

For each of the possible cases of \(s'' \rightarrow s'\), we prove all of these subgoals:

1. **Case assign:**
   Subgoals 2, 3, 5, and 6 are immediate after substitution by the induction hypothesis state\_universal\((P, s'')\).

   For subgoal 4, we apply the assumption offset\_oblivious\((P)\) (unfolding Definition 28), so our generated subgoal is immediate by the induction hypothesis state\_universal\((P, s'')\).

   For subgoal 1, we have \(s'.\text{Md} = s''.\text{Md}[c \leftrightarrow v]\) with \(\mathcal{E}_r, s''.\text{Md}, s''.\text{ddc}, s''.\text{stc}, s''.\text{pcc} \downarrow v\), and we distinguish two cases for an arbitrary \(a \in \text{dom}(s'.\text{Md})\):
   - **Case** \(a = c.\sigma + c.\text{off}\):
     Here, our goal \(P(s'.\text{Md}(a))\) follows by Lemma 44.
   - **Case** \(a \neq c.\sigma + c.\text{off}\):
     Here, our goal \(P(s'.\text{Md}(a))\) follows by the induction hypothesis state\_universal\((P, s'')\) (unfolding Definition 30).

2. **Case allocate:**
   Subgoals 2, 3, 5, and 6 are immediate after substitution by the induction hypothesis state\_universal\((P, s'')\).
For subgoal 4, we apply the assumption offset oblivious \((P)\) (unfolding Definition 28), so our generated subgoal is immediate by the induction hypothesis state universal \((P, s'')\).

For subgoal 1, we have:

\[
s'.M_d = s'.M_d[c \mapsto (\delta, s'.nalloc, s''.nalloc, 0)]\]

\[
\{i \mapsto 0 \mid i \in [s'.nalloc, s''.nalloc]\},
\]

and we distinguish three cases for an arbitrary \(a \in \text{dom}(s'.M_d)\):

- **Case** \(a = c.\sigma + c.\text{off}\):
  Here, our goal \(P(s'.M_d(a))\) follows by applying assumption allocation compatible \((P, s'.nalloc - 1)\) (unfolding Definition 29) to get the following subgoal:
  \[
  [s'.nalloc, s''.nalloc] \subseteq (s'.nalloc - 1, -1)
  \]
  for which it suffices to show that:
  \[
  s'.nalloc - 1 < s'.nalloc
  \]
  (immediate), and
  \[
  s''.nalloc \leq -1
  \]
  which is immediate by the induction hypothesis \(s''.nalloc < 0\).

- **Case** \(a \in [s'.nalloc, s''.nalloc]\):
  Here, our goal \(P(s'.M_d(a))\) follows by assumption z trivial \((P)\) (unfolding Definition 27).

- **Case** \(a \notin [s'.nalloc, s''.nalloc] \land a \neq c.\sigma + c.\text{off}\):
  Here, our goal follows by the induction hypothesis state universal \((P, s'')\) (unfolding Definition 30).

3. **Case jump0:**
   Subgoals 1, 2, 3, 5, and 6 follow immediately after substitution by the induction hypothesis state universal \((P, s'')\).

Subgoal 4 follows by Lemma 44.

4. **Case jump1:**
   Subgoals 1, 2, 3, 5, and 6 follow immediately after substitution by the induction hypothesis state universal \((P, s'')\).

Subgoal 4 follows after applying assumption offset oblivious \((P)\) (unfolding Definition 28) from the induction hypothesis state universal \((P, s'')\).

5. **Case cinvoke:**
   For subgoal 1, and by inversion of cinvoke-aux, we distinguish the following three cases for an arbitrary \(a \in \text{dom}(s'.M_d)\):

   - **Case** \(a \in [s + \text{off}, s + \text{off} + n\text{Args}]\):
     Here, our goal follows by applying Lemma 44 (The generated subgoals are available by the preconditions of rule cinvoke-aux).

   - **Case** \(a \in [s + \text{off} + n\text{Args}, s + \text{off} + n\text{Args} + n\text{Local}]\):
     Here, our goal follows from the assumption z trivial \((P)\) (unfolding Definition 27).

   - **Case** \(a \notin [s + \text{off}, s + \text{off} + n\text{Args} + n\text{Local}]\):
     Here, our goal follows from the induction hypothesis state universal \((P, s'')\) (unfolding Definition 30).

Subgoal 2 follows by applying the induction hypothesis state universal \((P, s'')\) (unfolding Definition 30 and applying conjunct 
\[
\forall \text{mid}. \ P(s''.\text{imp(mid)}.\text{pcc}) \land P(s''.\text{imp(mid)}.\text{dcc}) \land P(s''.\text{mstc(mid)}).
\]
The generated subgoals are immediate by the preconditions of \texttt{cinvoke-aux} defining \texttt{s'.ddc}.

Subgoal 3 follows by applying the induction hypothesis \texttt{state\_universal}(P, s'') (unfolding Definition 30 and applying conjunct
\[ \forall \text{mid}. \, P(s''.\text{imp(mid)}).\text{pcc} \land P(s''.\text{imp(mid)}.\text{ddc}) \land P(s''.\text{mstc(mid)}). \]
The generated subgoals are immediate by applying assumption \texttt{offset\_oblivious}(P) and the preconditions of \texttt{cinvoke-aux} defining \texttt{s'.stc}.

Subgoal 4 follows by applying the induction hypothesis \texttt{state\_universal}(P, s'') (unfolding Definition 30 and applying conjunct
\[ \forall \text{mid}. \, P(s''.\text{imp(mid)}.\text{pcc}) \land P(s''.\text{imp(mid)}).\text{ddc} \land P(s''.\text{mstc(mid)}). \]
The generated subgoals are immediate by applying assumption \texttt{offset\_oblivious}(P) and the preconditions of \texttt{cinvoke-aux} defining \texttt{s'.pcc}.

For subgoal 5, the first two conjuncts follow by applying the induction hypothesis \texttt{state\_universal}(P, s'') (unfolding Definition 30 and applying conjunct
\[ \forall \text{mid}. \, P(s''.\text{imp(mid)}.\text{pcc}) \land P(s''.\text{imp(mid)}).\text{ddc} \land P(s''.\text{mstc(mid)}). \]
The generated subgoals are immediate by substitution.

For the third conjunct, we distinguish two cases:
- \textbf{Case} \texttt{mid = mid\texttt{cinvoke}}:
  Here, the proof is the same as the proof of subgoal 3 above, after noticing the pre-condition \texttt{s'.mstc(mid) = s'.stc} of \texttt{cinvoke-aux}, and \texttt{cinvoke}.
- \textbf{Case} \texttt{mid \neq mid\texttt{cinvoke}}:
  Here, again the goal follows by applying the induction hypothesis \texttt{state\_universal}(P, s'').

For subgoal 6, we distinguish the following cases:
- \textbf{Case} \texttt{(cc, dc, \_\_\_, \_)} = \texttt{top(s'.stk)}:
  Here, the goal follows by applying the induction hypothesis \texttt{state\_universal}(P, s'') (the conjuncts about \texttt{s''.pcc} and \texttt{s''.ddc}).
- \textbf{Case} \texttt{(cc, dc, \_\_\_, \_)} \neq \texttt{top(s'.stk)}:
  Here, the goal follows by applying the induction hypothesis \texttt{state\_universal}(P, s'') (the conjunct about \texttt{s''.stk}).

\textbf{6. Case} \texttt{creturn}:
Subgoal 1 follows immediately after substitution from the induction hypothesis \texttt{state\_universal}(P, s'').

Subgoal 2 follows by applying the induction hypothesis \texttt{state\_universal}(P, s'') (the conjunct about \texttt{s''.stk}).

Subgoal 3 follows by applying the induction hypothesis \texttt{state\_universal}(P, s'') (the conjunct about \texttt{s''.mstc}).

Subgoal 4 follows by applying the induction hypothesis \texttt{state\_universal}(P, s'') (the conjunct about \texttt{s''.stk}).

Subgoal 5 follows by applying assumption \texttt{offset\_oblivious}(P) followed by applying the induction hypothesis \texttt{state\_universal}(P, s'') (the conjunct about \texttt{s''.mstc}).

Subgoal 6 follows from the corresponding conjunct of the induction hypothesis \texttt{state\_universal}(P, s'') after noticing that \texttt{elems(s'.stk) \subseteq elems(s''.stk)}.
7. Case cexit:
All subgoals are immediate after substitution by the induction hypothesis state_universal\((P, s')\).

**Definition 31** (Code capabilities have an imports origin).

\[
\kappa_{\text{has_origin}}_{\text{imp}}(v) \overset{\text{def}}{=} \vdash_{\kappa} v \implies \exists \text{mid} \in \text{dom}(\text{imp}). v \subseteq \text{imp}(\text{mid}).\text{pcc}
\]

**Lemma 46** (\(\kappa_{\text{has_origin}}_{\text{imp}}\) is sub-capability closed).

\[
\forall \text{imp}. \text{subcap\_closed}(\kappa_{\text{has_origin}}_{\text{imp}})
\]

**Proof.**
By unfolding Definition 26 of sub-capability closure, we assume for arbitrary \(\text{imp}, x, \sigma, e, \text{off}, \sigma', e'\) that \(\kappa_{\text{has_origin}}_{\text{imp}}(x, \sigma, e, \text{off})\), and that \([\sigma', e'] \subseteq [\sigma, e]\).

Our goal is: \(\kappa_{\text{has_origin}}_{\text{imp}}(x, \sigma', e', \text{off})\).

By unfolding Definition 31, our goal is:

\[
\vdash_{\kappa} (x, \sigma', e', \text{off}) \implies \exists \text{mid} \in \text{dom}(\text{imp}). (x, \sigma', e', \text{off}) \subseteq \text{imp}(\text{mid}).\text{pcc}
\]

Two cases arise (after unfolding Definition 1):

- **Case** \(x = \kappa\):
  Here, after unfolding Definition 3, our goal holds by applying the transitivity of \(\subseteq\) on intervals. The generated subgoals follow from the assumptions (after unfolding Definitions 3 and 31 in the assumption).

- **Case** \(x \neq \kappa\):
  Here, our goal holds vacuously.

**Lemma 47** (\(\kappa_{\text{has_origin}}_{\text{imp}}\) is \(\mathbb{Z}\)-trivial).

\[
\forall \text{imp}. \text{z\_trivial}(\kappa_{\text{has_origin}}_{\text{imp}})
\]

**Proof.**
Our goal, by unfolding Definitions 27 and 31, then Definition 1 holds vacuously.

**Lemma 48** (\(\kappa_{\text{has_origin}}_{\text{imp}}\) is offset oblivious).

\[
\forall \text{imp}. \text{offset\_oblivious}(\kappa_{\text{has_origin}}_{\text{imp}})
\]

**Proof.**
Our goal, after unfolding Definitions 28 and 31 follows by applying Lemma 1 about the offset obliviousness of \(\subseteq\).

**Lemma 49** (\(\kappa_{\text{has_origin}}_{\text{imp}}\) is allocation compatible).

\[
\forall \nabla, \text{imp}. \text{allocation\_compatible}(\kappa_{\text{has_origin}}_{\text{imp}}, \nabla)
\]

**Proof.**
By unfolding Definition 29 of allocation-compatibility, it suffices to show for arbitrary \(\text{imp}\) that \(\kappa_{\text{has_origin}}_{\text{imp}}((\delta, \_\_, \_\_, \_\_))\).

This latter goal is vacuously true after we unfold Definition 31 then Definition 1.
Lemma 50 ($\_\text{has\_origin}_\text{imp}$ is initial-state-universal).

$$\forall t, s. t \vdash_i s \implies \text{state\_universal}(\_\text{has\_origin}_\text{imp}, s)$$

Proof.
We assume $t \vdash_i s$ for arbitrary $t$ and $s$.
By Definition 30, we have the following subgoals:

- $\forall a. \_\text{has\_origin}_\text{imp}(s, \mathcal{M}_d(a))$
  By unfolding Definitions 1 and 31 and inverting the assumption using initial-state, this subgoal is vacuously true.
- $\_\text{has\_origin}_\text{imp}(s, \text{ddc})$
  By unfolding Definitions 1 and 31 and inverting the assumption using initial-state then exec-state (obtaining $\vdash_{s} s, \text{ddc}$), this subgoal is vacuously true.
- $\_\text{has\_origin}_\text{imp}(s, \text{stc})$
  By unfolding Definitions 1 and 31 and inverting the assumption using initial-state then exec-state (obtaining $\vdash_{s} s, \text{stc}$), this subgoal is vacuously true.
- $\_\text{has\_origin}_\text{imp}(s, \text{pcc})$
  By unfolding Definitions 1 and 31 and inverting the assumption using initial-state, our goal is satisfied by choosing $\_\text{mid} = \text{mainMod}$.  

- $\forall \_\text{mid}'. \_\text{has\_origin}_\text{imp}(s, \text{imp}(\_\text{mid'}), \text{pcc})$
  By unfolding Definitions 1 and 31, this subgoal holds by the reflexivity of $\subseteq$ (choosing $\_\text{mid} = \_\text{mid'}$).

- $\forall \_\text{mid}'. \_\text{has\_origin}_\text{imp}(s, \text{imp}(\_\text{mid'}), \text{ddc})$
  By unfolding Definitions 1 and 31, and inverting the assumption using initial-state then exec-state (obtaining $\vdash_{s} s, \text{imp}(\_\text{mid'}), \text{ddc}$), this subgoal is vacuously true.

- $\forall \_\text{mid}'. \_\text{has\_origin}_\text{imp}(s, \text{mstc}(\_\text{mid'}))$
  By unfolding Definitions 1 and 31, and inverting the assumption using initial-state then exec-state (obtaining $\vdash_{s} s, \text{mstc}(\_\text{mid'})$), this subgoal is vacuously true.

- $\forall (cc, dc, \_\text{___}) \in s, \text{stk} \_\text{has\_origin}_\text{imp}(cc) \land \_\text{has\_origin}_\text{imp}(dc)$
  By unfolding Definitions 1 and 31 and inverting the assumption using initial-state, this subgoal is vacuously true.

This concludes the proof of Lemma 50. \qed

Lemma 51 ($\_\text{has\_origin}_\text{imp}$ is universal for subsequent states).

$$\forall t, s, s'. t \vdash_i s \land s \rightarrow^* s' \implies \text{state\_universal}(\_\text{has\_origin}_\text{imp}, s')$$

Proof.
By Lemma 50, we know (*):

$\text{state\_universal}(\_\text{has\_origin}_\text{imp}, s)$

We apply Lemma 45 to our goal to get the following subgoals:

- $s.nalloc < 0$
  Immediate by inversion of assumption $t \vdash_i s$ using rule initial-state.
• \( \text{state\_universal}(\kappa\_\text{has\_origin}_{s,\text{imp}}, s) \)
  Immediate by (*)

• \( \forall \nabla. \text{allocation\_compatible}(\kappa\_\text{has\_origin}_{s,\text{imp}}, \nabla) \)
  Immediate by Lemma 49.

• \( \text{offset\_oblivious}(\kappa\_\text{has\_origin}_{s,\text{imp}}) \)
  Immediate by Lemma 48.

• \( \text{z\_trivial}(\kappa\_\text{has\_origin}_{s,\text{imp}}) \)
  Immediate by Lemma 47.

• \( s \to^* s' \)
  Immediate by assumption.

This concludes the proof of Lemma 51.

Corollary 1 (There is at least one module that is executing at any time).

\( \forall t : \text{TargetSetup}, s, s' : \text{TargetState}. t \vdash_i s \land s \to^* s' \implies \exists c \in \text{range}(s'.\text{imp}). s'.\text{pcc} \subseteq c.1 \)

Proof. Follows by applying Lemma 51 after unfolding Definition 30 and Definition 31.

Lemma 52 (Preservation of \( \vdash_{\text{exec}} \) by reduction).

\( \forall t, s, s'. t \vdash_{\text{exec}} s \land s \to^* s' \implies t \vdash_{\text{exec}} s' \)

Proof. We assume the antecedent \( t \vdash_{\text{exec}} s \land s \to^* s' \) for arbitrary \( t, s, s' \).

By inversion using rules \text{exec-state} and \text{valid-program}, we obtain the following assumptions:

\( t \) definition
\( t = (M_{c, \_}, \text{imp}, \text{mstc}_t, \phi) \)

\( s \) definition
\( s = (M_c, M_d, \text{stk}, \text{imp}, \phi, \text{ddc}, \text{stc}, \text{pcc}, \text{mstc}, \text{nalloc}) \)

\( \text{pcc} \) type
\( \models_\kappa \text{pcc} \)

\( \text{ddc} \) type
\( \models_\delta \text{ddc} \)

\( \text{stc} \) type
\( \models_\delta \text{stc} \)

\text{nalloc} is negative
\( \text{nalloc} < 0 \)

Domains are \text{modIDs}
\( \text{modIDs} = \text{dom}(\text{imp}) = \text{dom}(\text{mstc}) = \text{dom}(\text{mstc}_t) \)

Static memory is non-negative
\[ \bigcup_{\text{mid} \in \text{modIDs}} [\text{imp}(\text{mid}).\text{ddc}.\sigma, \text{imp}(\text{mid}).\text{ddc}.e] \cup [\text{mstc}(\text{mid}).\sigma, \text{mstc}(\text{mid}).e] \cap (-\infty, 0) = \emptyset \]
Types of $imp$ and $mstc$
\[
\forall mid \in modIDs. \vdash_k imp(mid).pcc \land \vdash_d imp(mid).ddc \land \vdash_d mstc(mid)
\]

$mstc$ capabilities are in-bounds
\[
\forall mid \in modIDs. \vdash_d mstc(mid)
\]

$mstc$ offsets correspond to the sizes of frames of the called functions
\[
\forall mid \in modIDs. mstc(mid).off = \sum_{(\_\_\_, mid, fid) \in stk} \phi(mid, fid).nArgs + \phi(mid, fid).nLocal +
\]
\[
(main \in \text{dom}(imp(mid).offs) \land \phi(mid, main).nArgs + \phi(mid, main).nLocal : 0)
\]

Capability registers describe a module
\[
\exists mid \in modIDs. pcc \doteq imp(mid).pcc \land ddc \doteq imp(mid).ddc \land stc \doteq mstc(mid)
\]

$stk$ frames describe a module
\[
\forall (dc, cc, \_\_, \_\_) \in \text{elems}(stk). \vdash_d dc \land \vdash_k cc \land \exists mid \in modIDs. cc \doteq imp(mid).pcc \land dc \doteq imp(mid).ddc
\]

Capabilities describe parts of the memory domains
\[
\forall mid \in modIDs. imp(mid).pcc \subseteq \text{dom}(M_c) \land imp(mid).ddc \subseteq \text{dom}(M_d)
\]

Stack region is pre-allocated statically
\[
\forall mid \in modIDs. mstc(mid) = mstc_s(mid)
\]

Data memory is addressable at static locations and newly-allocated ones
\[
\text{dom}(M_d) = \bigcup_{mid \in modIDs} \{imp(mid).ddc, imp(mid).ddc.e \cup mstc(mid).\sigma, mstc(mid).e \cup \text{[nalloc, -1]}
\]

Reachable addresses are addressable
\[
\text{reachable_addresses} = \bigcup_{mid \in modIDs} \{imp(mid).ddc, mstc(mid)\}, M_d \subseteq \text{dom}(M_d)
\]

A module does not have access to any other module’s stack
\[
\forall mid, a. a \in \text{reachable_addresses}(\{mstc(mid), imp(mid).ddc\}, M_d) \implies a \notin \bigcup_{mid \in modIDs \setminus \{mid\}} mstc(mid').\sigma, mstc(mid').e
\]

Stack capabilities do not leak outside the stack
\[
\forall a, mid \in modIDs. M_d(a) = (\delta, \sigma, e, \_\_) \land (|\sigma, e| \subseteq mstc(mid) \implies a \in mstc(mid).\sigma, mstc(mid).e)
\]

Stack regions and data segments are disjoint
\[
\forall sc \in \text{range}(mstc), c \in \text{range}(imp). sc \cap c.2 = \emptyset
\]

No code capability lives in memory
\[
\forall a. M_d(a) \neq (\kappa, \sigma, e, \_\_)
\]

Data capabilities in memory describe addressable locations
\[
\forall a. M_d(a) = (\delta, \sigma, e, \_\_) \implies (|\sigma, e| \subseteq \text{dom}(M_d)
\]

Top of the stack mentions currently-executing module
\[
stk \neq \text{nil} \implies pcc = imp(top(stk).mid).pcc
\]

Each stack frame describes the module-identity of the pcc of in the next frame
\[
\forall i \in [1, \text{length}(stk) - 1]. stk(i).pcc = imp(stk(i - 1).mid).pcc
\]

Our goal consists of similar subgoals about $s'$. For brevity, we use for the subgoals the same names that were used for the assumptions above.

Subgoals $t$ definition, $s'$ definition are immediate.
Subgoals Domains are modIDs, Types of imp and mstc, Stack region is pre-allocated statically, Stack regions and data segments are disjoint, and Static memory is non-negative follow from their corresponding assumptions by applying Lemmas 2 and 55 obtaining subgoals that are immediate by the assumption \( s \rightarrow s' \).

By case distinction on the assumption \( s \rightarrow s' \), we get the following cases. We prove our remaining subgoals separately for each of them:

1. Case assign:

   We obtain the following preconditions:
   
   - (S'-PCC-IN-BOUNDS):
     \[ \forall \kappa. \text{s.pcc} \]
   - (S'-PCC):
     \[ s'.\text{pcc} = \text{inc}(s.\text{pcc}, 1) \]
   - (S'-INSTR):
     \[ s.\text{M}_c(s.\text{pcc}) = \text{Assign } \mathcal{E}_L \mathcal{E}_R \]
   - (ER-EVAL-V):
     \[ \mathcal{E}_R, s.\text{M}_d, s.\text{ddc}, s.\text{stc}, s.\text{pcc}, v \]
   - (EL-EVAL-C):
     \[ \mathcal{E}_L, s.\text{M}_d, s.\text{ddc}, s.\text{stc}, s.\text{pcc}, c \]
   - (C-IN-BOUNDS):
     \[ \forall \delta. \text{c} \]
   - (STC-PROHIBITION):
     \[ \forall \delta. v \implies (v \cap s.\text{stc} = \emptyset \lor c \subseteq s.\text{stc}) \]
   - (S'-MEM):
     \[ s'.\text{M}_d = s.\text{M}_d[c \mapsto v] \]
   - (S'-DDC):
     \[ s'.\text{ddc} = s.\text{ddc} \]
   - (S'-STC):
     \[ s'.\text{stc} = s.\text{stc} \]
   - (S'-NALLOC):
     \[ s'.\text{nalloc} = s.\text{nalloc} \]
   - (S'-STK):
     \[ s'.\text{stk} = s.\text{stk} \]
   - (S'-MSTC):
     \[ s'.\text{mstc} = s.\text{mstc} \]

   Subgoal \( s'.\text{pcc} \text{ type} \) follows from the corresponding assumption after unfolding using (S'-PCC) and the definition of \text{inc}.

   Subgoal \( s'.\text{ddc} \text{ type} \) is immediate from the corresponding assumption after substitution using (S'-DDC).

   Subgoal \( s'.\text{stc} \text{ type} \) is immediate from the corresponding assumption after substitution using (S'-STC).

   Subgoal \( s'.\text{nalloc} \text{ is negative} \) is immediate from the corresponding assumption after substitution using (S'-NALLOC).

   Subgoal mstc capabilities are in-bounds is immediate from the corresponding assumption after substitution using (S'-MSTC).

   Subgoal mstc offsets correspond to the sizes of frames of the called functions is immediate from the corresponding assumption after substitution using (S'-MSTC).
Subgoal **Capability registers describe a module** follows easily from the corresponding assumption after substitution using (S’-PCC), (S’-DDC), and (S’-STC) by the definition of inc and by instantiating Lemma 2.

Subgoal s’.stk frames describe a module follows easily from the corresponding assumption after substitution using (S’-STK) and instantiation of Lemma 2.

Subgoal Capabilities describe parts of the memory domains follows easily from the corresponding assumption after substitution using (S’-MEM) and noticing that dom(s’.Md) ⊇ dom(s.Md) and instantiation of Lemma 2.

For subgoal Data memory is addressable at static locations and newly-allocated ones, we have to prove:

\[ \text{dom}(s'.Md) = \bigcup_{\text{modIDs}} (s'.\text{imp}(\text{mid}),d\text{dc}.\sigma,s'.\text{imp}(\text{mid}),d\text{dc}.e) \cup \{ s'.\text{mstc}(\text{mid}).\sigma,s'.\text{mstc}(\text{mid}).e \} \cup \{ s'.\text{nalloc},-1 \} \]

By applying transitivity, it suffices to prove the following subgoals:

- We pick an arbitrary \( a \in \text{dom}(s.Md) \), and we show that \( a \in \text{dom}(s'.Md) \). This is immediate by (S’-MEM).
- We pick an arbitrary \( a \in \text{dom}(s'.Md) \), and we show that \( a \in \text{dom}(s.Md) \). We distinguish the following two cases:
  - Case \( c.\sigma + c.\text{off} \):
    Here, by applying the definition of \( \subseteq \) instantiated with assumption Reachable addresses are addressable, it suffices to instead show that:
    \( c.\sigma + c.\text{off} \in \text{reachable_addresses}(\bigcup_{\text{modIDs}} \{ s.\text{imp}(\text{mid}).d\text{dc}, s.\text{mstc}(\text{mid}) \}, s.Md) \)
    By applying Lemma 18, it suffices by easy set identities to show that:
    \( \exists \text{mid} \in \text{modIDs}. c.\sigma + c.\text{off} \in \text{reachable_addresses}(\{ s.\text{imp}(\text{mid}).d\text{dc}, s.\text{mstc}(\text{mid}) \}, s.Md) \)
    We then apply Lemma 19 obtaining the following subgoal (after applying some set identities):
    \( \exists \text{mid} \in \text{modIDs}, C. \)
    \( \text{addr}(C) \cup \text{addr}(\{ s.d\text{dc}, s.stc \}) = \text{addr}(\{ s.\text{imp}(\text{mid}).d\text{dc}, s.\text{mstc}(\text{mid}) \}) \) ∧
    \( c.\sigma + c.\text{off} \in \text{reachable_addresses}(\{ s.d\text{dc}, s.stc \}, s.Md) \)
    We choose the \( \text{mid} \) given by assumption Capability registers describe a module.
    And choose \( C := \{ \)
    \( (\delta, s.\text{imp}(\text{mid}).d\text{dc}.\sigma, s.d\text{dc}.\sigma, \_), \)
    \( (\delta, s.d\text{dc}.e, s.\text{imp}(\text{mid}).d\text{dc}.e, \_), \)
    \( (\delta, s.\text{mstc}(\text{mid}).\sigma, s.stc.\sigma, \_), \)
    \( (\delta, s.stc.e, s.\text{mstc}(\text{mid}).e, \_)) \}

The first conjunct is thus immediate by assumption Capability registers describe a module after unfolding the definition of addr in the goal and the Definition 3 of \( \subseteq \) in the assumption.

For the second conjunct, we apply Lemma 25, and some set identities obtaining the following subgoals:
\[ \mathcal{E}, s.\mathcal{M}_d, s.\text{ddc}, s.\text{stc}, s.\text{pcc} \Downarrow (\delta, c.\sigma, c.e, c.\text{off}) \]

Immediate by (EL-EVAL-C) and (C-IN-BOUNDS), after unfolding Definition 2.

\[ c.\sigma + c.\text{off} \in [c.\sigma, c.e) \]

Immediate by (C-IN-BOUNDS), after unfolding Definition 2.

\[ s.\text{pcc} = (\kappa, \_\_, \_\_, \_\_) \]

Immediate by assumption \text{pcc type}.

\[ s.\text{ddc} = (\delta, \_\_, \_\_, \_\_) \]

Immediate by assumption \text{ddc type}.

\[ s.\text{stc} = (\delta, \_\_, \_\_, \_\_) \]

Immediate by assumption \text{stc type}.

* Case \( a \neq c.\sigma + c.\text{off} \):

Here, by (S'-MEM), our goal is immediate.

\[ \text{dom}(s.\mathcal{M}_d) = \bigcup_{\text{mid} \in \text{modIDs}} \{ s.\text{imp}(\text{mid}).\text{ddc}, s.\text{imp}(\text{mid}).\text{ddc}.e \} \cup \{ s.\text{mstc}(\text{mid}).\sigma, s.\text{mstc}(\text{mid}).e \} \cup \{ s.\text{nalloc}, -1 \} \]

This is immediate by the assumption Data memory is addressable at static locations and newly-allocated ones.

For subgoal \text{Reachable addresses are addressable}, we have to prove that:

\[
\text{reachable_addresses}( \bigcup_{\text{mid} \in \text{modIDs}} \{ s'.\text{imp}(\text{mid}).\text{ddc}, s'.\text{mstc}(\text{mid}) \}, s'.\mathcal{M}_d) \subseteq \text{dom}(s'.\mathcal{M}_d)
\]

By applying the corresponding assumption, we are left with the following two subgoals:

- \( \text{dom}(s.\mathcal{M}_d) = \text{dom}(s'.\mathcal{M}_d) \)
  Proved above.
- \( \text{reachable_addresses}( \bigcup_{\text{mid} \in \text{modIDs}} \{ s.\text{imp}(\text{mid}).\text{ddc}, s.\text{mstc}(\text{mid}) \}, s.\mathcal{M}_d) = \text{reachable_addresses}( \bigcup_{\text{mid} \in \text{modIDs}} \{ s'.\text{imp}(\text{mid}).\text{ddc}, s'.\text{mstc}(\text{mid}) \}, s'.\mathcal{M}_d) \)
  By substitution using \( s'.\text{mstc} = s.\text{mstc} \) and \( s'.\text{imp} = s.\text{imp} \), it suffices to show that:
    \[
    \text{reachable_addresses}( \bigcup_{\text{mid} \in \text{modIDs}} \{ s.\text{imp}(\text{mid}).\text{ddc}, s.\text{mstc}(\text{mid}) \}, s.\mathcal{M}_d) = \text{reachable_addresses}( \bigcup_{\text{mid} \in \text{modIDs}} \{ s.\text{imp}(\text{mid}).\text{ddc}, s.\text{mstc}(\text{mid}) \}, s'.\mathcal{M}_d)
    \]
  Here, we apply Lemma 38.
  The generated subgoals are easy by (S'-MEM), (EL-EVAL-C) and by Lemma 25 using (ER-EVAL-V), and (C-IN-BOUNDS), unfolding Definition 23.

For subgoal \text{No code capability lives in memory}, we pick an arbitrary \( a \) where \( a \in \text{dom}(s'.\mathcal{M}_d) \).

Using (S'-MEM), we distinguish the following two cases:

- **Case** \( a \neq c.\sigma + c.\text{off} \):
  Here, our goal follows from assumption \text{No code capability lives in memory}.

- **Case** \( a = c.\sigma + c.\text{off} \):
  Here, our goal follows by applying Lemma 4 obtaining subgoals that are immediate by assumption \text{ddc type}, assumption \text{stc type}, assumption \text{No code capability lives in memory}, and by (ER-EVAL-V).

For subgoal \text{Data capabilities in memory describe addressable locations}, we pick an arbitrary \( a \) where \( a \in \text{dom}(s'.\mathcal{M}_d) \).

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Using (S’-MEM), we distinguish the following two cases:

- **Case a ≠ c.σ + c.aff:**
  Here, our goal follows from assumption **Data capabilities in memory describe addressable locations.**

- **Case a = c.σ + c.aff:**
  Here, instantiate Lemma 25 using (ER-EVAL-V) and using assumptions pcc **type, ddc type** and stc **type** obtaining:
  \[ v = (\delta, \sigma, e, _) \implies [\sigma, e] \subseteq \text{reachable_addresses}(\{\text{stc, ddc}\}, M_d) \]
  Instantiating this using our assumption above, we obtain:
  \[ [\sigma, e] \subseteq \text{reachable_addresses}(\{\text{stc, ddc}\}, M_d) \]
  By transitivity of \(\subseteq\) and using assumption **Reachable addresses are addressable**, we know:
  \[ [\sigma, e] \subseteq \text{dom}(M_d) \]
  which is our goal.

For subgoal **A module does not have access to any other module’s stack**, we have to prove:
\[ \forall \text{mid}, a. a \in \text{reachable_addresses}(\{s’.\text{mstc(mid)}, \text{imp(mid)}.\text{ddc}\}, s’.M_d) \implies a \notin \bigcup_{\text{mid’} \in \text{modIDs} \setminus \{\text{mid}\}} [s’.\text{mstc(mid’)}.\sigma, s’.\text{mstc(mid’)}.e] \]
Fix arbitrary \(\text{mid}, a\).
Assume \(a \in \text{reachable_addresses}(\{s.\text{mstc(mid)}, \text{imp(mid)}.\text{ddc}\}, s’.M_d)\) (applied (S’-MSTC))
Our goal is: \(a \notin \bigcup_{\text{mid’} \in \text{modIDs} \setminus \{\text{mid}\}} [s.\text{mstc(mid’)}.\sigma, s.\text{mstc(mid’)}.e]\) (applied (S’-MSTC))
By instantiating Lemma 38, we know that:
\(a \in \text{reachable_addresses}(\{s.\text{mstc(mid)}, \text{imp(mid)}.\text{ddc}\}, s.M_d)\)
which we use to instantiate the corresponding assumption (**A module does not have access to any other module’s stack**) immediately obtaining our goal.

For subgoal **Stack capabilities do not leak outside the stack**, we have to prove:
\[ \forall a, \text{mid} \in \text{modIDs}. s’.M_d(a) = (\delta, \sigma, e, _) \land [\sigma, e] \subseteq s’.\text{mstc(mid)} \implies a \in [s’.\text{mstc(mid)}.\sigma, s’.\text{mstc(mid)}.e] \]
Pick arbitrary \(a, \text{mid}\) where \(a \in \text{dom}(s’.M_d)\) and \(\text{mid} \in \text{modIDs}\).
Assume \(s’.M_d(a) = (\delta, \sigma, e, _)\),
and assume \([\sigma, e] \subseteq s’.\text{mstc(mid)}\).
Our goal is: \(a \in [s’.\text{mstc(mid)}.\sigma, s’.\text{mstc(mid)}.e]\).
By (S’-MSTC), it suffices to prove:
\(a \in [s.\text{mstc(mid)}.\sigma, s.\text{mstc(mid)}.e]\)
Using (S’-MEM), distinguish the following cases:

- **Case a = c.σ + c.aff:**
  By instantiating (STC-PROHIBITION) using the first assumption, we know (*):
  \[ v \cap s.\text{stc} \neq \emptyset \implies c \subseteq s.\text{stc} \]
  We claim: \([\sigma, e] \subseteq s’.\text{mstc(mid)} \implies s.\text{stc} \supseteq \text{mstc(mid)}\)
− Using assumption **Capability registers describe a module**, obtain \( \text{mid}^\ast \) with:
\[
s_{stc} = mstc(\text{mid}^\ast)
\]
− Thus, our claim becomes:
\[
\forall \text{mid} \quad [\sigma, e) \subseteq s_{mstc}(\text{mid}) \implies \text{mid} = \text{mid}^\ast
\]
− By Lemma 25, we know \([\sigma, e) \subseteq \text{reachable\_addresses}(\text{mstc}(\text{mid}^\ast), \text{imp}(\text{mid}^\ast), \text{ddc}); s, M_d)\) Thus, by instantiating assumption **A module does not have access to any other module’s stack**, we know:
\[
[\sigma, e) \subseteq s_{stc}
\]
Thus, by instantiating (*) obtaining:
\[
c \subseteq s_{stc}
\]
But by (C-IN-BOUNDS), we know:
\[
c.\sigma + c.\text{off} \in s_{stc}
\]
Thus, by easy substitutions using our case condition, and using the claim above about \(\text{mid}\), we obtain:
\[
a \in s_{mstc}(\text{mid})
\]
which is our goal.

**Case** \(a \neq c.\sigma + c.\text{off}\):

Here, by (S'-MEM), know \(s.M_d(a) = s'.M_d(a)\).

By instantiating the corresponding assumption about \(s.M_d\), we know:
\[
s'.M_d(a) = (\delta, \sigma, e, \_\_) \land \exists \text{mid} \in \text{modIDs} \quad [\sigma, e) \subseteq s_{mstc}(\text{mid}) \implies a \in s_{mstc}(\text{mid}).\sigma, s_{mstc}(\text{mid}).e
\]
By instantiation using the assumptions above, we immediately have our goal.

Subgoal **Top of the stack mentions currently-executing module** is immediate by substitution using (S'-STK) and (S'-PCC).

Subgoal **Each stack frame describes the module-identity of the pcc of in the next frame** is immediate by substitution using (S'-STK) and (S'-PCC).

This concludes the proof of **case assign**.

2. **Case allocate**:

We obtain the following preconditions:

(S-PCC-IN-BOUNDS):
\[
\vdash_{s} s.pcc
\]
(S'-PCC):
\[
s'.pcc = \text{inc}(s.pcc, 1)
\]
(S-INST):
\[
s.M_c(s.pcc) = \text{Alloc}\ \mathcal{E}_L, \mathcal{E}_R
\]
(ESIZE-EVAL-V):
\[
\mathcal{E}_{\text{size}}, s.M_d, s.ddc, s.stc, s.pcc \Downarrow v
\]
(EL-EVAL-C):
\[
\mathcal{E}_L, s.M_d, s.ddc, s.stc, s.pcc \Downarrow c
\]
(C-IN-BOUNDS):
\[
\vdash_{s} c
\]
(V-POSITIVE):
\[
v \in \mathbb{Z}^+
\]
(S'-MEM):
\( s'.M_d = s.M_d[i \mapsto (\delta, nalloc - v, nalloc, 0), i \mapsto 0 \; \forall i \in [nalloc - v, nalloc)] \)

(S'-DDC):
\( s'.ddc = s.ddc \)

(S'-STC):
\( s'.stc = s.stc \)

(S'-NALLOC):
\( s'.nalloc = s.nalloc - v \)

(S'-NALLOC-INF):
\( s'.nalloc > \nabla \)

(S'-STK):
\( s'.stk = s.stk \)

(S'-MSTC):
\( s'.mstc = s.mstc \)

Subgoal \( s'.pcc\) type follows from the corresponding assumption after unfolding using (S'-PCC) and the definition of inc.

Subgoal \( s'.ddc\) type is immediate from the corresponding assumption after substitution using (S'-DDC).

Subgoal \( s'.stc\) type is immediate from the corresponding assumption after substitution using (S'-STC).

Subgoal \( s'.nalloc\) is negative is immediate from the corresponding assumption after substitution using (S'-NALLOC) and noting (V-POSITIVE).

Subgoal mstc capabilities are in-bounds is immediate from the corresponding assumption after substitution using (S'-MSTC).

Subgoal mstc offsets correspond to the sizes of frames of the called functions is immediate from the corresponding assumption after substitution using (S'-MSTC).

Subgoal Capability registers describe a module follows easily from the corresponding assumption after substitution using (S'-PCC), (S'-DDC), and (S'-STC) by the definition of inc and by instantiating Lemma 2.

Subgoal s'.stk frames describe a module follows easily from the corresponding assumption after substitution using (S'-STK) and instantiation of Lemma 2.

Subgoal Capabilities describe parts of the memory domains follows easily from the corresponding assumption after substitution using (S'-MEM) and noticing that \( \text{dom}(s'.M_d) \supseteq \text{dom}(s.M_d) \) and instantiation of Lemma 2.

For subgoal Data memory is addressable at static locations and newly-allocated ones, we have to prove:
\[
\text{dom}(s'.M_d) = \bigcup_{\text{mid} \in \text{modIDs}} \left[ s'.\text{imp}(\text{mid}).\text{ddc}.\sigma, s'.\text{imp}(\text{mid}).\text{ddc}.\varepsilon \right] \cup \left[ s'.\text{mstc}(\text{mid}).\sigma, s'.\text{mstc}(\text{mid}).\varepsilon \right] \cup \left[ s'.\text{nalloc}, -1 \right]
\]
Using (S’-MEM) and properties about the map update operator, we know that (*):
\[ \text{dom}(s'.M_d \mid c \mapsto \_ ) = \text{dom}(s.M_d[c \mapsto \_] \cup [s'.\text{nalloc}, s.\text{nalloc}] \) \]

Thus, from (*) and (S’-NALLOC) and (V-POSITIVE) and by set identities, it suffices for our goal to show:
\[ \text{dom}(s.M_d[c \mapsto \_ ] ) = \bigcup_{\text{mid} \in \text{modIDs}} [s'.\text{imp}(\text{mid}).\text{ddc}.\sigma, s'.\text{imp}(\text{mid}).\text{ddc}.e] \cup [s'.\text{mstc}(\text{mid}).\sigma, s'.\text{mstc}(\text{mid}).e] \cup [s.nalloc, -1] \]

This is now exactly the same as the corresponding goal in case assign. We omit the proof here.

For subgoal \textbf{Reachable addresses are addressable}, we have to prove that:
\[ \text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{s'.\text{imp}(\text{mid}).\text{ddc}, s'.\text{mstc}(\text{mid})\}, s'.M_d) \subseteq \text{dom}(s'.M_d) \]

It suffices to show that:
\[ \text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{s.\text{imp}(\text{mid}).\text{ddc}, s.\text{mstc}(\text{mid})\}, s'.M_d) \subseteq \text{dom}(s'.M_d) \]

By instantiating Lemma 40 using \( M_d := s.M_d[i \mapsto 0 \mid i \in [s.nalloc - v, s.nalloc]] \), and \( \hat{a} := c.\sigma + c.\text{off} \) from (S’-MEM), we know (*):
\[ \text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{s.\text{imp}(\text{mid}).\text{ddc}, s.\text{mstc}(\text{mid})\}, s'.M_d) = \text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{s.\text{imp}(\text{mid}).\text{ddc}, s.\text{mstc}(\text{mid})\}, s.M_d) \cup [s'.\text{nalloc}, s.nalloc] \]

And by assumption \textbf{Reachable addresses are addressable}, we know (**):
\[ \text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{s.\text{imp}(\text{mid}).\text{ddc}, s.\text{mstc}(\text{mid})\}, s.M_d) \subseteq \text{dom}(s.M_d) \]

From (**) and (*) using set identities, we have:
\[ \text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{s.\text{imp}(\text{mid}).\text{ddc}, s.\text{mstc}(\text{mid})\}, s'.M_d) \subseteq \text{dom}(s.M_d) \cup [s'.\text{nalloc}, s.nalloc] \]

Thus, it suffices for our goal by substitution to show that:
\[ \text{dom}(s'.M_d) = \text{dom}(s.M_d) \cup [s'.\text{nalloc}, s.nalloc] \]

For this, it suffices to show that:
\[ \text{dom}(s.M_d[c \mapsto \_ ] ) = \text{dom}(s.M_d) \]

That has been proved for the previous subgoal. We avoid repetition.

For subgoal \textbf{No code capability lives in memory}, we pick an arbitrary \( a \) where \( a \in \text{dom}(s'.M_d) \).

Our goal is: \( s'.M_d(a) \neq (\kappa, \_, \_, \_, \_) \).

Using (S’-MEM), we distinguish the following three cases:

- **Case** \( a = c.\sigma + c.\text{off} \):
  Immediate by (S’-MEM).

- **Case** \( a \in [s'.\text{nalloc}, s.nalloc] \):
  Immediate by (S’-MEM).

- **Case** \( a \notin \{c.\sigma + c.\text{off}\} \cup [s'.\text{nalloc}, s.nalloc] \):
  Immediate by assumption \textbf{No code capability lives in memory}.  

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For subgoal **Data capabilities in memory describe addressable locations**, we pick an arbitrary \( a \) where \( a \in \text{dom}(s'.M_d) \).

Assume \( s'.M_d(a) = (\delta, \sigma, e, \_). \)

Our goal is: \( [\sigma, e] \subseteq \text{dom}(s'.M_d) \).

Using (S'-MEM), we distinguish the following three cases:

- **Case** \( a = c.\sigma + c.\text{off} \):
  Here, our goal follows by the map update operator in (S'-MEM).

- **Case** \( a \in [s'.\text{nalloc}, s.\text{nalloc}] \):
  Here, our goal is true after deriving a contradiction to assumption \( s'.M_d(a) = (\delta, \_ , \_ , \_). \)

- **Case** \( a \notin \{c.\sigma + c.\text{off}\} \cup [s'.\text{nalloc}, s.\text{nalloc}] \):
  Here, our goal follows by instantiating assumption **Data capabilities in memory describe addressable locations**.

For subgoal **A module does not have access to any other module’s stack**, we have to prove:

\[
\forall \text{mid}, a. \ a \in \text{reachable_addresses}(\{s'.\text{mstc}(	ext{mid}), \text{imp}(	ext{mid}).\text{ddc}\}, s'.M_d) \implies a \notin \bigcup_{\text{mid}' \in \text{modIDs} \setminus \{\text{mid}\}} [s'.\text{mstc}(	ext{mid'}).\sigma, s'.\text{mstc}(	ext{mid'}).e)
\]

Fix arbitrary \( \text{mid}, a \).

Assume \( a \in \text{reachable_addresses}(\{s.\text{mstc}(	ext{mid}), \text{imp}(	ext{mid}).\text{ddc}\}, s'.M_d) \) (applied (S’-MSTC))

Our goal is: \( a \notin \bigcup_{\text{mid}' \in \text{modIDs} \setminus \{\text{mid}\}} [s.\text{mstc}(	ext{mid'}).\sigma, s.\text{mstc}(	ext{mid'}).e) \) (applied (S’-MSTC))

By instantiating Lemma 40 using \( M_d := s.\text{M}_d[i \mapsto 0 \mid i \in [s.\text{nalloc} - v, s.\text{nalloc}], \text{and } \hat{a} := c.\sigma + c.\text{off} \) from (S'-MEM), we know (*):

\[
\text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{s.\text{imp}(	ext{mid}).\text{ddc}, s.\text{mstc}(	ext{mid})\}, s'.M_d) = \\
\text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{s.\text{imp}(	ext{mid}).\text{ddc}, s.\text{mstc}(	ext{mid})\}, s.M_d) \cup [s'.\text{nalloc}, s.\text{nalloc}]
\]

Thus, distinguish two cases:

- **Case** \( a \in \text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{s.\text{imp}(	ext{mid}).\text{ddc}, s.\text{mstc}(	ext{mid})\}, s.M_d) \):
  Here, instantiate the corresponding assumption, **A module does not have access to any other module’s stack**, obtaining our goal.

- **Case** \( a \in [s'.\text{nalloc}, s.\text{nalloc}] \):
  Here, our goal follows from both assumptions **Static memory is non-negative and nalloc is negative**.

For subgoal **Stack capabilities do not leak outside the stack**, we have to prove:

\[
\forall a, \text{mid} \in \text{modIDs}. \ s'.M_d(a) = (\delta, \sigma, e, \_ ) \land [\sigma, e] \subseteq s'.\text{mstc}(\text{mid}) \implies a \in [s'.\text{mstc}(\text{mid}).\sigma, s'.\text{mstc}(\text{mid}).e)
\]

Pick arbitrary \( a, \text{mid} \) where \( a \in \text{dom}(s'.M_d) \) and \( \text{mid} \in \text{modIDs} \).

Assume \( s'.M_d(a) = (\delta, \sigma, e, \_ ) \), and assume \( [\sigma, e] \subseteq s'.\text{mstc}(\text{mid}) \).

Our goal is: \( a \in [s'.\text{mstc}(\text{mid}).\sigma, s'.\text{mstc}(\text{mid}).e) \).
By (S'-MSTC), it suffices to prove:
\[ a \in [s.\text{mstc}(\text{mid}).\sigma, s.\text{mstc}(\text{mid}).e) \]

Using (S'-MEM), distinguish the following cases:

- **Case** \( a = c.\sigma + c.\text{off} \):
  Here, our goal is provable after deriving a contradiction to assumption \( [s'.\text{nalloc}, s.\text{nalloc}] \subseteq s'.\text{mstc}(\text{mid}) \) from assumptions *Static memory is non-negative* and *\text{nalloc is negative}.*

- **Case** \( a \in [s'.\text{nalloc}, s.\text{nalloc}] \):
  Here, our goal is provable after deriving a contradiction to assumption \( s'.\mathcal{M}_d(a) = (\delta, _, _, _) \) using (S'-MEM).

- **Case** \( a \notin [s'.\text{nalloc}, s.\text{nalloc}] \cup \{c.\sigma + c.\text{off}\} \):
  Follows from the corresponding assumption. **Stack capabilities do not leak outside the stack** using (S'-MEM).

Subgoal **Top of the stack mentions currently-executing module** is immediate by substitution using (S'-STK) and (S'-PCC).

Subgoal **Each stack frame describes the module-identity of the pcc of in the next frame** is immediate by substitution using (S'-STK) and (S'-PCC).

This concludes the proof of **case allocate**.

3. **Case** \( \text{jump0} \):

We obtain the following preconditions:

(S-PCC-IN-BOUNDS):
\[ \vdash_{\kappa} s.\text{pcc} \]

(S-INSTR):
\[ s.\mathcal{M}_c(s.\text{pcc}) = \text{JumpIfZero} \mathcal{E}_{\text{cond}} \mathcal{E}_{\text{off}} \]

(ECOND-EVAL-V):
\[ \mathcal{E}_{\text{cond}}, s.\mathcal{M}_d, s.\text{ddc}, s.\text{stc}, s.\text{pcc} \Downarrow v \]

(V-ZERO):
\[ v = 0 \]

(EOFF-EVAL-OFF):
\[ \mathcal{E}_{\text{off}}, s.\mathcal{M}_d, s.\text{ddc}, s.\text{stc}, s.\text{pcc} \Downarrow \text{off} \]

(OFF-INTEGER):
\[ \text{off} \in \mathbb{Z} \]

(S'-PCC):
\[ s'.\text{pcc} = \text{inc}(s.\text{pcc}, \text{off}) \]

(S'-MEM):
\[ s'.\mathcal{M}_d = s.\mathcal{M}_d \]

(S'-DDC):
\[ s'.\text{ddc} = s.\text{ddc} \]

(S'-STC):
\[ s'.\text{stc} = s.\text{stc} \]

(S'-NALLOC):
\[ s'.\text{nalloc} = s.\text{nalloc} \]

(S'-STK):
\[ s'.\text{stk} = s.\text{stk} \]
(S’-MSTC):
\[ s'.mstc = s.mstc \]
Subgoal \( s'.pcc \) follows from the corresponding assumption after unfolding using (S’-PCC) and the definition of inc.

Subgoal **Capability registers describe a module** follows easily from the corresponding assumption after substitution using (S’-PCC), (S’-DDC), and (S’-STC) by the definition of inc and by instantiating Lemma 2.

All other subgoals are immediate by the corresponding assumptions after substitution from the preconditions.

4. **Case jump1:**

We obtain the following preconditions:

(S-PCC-IN-BOUNDS):
\[ \vdash_s \mathcal{S}.pcc \]

(S-INST):
\[ s.\mathcal{M}_c(s.pcc) = \text{JumpIfZero } \mathcal{E}_{\text{cond}} \mathcal{E}_{\text{off}} \]

(ECOND-EVAL-V):
\[ \mathcal{E}_{\text{cond}}, s.\mathcal{M}_d, s.\mathcal{D}_c, s.\mathcal{S}_c, s.\mathcal{P}_c \vdash v \]

(V-NON-ZERO):
\[ v \neq 0 \]

(S’-PCC):
\[ s'.\mathcal{P}_c = \text{inc}(s.\mathcal{P}_c, 1) \]

(S’-MEM):
\[ s'.\mathcal{M}_d = s.\mathcal{M}_d \]

(S’-DDC):
\[ s'.\mathcal{D}_c = s.\mathcal{D}_c \]

(S’-STC):
\[ s'.\mathcal{S}_c = s.\mathcal{S}_c \]

(S’-NALLOC):
\[ s'.\mathcal{N}_{\text{alloc}} = s.\mathcal{N}_{\text{alloc}} \]

(S’-STK):
\[ s'.\mathcal{S}_{\text{tk}} = s.\mathcal{S}_{\text{tk}} \]

(S’-MSTC):
\[ s'.\mathcal{M}_{\text{stc}} = s.\mathcal{M}_{\text{stc}} \]

Subgoal \( s'.pcc \) follows from the corresponding assumption after unfolding using (S’-PCC) and the definition of inc.

Subgoal **Capability registers describe a module** follows easily from the corresponding assumption after substitution using (S’-PCC), (S’-DDC), and (S’-STC) by the definition of inc and by instantiating Lemma 2.

All other subgoals are immediate by the corresponding assumptions after substitution from the preconditions.
5. Case \texttt{cinvoke}:

We obtain the following preconditions (after inversion using \texttt{cinvoke-aux}):

\begin{enumerate}
\item [(S-PCC-IN-BOUNDS)]
\begin{align*}
\vdash \kappa_{s.pcc}
\end{align*}
\item [(S-INSTR)]
\begin{align*}
{s.M_c(s.pcc)} = \texttt{Cinvoke mid} \text{call fid} \text{call }\tau
\end{align*}
\item [(S'-STK)]
\begin{align*}
s'.stk = \texttt{push}(s.stk, (s.ddc, s.pcc, mid} \text{call, fid} \text{call})
\end{align*}
\item [(PHI-MID-FID)]
\begin{align*}
\phi(mid} \text{call, fid} \text{call}) = (nArugs, nLocal)
\end{align*}
\item [(MSTC-MID)]
\begin{align*}
s.mstc(mid} \text{call}) = (\delta, \sigma, e, off)
\end{align*}
\item [(S'-STC)]
\begin{align*}
s'.stc = (\delta, \sigma, e, off + nArugs + nLocal)
\end{align*}
\item [(Es-EVAL)]
\begin{align*}
\forall i \in [0, nArugs]. \overline{\tau(i), s.M_d, s.ddc, s.stc, s.pcc} \downarrow v_i
\end{align*}
\item [(NO-STC-LEAK)]
\begin{align*}
\forall i \in [0, nArugs]. \vdash_{\delta} v_i \implies v_i \cap s.stc = \emptyset
\end{align*}
\item [(S'-MEM)]
\begin{align*}
{s'.M_d} = s.M_d[\sigma + off + i \mapsto v_i \forall i \in [0, nArugs]][\sigma + off + nArugs + i \mapsto 0 \forall i \in [0, nLocal]]
\end{align*}
\item [(S'-MSTC)]
\begin{align*}
mstc' = mstc[mid} \text{call} \mapsto stc'
\end{align*}
\item [(IMP-MID)]
\begin{align*}
(c, d, offs) = \texttt{imp(mid} \text{call})
\end{align*}
\item [(S'-DDC)]
\begin{align*}
s'.ddc = d
\end{align*}
\item [(S'-PCC)]
\begin{align*}
s'.pcc = \texttt{inc}(c, offs(fid))
\end{align*}
\item [(S'-STC-IN-BOUNDS)]
\begin{align*}
\vdash_{\delta} s'.stc
\end{align*}
\end{enumerate}

Subgoal \textit{s'.pcc type} follows from assumption \textbf{Types of} \texttt{imp and mstc} instantiated with \texttt{mid} \text{call} after substitution from (IMP-MID) in (S'-PCC) and unfolding the definition of \texttt{inc}.

Subgoal \textit{s'.ddc type} follows from assumption \textbf{Types of} \texttt{imp and mstc} instantiated with \texttt{mid} \text{call} after substitution from (IMP-MID) in (S'-DDC).

Subgoal \textit{s'.stc type} is immediate from the corresponding assumption and (S'-STC).

Subgoal \textit{s'.nalloc is negative} is immediate from the corresponding assumption after substitution using (S'-NALLOC).

Subgoal \textit{mstc capabilities are in-bounds} follows from (S'-MSTC) and (S'-STC-IN-BOUNDS).

Subgoal \textit{mstc offsets correspond to the sizes of frames of the called functions} follows by easy arithmetic after substitution using (S'-MSTC), (S'-STC), and (S'-STK).
Subgoal **Capability registers describe a module** follows easily from (S'-PCC), (S'-DDC), and (S'-STC) after substitution using (MSTC-MID) and (IMP-MID).

For subgoal **s'.stk frames describe a module**, we distinguish two cases for arbitrary \( dc, cc \) with \( (dc, cc, _, _) \in \text{elems}(s'.stk) \):

- **Case** \( \text{top}(s'.stk) = (dc, cc, _, _) \):
  
  Here, our goal follows from assumptions pcc type, ddc type, and Capability registers describe a module after unfolding (S'-STK).

- **Case** \( \text{top}(s'.stk) \neq (dc, cc, _, _) \):
  
  Here, our goal follows from the corresponding assumption, stk frames describe a module.

Subgoal **Capabilities describe parts of the memory domains** follows easily from the corresponding assumption after substitution using (S'-MEM) and noticing that \( \text{dom}(s'.M_d) \supseteq \text{dom}(s.M_d) \) and instantiation of Lemma 2.

For subgoal **Data memory is addressable at static locations and newly-allocated ones**, we have to prove:

\[
\text{dom}(s'.M_d) = \bigcup_{mid \in \text{modIDs}} \{ s'.\text{imp}(mid).ddc.\sigma, s'.\text{imp}(mid).ddc.e \} \cup \{ s'.\text{mstc}(mid).\sigma, s'.\text{mstc}(mid).e \} \cup \{ s'.\text{nalloc}, -1 \}
\]

Notice by Lemma 2 and by substitution using (S'-MSTC), (S'-STC), and (S'-NALLOC) that it suffices to prove:

\[
\text{dom}(s'.M_d) = \bigcup_{mid \in \text{modIDs}} \{ s.\text{imp}(mid).ddc.\sigma, s.\text{imp}(mid).ddc.e \} \cup \{ s.\text{mstc}(mid).\sigma, s.\text{mstc}(mid).e \} \cup \{ s.\text{nalloc}, -1 \}
\]

Thus, by substitution using assumption Data memory is addressable at static locations and newly-allocated ones, it suffices to prove:

\[
\text{dom}(s'.M_d) = \text{dom}(s.M_d)
\]

Thus, it suffices by (S'-MEM) to prove \([\sigma + \text{off}, \sigma + \text{off}'] \subseteq \text{dom}(s.M_d)\).

By substitution again using assumption Data memory is addressable at static locations and newly-allocated ones, it suffices to prove:

\([\sigma + \text{off}, \sigma + \text{off}'] \subseteq [s.\text{mstc}(mid_{\text{call}}).\sigma, s.\text{mstc}(mid_{\text{call}}).e]\).

This follows from (S'-STC-IN-BOUNDS) and from assumption mstc capabilities are in-bounds.

For subgoal **Reachable addresses are addressable**, we have to prove that:

\[
\text{reachable_addresses}( \bigcup_{mid \in \text{modIDs}} \{ s'.\text{imp}(mid).ddc, s'.\text{mstc}(mid) \}, s'.M_d) \subseteq \text{dom}(s'.M_d)
\]

By Lemmas 6 and 18 instantiated using (S'-MSTC), it suffices to show that:

\[
\text{reachable_addresses}( \bigcup_{mid \in \text{modIDs}} \{ s.\text{imp}(mid).ddc, s.\text{mstc}(mid) \}, s'.M_d) \subseteq \text{dom}(s'.M_d)
\]

This follows similarly as in case assign.

Subgoal **No code capability lives in memory** follows similarly as in case assign.
Subgoal **Data capabilities in memory describe addressable locations** follows similarly as in case assign.

Subgoal **A module does not have access to any other module’s stack** is similar to the same subgoal of case assign.

For subgoal **Stack capabilities do not leak outside the stack**, we have to prove:

\[ \forall a, mid \in \text{modIDs}. \ s'.M_d(a) = (\delta, \sigma, e, _) \land [\sigma, e) \subseteq s'.\text{mstc}(mid) \implies a \in [s'.\text{mstc}(mid)].\sigma, s'.\text{mstc}(mid).e) \]

Pick arbitrary \( a, mid \) where \( a \in \text{dom}(s'.M_d) \) and \( mid \in \text{modIDs} \).

Assume \( s'.M_d(a) = (\delta, \sigma, e, _) \),

and assume \([\sigma, e) \subseteq s'.\text{mstc}(mid) \).

Our goal is: \( a \in [s'.\text{mstc}(mid)].\sigma, s'.\text{mstc}(mid).e) \).

By (S'-MSTC) and (S'-STC), it suffice to prove:

\( a \in [s.\text{mstc}(mid)].\sigma, s.\text{mstc}(mid).e) \)

Using (S'-MEM), distinguish the following cases:

- **Case \( a \in [\sigma + \text{off}, \sigma + \text{off} + \text{off} + \text{Args}) \):**
  - This is similar, after instantiating (NO-STC-LEAK) to the corresponding sub-case of case assign.

- **Case \( a \in [\sigma + \text{off} + \text{Args}, \sigma + \text{off} + \text{Args} + \text{Local}) \):**
  - Here, by contradiction from (S'-MEM) to assumption \( s'.M_d(a) = (\delta, \sigma, e, _) \), our goal follows vacuously.

- **Case \( a \notin [\sigma + \text{off}, \sigma + \text{off} + \text{Args} + \text{Local}) \):**
  - Here, have \( s'.M_d(a) = s.M_d(a) \) by (S'-MEM).
  - Thus, goal follows by instantiating the corresponding assumption Stack capabilities do not leak outside the stack.

Subgoal **Top of the stack mentions currently-executing module** follows immediately from the preconditions (S'-STK), (S'-PCC), and (IMP-MID).

Subgoal **Each stack frame describes the module-identity of the pcc of in the next frame** follows in one case from assumption Top of the stack mentions currently-executing module after noticing the precondition (S'-STK), and in the other cases from the corresponding assumption.

This concludes the proof of case cinvoke.

6. **Case creturn:**

We obtain the following preconditions:

- **(S-PCC-IN-BOUNDS):**
  \[ \vdash_s \text{pcc} \]

- **(S-INSTR):**
  \[ s.M_c(s.\text{pcc}) = \text{Creturn} \]

- **(S'-STK-DDC-PCC):**
  \[ stk', (ddc', \text{pcc'}, mid, fid) = \text{pop}(stk) \]

- **(PHI-MID-FID):**
  \[ \phi(mid, fid) = (n\text{Args}, n\text{Local}) \]
(MSTC-MID):
\((\delta, s, e, \text{off}) = \text{mstc}(\text{mid})\)

(OFF'):
\(\text{off}' = \text{off} - n\text{Args} - n\text{Local}\)

(S'-MSTC-MID):
\(\text{mstc}' = \text{mstc}[\text{mid} \mapsto (\delta, s, e, \text{off}')]\)

(S'-STC):
\(\exists \text{mid}', \text{pcc}' \doteq \text{imp}(\text{mid}').\text{pcc} \land \text{stc}' = \text{mstc}(\text{mid}')\)

(S'-MEM):
\(s'.\mathcal{M}_d = s.\mathcal{M}_d\)

(S'-NALLOC):
\(s'.\text{nalloc} = s.\text{nalloc}\)

Subgoal \(s'.\text{pcc} \text{ type}\) follows from assumption \(\text{stk frames describe a module}\) after substitution using (S'-STK-DDC-PCC).

Subgoal \(s'.\text{ddc} \text{ type}\) follows from assumption \(\text{stk frames describe a module}\) after substitution using (S'-STK-DDC-PCC).

Subgoal \(s'.\text{stc} \text{ type}\) follows from assumption \(\text{Types of imp and mstc after substitution using (S'-STC)}\).

Subgoal \(s'.\text{nalloc} \text{ is negative}\) is immediate from the corresponding assumption after substitution using (S'-NALLOC).

For subgoal \(\text{mstc capabilities are in-bounds}\), we fix an arbitrary \(\text{mid}'\) such that \(\text{mid}' \in \text{modIDs}\).

Our goal (after unfolding Definition 2, applying arithmetic, and removing the already proven conjunct, \(\models s'.mstc(\text{mid}')\)) is:
\(s'.mstc(\text{mid}').\text{off} \in [0, s'.mstc(\text{mid}').e - s'.mstc(\text{mid}').\sigma]\)

Distinguish two cases:

- **Case \(\text{mid}' = \text{mid}\):**
  Here, our goal follows by arithmetic after substitutions using (S'-STK-DDC-PCC), (PHI-MID-FID), (OFF'), (S'-MSTC-MID), and assumption \(\text{mstc offsets correspond to the sizes of frames of the called functions}\).

- **Case \(\text{mid}' \neq \text{mid}\):**
  Here, goal follows from the corresponding assumption \(\text{mstc capabilities are in-bounds}\).

Subgoal \(\text{mstc offsets correspond to the sizes of frames of the called functions}\) follows by arithmetic after substitutions using (S'-STK-DDC-PCC), (S'-MSTC-MID), (OFF)', and (PHI-MID-FID).

Subgoal **Capability registers describe a module** follows from assumptions \(\text{stk frames describe a module}\), and (S'-STC) after substitution using (S'-STK-DDC-PCC).
Subgoal **stk frames describe a module** follows by instantiating the corresponding assumption after noticing from \((S'\text{-STK-DDC-PCC})\) that \(\text{elems}(s'.\text{stk}) \subset \text{elems}(s.\text{stk})\).

Subgoal **Capabilities describe parts of the memory domain** follows by substitution using \((S'\text{-MEM})\) and Lemma 2 from the corresponding assumption.

Subgoal **Data memory is addressable at static locations and newly-allocated ones** follows from the corresponding assumption after substitution using \((S'\text{-MEM})\) and \((S'\text{-NALLOC})\).

Subgoal **Reachable addresses are addressable** follows from the corresponding assumption after substitution using \((S'\text{-MEM})\).

Subgoal **A module does not have access to any other module’s stack** follows from the corresponding assumption after substitution using \((S'\text{-MEM})\).

Subgoal **Stack capabilities do not leak outside the stack** follows from the corresponding assumption after substitution using \((S'\text{-MEM})\).

Subgoal **No code capability lives in memory** follows from the corresponding assumption after substitution using \((S'\text{-MEM})\).

Subgoal **Data capabilities in memory describe addressable locations** follows from the corresponding assumption after substitution using \((S'\text{-MEM})\).

Subgoal **Top of the stack mentions currently-executing module** follows from assumption **Each stack frame describes the module-identity of the pcc of in the next frame** by noticing the precondition \((S'\text{-STK-DDC-PCC})\).

Subgoal **Each stack frame describes the module-identity of the pcc of in the next frame** follows immediately from the corresponding assumption after noticing the precondition \((S'\text{-STK})\).

This concludes the proof of case **creturn**.

7. **Case cexit:**

   All goals are immediate by substitution. Notice that \(s' = s\).

This concludes the proof of Lemma 52.

**Corollary 2** (Preservation of \(\vdash_{\text{exec}}\) by \(\rightarrow^*\)).

\[\forall t, s, s'. t \vdash_{\text{exec}} s \land s \rightarrow^* s' \implies t \vdash_{\text{exec}} s'\]

*Proof.* Easy by Lemma 52.

**Corollary 3** (Data and stack capabilities always hold a data-capability value).

\[\forall t : \text{TargetSetup}, s, s' : \text{TargetState}. t \vdash_{\text{exec}} s \land s \rightarrow^* s' \implies (s'.\text{ddc} \in \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \land s'.\text{stc} \in \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})\]

*Proof.*

Follows by Lemma 52.
Lemma 53 (Preservation of ⊢_{exec} by ≃).
∀t,s,s'. t ⊢_{exec} s ∧ s ≃ s' =⇒ t ⊢_{exec} s'

Proof. After inversion of the assumptions using rules cinvoke-aux and exec-state, the proof proceeds similarly to case cinvoke in the proof of Lemma 52. We avoid repetition.

Lemma 54 (At the initial state, the program counter capability pcc and the data capability ddc are prescribed by some capability object).
∀t,s. t ⊬ i s =⇒ ∃(cc, dc, _) ∈ range(s.imp). pcc ⊆ cc ∧ ddc ⊆ dc

Proof. Immediate by inversion using rules initial-state then exec-state.

Claim 2 (At the initial state, the data and stack capabilities are disjoint).
∀t,s. t ⊬ i s =⇒ s.stc ∩ s.ddc = ∅

Proof. Immediate by rules initial-state and exec-state.

Claim 3 (Uniqueness of the initial state (Existence of at most one initial state for a given TargetSetup)).

∀t : TargetSetup, funIDs.
funIDs = [fid | fid ∈ dom(offs) ∧ (_, _, off) ∈ range(t.imp) ] ∧
all_distinct(funIDs) ∧
∃s,s'. t ⊬ i s ∧ t ⊬ i s'
⇒ s = s'

Proof. Follows from rules initial-state and exec-state.

Lemma 55 (Preservation of the bounds of stack capabilities).
∀s. s → s' =⇒ (∀mid, σ, e. s.mstc(mid) = (δ, σ, e, _) =⇒ s'.mstc(mid) = (δ, σ, e, _))

Proof. We fix an arbitrary state s, assume the antecedent s → s' and consider all the possible cases for s → s':

1. Case assign,
2. Case allocate,
3. Case jump1, and
4. Case jump0:

In all of these cases, we notice that s.mstc = s'.mstc, and so our goal follows by definition of equality on maps.

5. Case cinvoke:

Here, we obtain the necessary precondition s ≃ s', from which by rule cinvoke-aux, we obtain the following necessary conditions for some fixed mid:

- s.mstc(mid) = (δ, σ, e, off)
- stc' = (δ, σ, e, off')
- s'.mstc = s.mstc[mid → stc']

Thus, we can show our goal for an arbitrary mid' ∈ dom(s.mstc) by case distinction on mid':

- Case mid' = mid:
  In this case, our goal follows from stc'.σ = s.mstc(mid).σ and stc'.e = s.mstc(mid).e.
• Case $mid' \neq mid$:
  In this case, the value in the $s'.mstc$ map was not updated, so our goal follows from $s'.mstc(mid') = s.mstc(mid')$.

6. Case creturn:
  This case is similar to cinvoke.

1.4 Summary of target language features

Our model, CHERIExpress, aims to model the essential security features provided by the CHERI hardware architecture and its runtime library, libcheri. In particular, call invocations between mutually distrustful components is a core feature of CHERI, which can be used to attain compartmentalized execution [3]. Passing parameters of function calls while ensuring non-retention of access to the stack frame of the callee after the call has returned is also a core feature of CHERI that we model in our language using the stack capability, and a restriction on storing the stack capability in memory (note that the rule assign categorically prohibits storing the stack capability in memory). In the actual CHERI architecture, these restrictions can be implemented using what is called the “permissions field” on capabilities. Here, we abstract a bit by modeling specific uses of this field rather than the field itself. Formal arguments showing that the permissions field can actually be used to attain our abstractions already exist in prior work [3,4].

One limitation (to attacker strength) in our CHERIExpress model is that the default data capability ($ddc$), and the stack capability ($stc$) are managed by the trusted call (cinvoke) and return (creturn) instructions, but there is no way to assign them directly. While in the actual CHERI architecture, only system-reserved registers are protected from arbitrary load operations [2], we still claim that our additional reservation on the root data and stack capability registers does not significantly weaken the attacker model. In particular, rather than being able to change the view of the memory by changing the values of $ddc$ and $stc$, an attacker code that gets access to unlawful data-capabilities can still use them to load data from the unlawful memory region and store it in the region referenced by the current fixed $ddc$ and $stc$. This way, it (the malicious code) can effectively change the view of the memory by copying the actual data rather than by directly installing the stolen data capabilities into the $ddc$ or $stc$ registers.

This built-in trust though (in how $ddc$ and $stc$ are managed) admittedly weakens the attacker model a bit because it enables for honest code the defense mechanism of checking the integrity of the data capabilities before executing sensitive code. So, subverting control flow attacks are allowed, but they are constrained in the sense that data capability registers are not arbitrarily loadable.
2  A source language (*ImpMod*) with pointers and modules

The source language of our transformation is a simple imperative language *ImpMod* that features modules and functions with conditional goto statements. By design, *ImpMod* features protection of module-private variables.

2.1 Program and module representation, and well-formedness

A program in *ImpMod* consists of a list of modules. Each module consists of a list of function definitions, and a list of module-private variables. We skip the syntax of module and function definitions, and we directly represent them as structures (tuples of lists) that are output by the parser. We refer to the set of module identifiers as *ModID*, function identifiers as *FunID*, variable identifiers as *VarID*, and commands as *Cmd*. We give the syntax for commands and expressions later. We define the set of functions as *FunDef* = *ModID* × *FunID* × *VarID* × *VarID* × *Cmd* where a function specifies argument names *args*, local variable names *localIDs*, and a body (list of commands). Modules *Mod* = *ModID* × *VarID* × *FunDef* where a module specifies a list of module-private variable names, and a list of function definitions. Programs *Prog* = *Mod* are lists of modules subject to the following well-formedness conditions (formally stated in fig. 4):

1. Module identifiers are unique across the program.
2. Function identifiers are unique across the program.
3. Programs are closed (i.e., the set of all function identifiers existing in a program contains all the function identifiers that are called by any command in the program).
4. The last command of every function is a *Return*.

We refer to the operation of linking two lists of modules *mods*1 and *mods*2 into one well-formed program *P* as *P* = *mods*1 × *mods*2 where × reorders and concatenates the two lists of modules only if they form a well-formed program *P*, and is not defined otherwise.

**Definition 32** (Valid linking). Two programs (lists of modules) can be linked if there exists *m* where judgment *m*1 × *m*2 = [*m*] holds according to rule Valid-linking-src in Figure 7. If that is the case, then we sometimes write *m*1[*m*2] for such *m*.

2.2 Values, expressions, and commands

Expressions *E* ::= *addr*(*VarID*) | *derref*(*E*) | *E* + *E* | *Z* | *VarID* | *E* | *addr*(*E*[*E*]) | *start*(*E*) | *end*(*E*) | *offset*(*E*) | *limRange*(*E*, *E*, *E*) | capType(*E*) in *ImpMod* manipulate integer values and a bounds-checked version of C pointers. Expressions allow reading and storing addresses of variables and they allow basic pointer arithmetic (addition) and by definition of the evaluation semantics, they allow only safe dereferencing. Evaluation of an expression that performs an unsafe memory dereference gets stuck. Values *V* = *Z* ⊔ (∅, δ, κ) × *Z* × *Z* × *Z* are integers, or fat pointer values (i.e., values that represent the bounds and offset of a memory entity). The labels δ and κ on fat pointers indicate that the permissions available on the memory entity (the pointee) are data or code permissions respectively. The availability of code permissions still does not allow the source language semantics to execute this code; only code that is part of the program definition is executable (see *Jump-zero*, *Jump-non-zero* and *Call*). The ability to distinguish code pointers from data pointers though is important for defensive programming (and hence, for enhancing the expressiveness of the source programs as compared to the target ones, which is needed for proving that the translation between the two languages is fully abstract). Evaluation of expressions is given by the rules of the form *E*, *Σ*, *Δ*, *β*, *MVar*, *Fd*, *Mem*, *Φ*, *pc* ↓ *V*.

The syntax of commands is given by the grammar

*Cmd* ::= *Assign* *E*1 *E*2 | *Alloc* *E*1 *E*size | *Call* *FunID* *E* | *Return* | *JumpIfZero* *E* ε *E*off | Exit.
Figure 4: Well-formed programs of ImpMod

\[
\begin{align*}
\text{(Whole program)} \\
\text{wfp}(P) \\
\forall \text{cmd}. \ (\text{cmd} = \text{Call} \ fid \ _\_ \ \land \ \exists \text{mod}, \_ \_ \_ \ \in \ \text{mods} \ \land \ fid \ \in \ \text{funDefs(mod)} \ \land \\
\text{cmd} \ \in \ \text{commands(fd)} \ \implies \ \exists \text{mod'}, \_ \_ \_ \_ \_ \ \in \ \text{mods} \ \land \ \text{funDef'(mod')} \ \land \ \_ \_ \_ \_ \_ \ = \ \text{funID(fd')} \\
\end{align*}
\]

\[
\begin{align*}
\text{(Well-formed program)} \\
P = \overline{\text{mods}} \ \forall \text{mod} \in \overline{\text{mods}}. \ \text{MVar(mid)} \cap \{ \text{localIDs(fd)} \ \cup \ \text{args(fd)} \ \mid \ fd \in \text{funDefs(mod)} \} = \emptyset \\
\forall \text{mod}, \text{mod'} \in \overline{\text{mods}}. \ \text{moduleID(mod)} = \text{moduleID(mod')} \ \implies \ \text{mod} = \text{mod'} \\
\forall \text{mod}, \text{mod'}, \text{mod''}. \ (\text{mod}, \text{mod'} \in \overline{\text{mods}} \ \land \ \text{funDef'(mod')} \ \land \ \text{funID(fd)} = \text{funID(fd')} \ \implies \ (\text{fd} = \text{fd'} \ \land \ \text{mod} = \text{mod'}) \\
\end{align*}
\]

\[
\begin{align*}
\text{(Well-formed program and parameters)} \\
\text{wfp}(P) \\
\text{wfp(mods)} \\
\text{modIDs} = \{ \text{modID} \mid (\text{modID}, \_\_\_, \_) \in \overline{\text{mods}} \} \\
\forall \text{mid}, \text{mid'} \in \overline{\text{modIDs}}. \ \text{mid} \neq \text{mid'} \ \implies \\
\Delta(\text{mid}) \cap \Delta(\text{mid'}) = \emptyset \ \land \ \text{Kmod(mid)} \cap \text{Kmod(mid')} = \emptyset \ \land \ \Sigma(\text{mid}) \cap \Sigma(\text{mid'}) = \emptyset \\
\bigcup(\Delta(\text{mid})) \cap \bigcup(\Sigma(\text{mid})) \cap (-\infty, 0) = \emptyset \\
\text{dom(Kmod)} = \text{dom(MVar)} = \text{dom(Σ)} = \text{dom(Δ)} = \overline{\text{modIDs}} \\
\text{Fd} = \text{fd_map(mods)} \ \quad \text{MVar} = \text{mvar(mods)} \\
\text{dom(β)} = \{ (\text{vid}, \text{fid}, \text{mid}) \mid \text{mid} \in \overline{\text{modIDs}} \ \land \\
(\text{vid} \in \text{MVar(mid)} \ \land \ \text{fid} = \bot \lor \text{fid} \in \text{dom(Fd)} \ \land \ \text{vid} \in \text{localIDs(Fd(fid))} \ \cup \ \text{args(Fd(fid))}) \\
\forall \text{mid}, \text{fid}, \text{vid}. \ \text{vid} \in \text{args(Fd(fid))} \ \land \ \text{β(vid, fid, mid)} = (s, e) \ \implies \ |s - e| = 1 \\
\forall \text{fid} \in \text{dom(Fd)}. \ \text{frameSize(Fd(fid))} \geq 0 \\
\forall \text{mid} \in \overline{\text{modIDs}}, \text{fid} \in \text{dom(Fd)}. \ \bigcup_{\text{vid} \in \text{localIDs(Fd(fid))} \cup \text{args(Fd(fid))}} \ \text{β(vid, fid, mid)} = \text{frameSize(Fd(fid))} \\
\forall \text{mid} \in \overline{\text{modIDs}}. \ \bigcup_{\text{mid} \in \text{MVar(mid)}} \ \text{β(vid, fid, mid)} = |\text{commands(Fd(fid))}| \\
\forall \text{mid} \in \overline{\text{modIDs}}, \text{fid} \in \text{dom(Fd)}. \ |\text{Kfun(Fd)}| = |\text{commands(Fd(fid))}| \\
\forall \text{mid} \in \overline{\text{modIDs}}. \ \bigcup_{\text{mid} \in \text{MVar(mid)}} \ \text{Kfun(Fd)} = |\text{Kmod(mid)}| \\
\end{align*}
\]

\[
\begin{align*}
\text{wfp\_params(mods, Δ, Σ, β, Kmod, Kfun)}
\end{align*}
\]
2.3 Program state

A program state \((\text{Mem}, stk, pc, \Phi, nalloc)\) whose type is denoted by \textit{SourceState} consists of:

- a data memory \(\text{Mem} : \mathbb{Z} \xrightarrow{\text{fun}} \mathcal{V}\) which is a map from addresses \(\mathbb{Z}\) to values \(\mathcal{V}\).
- a call stack \(\text{stk} : \text{FunID} \times \mathbb{N}\) which is a list of program counters that record the function calls history (see \(pc\) below),
- \(\Phi : \text{ModID} \rightarrow \mathbb{Z}\) which maintains for every module a pointer to its top-most stack frame,
- a program counter \(pc : \text{FunID} \times \mathbb{N}\) modeling the index of the executing command within the list of commands of the current function. We define \(\text{inc}((\text{funId}, n)) \overset{\text{def}}{=} (\text{funId}, n + 1)\).
- and an allocation status \(nalloc : \mathbb{Z}\) which simply represents the first (in descending order) free memory address (i.e., the first address that was never allocated before).

A program evaluation context \(\Sigma; \Delta; \beta; MVar; Fd\) consists of:

- \(\Sigma : \text{ModID} \rightarrow \mathbb{Z}^2\) which maintains for every module the start and end addresses of its stack region. Recall that each module in \textit{ImpMod} has its own stack which stores the local variables when this module is callee. Notice that return pointers on the other hand are stored on the trusted stack \(\text{stk}\) rather than on a module’s own stack. The latter only stores arguments and local variables,
- \(\Delta : \text{ModID} \rightarrow \mathbb{Z}^2\) which maps each module to a range of addresses representing the data segment in which the static data of the module lives. Offsets from \(\beta\) are added to the first component of the range that is output by this map in order to compute the location in memory of module-global variables.
- \(\beta : (\text{VarID} \times (\text{FunID} \sqcup \bot) \times \text{ModID}) \rightarrow \mathbb{Z}^2\) which maps each variable identifier to bounds that represent the offsets within the data segment or the stack frame to which the (module-global or function-local) variable is mapped,
- an immutable map \(MVar : \text{ModID} \rightarrow \text{VarID}\) of module IDs to module-private variable identifiers,
- and an immutable map \(Fd : \text{FunID} \rightarrow \text{FunDef}\) of function identifiers to function definitions.

The following are useful representations of a program:

**Definition 33** (Set of function definitions of a list of modules).

\[
\text{fun-defs}^{\overline{\text{mods}}} \overset{\text{def}}{=} \{ \text{mdef} \mid \text{mdef} \in \text{mdefs} \land (\_, \_, \text{mdefs}) \in \overline{\text{mods}} \}
\]

**Definition 34** (Function ID to function definition map).

\[
\text{fd-map}^{\overline{\text{mods}}} \overset{\text{def}}{=} \{ \text{fid} \mapsto \text{fdef} \mid \text{fdef} \in \text{fun-defs}^{\overline{\text{mods}}} \land \text{fdef} = (\_, \text{fid}, \_, \_, \_)) \}
\]

**Definition 35** (Module variables map).

\[
\text{mvar}^{\overline{\text{mods}}} \overset{\text{def}}{=} \{ \text{mid} \mapsto \overline{\text{vids}} \mid (\text{mid}, \overline{\text{vids}}, \_) \in \overline{\text{mods}} \}
\]

The semantics of expressions and commands are given in fig. 5 and fig. 6.
Figure 5: Evaluation of expressions $E$ in ImpMod

$z, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow z$

$e, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow (-, z, -, -)$

$\text{(Evaluate-expr-const)}$

$z \in \mathbb{Z}$

$\text{(Evaluate-expr-to-integer-start)}$

$\Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow z$

$\text{(Evaluate-expr-to-integer-end)}$

$e, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow (-, z, -, -)$

$\text{Offset}(e), \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow z$

$\text{(Evaluate-expr-cap-type)}$

$\text{capType}(e), \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow v$

$e_1, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow z_1$

$e_2, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow z_2$

$z_r = z_1 \oplus z_2$

$\text{(Evaluate-expr-binop)}$

$x \in \mathbb{Z} \implies v = 0$

$x \in \{k\} \times \mathbb{Z} \times \mathbb{Z} \implies v = 1$

$x \in \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \implies v = 2$

$\text{(Evaluate-expr-addr-local)}$

$(\text{fid}, -) = \text{pc}$

$\text{vid} \in \text{localIds}(\text{Fd}(\text{fid})) \cup \text{args}(\text{Fd}(\text{fid}))$

$\beta(\text{vid}, \text{fid}, \text{mid}) = [s, e]$

$\delta, \phi + s, \phi + e, 0)$

$\phi = \Sigma(\text{mid}).1 + \Phi(\text{mid})$

$\text{addr}(\text{vid}), \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow (\delta, \Delta(\text{mid}).1 + s, \Delta(\text{mid}).1 + e, 0)$

$\text{(Evaluate-expr-var)}$

$\text{vid} \in \text{VarID}$

$\text{addr}(\text{vid}), \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow (\delta, s, e, \text{off})$

$s \leq s + \text{off} < e$

$\text{Mem}(s + \text{off}) = v$

$\text{addr}(e_{\text{arr}}, \text{MVar}, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow (\delta, s, e, \text{off})$

$e_{\text{idz}}, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow \text{off}'$

$\text{off}' \in \mathbb{Z}$

$\text{addr}(e_{\text{arr}}[e_{\text{idz}}], \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow (\delta, s, e, \text{off} + \text{off}')$

$\text{(Evaluate-expr-arr)}$

$s \leq s + \text{off} < e$

$\text{Mem}(s + \text{off}) = v$

$\text{addr}(e_{\text{arr}}[e_{\text{idz}}], \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow (\delta, s, e, \text{off})$

$e_{\text{arr}}[e_{\text{idz}}], \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow v$

$\text{Evaluate-expr-deref}$

$s \leq s + \text{off} < e$

$\text{Mem}(s + \text{off}) = v$

$\text{deref}(e), \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow v$

$\text{(Evaluate-expr-limrange)}$

$e_x, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow s$

$s' \in \mathbb{Z}$

$e_{ee}, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow e'$

$e' \in \mathbb{Z}$

$\{s', e'\} \subseteq [s, e)$

$\text{limRange}(e, e_x, e_e), \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow (x, s', e', 0)$
Figure 6: Evaluation of commands \( \text{Cmd} \) in \( \text{ImpMod} \)

<table>
<thead>
<tr>
<th>(Assign-to-var-or-arr)</th>
<th>( \text{commands}(Fd(fid))(n) = \text{Assign} , e_l , e_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{fid}, \Sigma, \Delta, \beta, \text{MVar}, Fd, \text{Mem}, \Phi, pc \downarrow (\delta, s, e, off) )</td>
<td>( e_r, \Sigma, \Delta, \beta, \text{MVar}, Fd, \text{Mem}, \Phi, pc \downarrow v )</td>
</tr>
<tr>
<td>( \text{modID} = \text{moduleID}(Fd(fid)) )</td>
<td>( \forall s', e'. , v = (\delta, s', e', _ ) \implies ([s', e'] \cap \Sigma(\text{modID}) = \emptyset \lor [s, e] \subseteq \Sigma(\text{modID})) )</td>
</tr>
<tr>
<td>( s \leq s + \text{off} &lt; e )</td>
<td>( \text{Mem}' = \text{Mem}[s + \text{off}] \mapsto v )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(Allocate)</th>
<th>( \text{commands}(Fd(fid))(n) = \text{Allo}c e_l , \text{size} e_t, \Sigma, \Delta, \beta, \text{MVar}, Fd, \text{Mem}, \Phi, pc \downarrow (\delta, s, e, off) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma, \Delta, \beta, \text{MVar}, Fd \vdash \langle \text{Mem}, \text{stk}, pc, \Phi, \text{nalloc} \rangle \rightarrow \langle \text{Mem}', \text{stk}, \text{inc}(pc), \Phi, \text{nalloc}' \rangle )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(Call)</th>
<th>( \text{commands}(Fd(fid))(n) = \text{Call} , f_{\text{call}} , v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{fid}, n = pc )</td>
<td>( \text{modID} = \text{moduleID}(Fd(fid)) )</td>
</tr>
<tr>
<td>( \text{argNames} = \text{args}(Fd(f_{\text{call}})) )</td>
<td>( \text{localIDs} = \text{localIDs}(Fd(fid)) )</td>
</tr>
<tr>
<td>( n\text{Args} = \text{length}([\text{argNames}]) = \text{length}(v) )</td>
<td>( \text{curModID} = \text{moduleID}(Fd(fid)) )</td>
</tr>
<tr>
<td>( \text{curFrameSize} = \text{frameSize}(Fd(fid)) )</td>
<td>( \text{frameSize} = \text{frameSize}(Fd_{\text{call}}) )</td>
</tr>
<tr>
<td>( \Sigma(\text{modID}).1 + \Phi(\text{modID}) + \text{frameSize} &lt; \Sigma(\text{modID}).2 )</td>
<td>( \text{stk} = \text{push}(\text{stk}, \text{pc}) )</td>
</tr>
<tr>
<td>( \Phi' = \Phi[\text{modID} \mapsto \Phi(\text{modID}) + \text{frameSize}] )</td>
<td>( \text{stk}' = \text{push}(\text{stk}, \text{pc}) )</td>
</tr>
<tr>
<td>( \forall i \in [0, n\text{Args}]) )</td>
<td>( \text{stk}' = (\text{fid}', _ ) )</td>
</tr>
<tr>
<td>( v_i = (\delta, s', e', _ ) \implies [s', e'] \cap \Sigma(\text{curModID}) = \emptyset )</td>
<td>( \text{stk} = \text{stk} + 1 )</td>
</tr>
<tr>
<td>( s_i \in \beta(\text{argNames}(i), \text{fid}_{\text{call}}, \text{modID}) \land i \in [0, n\text{Args}] )</td>
<td>( s_i \in \beta(\text{localIDs}(i), \text{fid}_{\text{call}}, \text{modID}) \land i \in [0, n\text{Local}] )</td>
</tr>
<tr>
<td>( \phi' = s + v_i \mapsto v_i )</td>
<td>( \text{stk}' = \text{push}(\text{stk}, \text{pc}) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(Return)</th>
<th>( \text{commands}(Fd(fid))(n) = \text{Return} , (pc', \text{stk}') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{fid}, n = pc )</td>
<td>( \text{curFrameSize} = \text{frameSize}(Fd(fid)) )</td>
</tr>
<tr>
<td>( \text{curModID} = \text{moduleID}(Fd(fid)) )</td>
<td>( \text{stk}' = (\text{stk}', _ ) )</td>
</tr>
<tr>
<td>( \Phi' = \Phi[\text{curModID} \mapsto \Phi(\text{curModID}) - \text{curFrameSize}] )</td>
<td>( \text{stk}' = \text{push}(\text{stk}, \text{pc}) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(Jump-non-zero)</th>
<th>( \text{commands}(Fd(fid))(n) = \text{JumpIfZero} , e_c , e_{\text{off}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{fid}, n = pc )</td>
<td>( \Sigma, \Delta, \beta, \text{MVar}, Fd, \text{Mem}, \Phi, pc \downarrow v )</td>
</tr>
<tr>
<td>( v \neq 0 )</td>
<td>( \text{stk}' = \text{push}(\text{stk}, \text{pc}) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(Jump-zero)</th>
<th>( \text{commands}(Fd(fid))(n) = \text{JumpIfZero} , e_c , e_{\text{off}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{fid}, n = pc )</td>
<td>( \Sigma, \Delta, \beta, \text{MVar}, Fd, \text{Mem}, \Phi, pc \downarrow v )</td>
</tr>
<tr>
<td>( e_{\text{off}}, \Sigma, \Delta, \beta, \text{MVar}, Fd, \text{Mem}, \Phi, pc \downarrow v )</td>
<td>( off \in \mathbb{Z} )</td>
</tr>
<tr>
<td>( v = 0 )</td>
<td>( \text{stk}' = \text{push}(\text{stk}, \text{pc}) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(Exit)</th>
<th>( \text{commands}(Fd(fid))(n) = \text{Exit} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{fid}, n = pc )</td>
<td>( \Sigma, \Delta, \beta, \text{MVar}, Fd \vdash \langle \text{Mem}, \text{stk}, pc, \Phi, \text{nalloc} \rangle \rightarrow \langle \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc} \rangle )</td>
</tr>
</tbody>
</table>

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2.4 Initial, terminal and execution states

Definition 36 (Valid execution state of a program).
A state \( \langle \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc} \rangle \) is a valid execution state of a program \( \text{mods} \) if it satisfies the judgment \( \text{mods}; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd} \vdash_{\text{exec}} \langle \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc} \rangle \) defined in rule Exec-state-src of Figure 7.

Definition 37 (Initial state).
An initial state of a program \( \text{mods} \) is any state \( \langle \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc} \rangle \) satisfying \( \text{mods}; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd} \vdash_{\text{i}} \langle \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc} \rangle \) which is defined in rule Initial-state-src in Figure 7.

Definition 38 (Initial state function).
\[
\text{initial\_state}(\overline{m}, \Delta, \Sigma, \text{mainModID}) \stackrel{\text{def}}{=} \\
\{ a \mapsto 0 \mid a \in \bigcup_{m \in \overline{m}} \Delta(m.\text{mid}) \cup \Sigma(m.\text{mid}) \},
\]
nil, (main, 0),
\[
\{ \text{mainModID} \mapsto \text{frameSize}(\overline{m}(\text{mainModID}), \text{fds}(\text{main})) \} \cup \\
\bigcup_{\text{mid} \in \{ m.\text{mid} \mid m \in \overline{m} \} \setminus \{ \text{mainModID} \}} \{ \text{mid} \mapsto 0 \},
\]
\(-1\)

Definition 39 (Main module).
\[
\text{main\_module}(\overline{m}) = \text{main} \iff \exists m, \text{fd}. m \in \overline{m} \land \text{fd} \in m.\text{fds} \land \text{main} = \text{funID}(\text{fd}) \land \text{moduleId}(m) = \text{mid}
\]

Claim 4 (The function initial\_state and the judgment \( \vdash_{\text{i}} \) are compatible).
\[
\forall K_{\text{mod}}; K_{\text{fun}}: \overline{m}, \Sigma, \Delta, \beta \ 
\text{main\_module}(\overline{m}) = \text{mainModID} \land 
\text{wfp\_params}(\overline{m}, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}) \land 
\text{initial\_state}(\overline{m}, \Delta, \Sigma, \text{mainModID}) = s_i 
\implies 
\exists \text{MVar}, \text{Fd}. K_{\text{mod}}; K_{\text{fun}}; \overline{m}; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd} \vdash_{\text{i}} s_i
\]

Definition 40 (Terminal state).
A terminal state of a program \( \text{mods} \) is any state \( \langle \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc} \rangle \) satisfying \( \text{fd\_map}(\text{mods}) \vdash_{\text{t}} \langle \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc} \rangle \) which is defined in rule Terminal-state-src-exit in Figure 7.

We now define convergence of a program \( \overline{m} \) running with a context \( \mathcal{C} \) as successful linking, successful loading, and reachability of a terminal state from every loadable initial state.

Definition 41 (Layout places \( \overline{m} \) before \( \mathcal{C} \)).
\[
\overline{m} \triangleright_{L_1, L_2} \mathcal{C} \stackrel{\text{def}}{=} \\
\max_{\text{mod} \in \overline{m}} \{ L_1(\text{moduleId}(\text{mod})).2 \} \cup \{ L_2(\text{moduleId}(\text{mod})).2 \} < \\
\min_{\text{mod} \in \mathcal{C}} \{ L_1(\text{moduleId}(\text{mod})).1 \} \cup \{ L_2(\text{moduleId}(\text{mod})).1 \}
\]

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Figure 7: Valid execution and initial states in \textbf{ImpMod}

\[
\begin{align*}
\text{(Valid-linking-src)} & \quad \overline{m} = m_1 \cup m_2 \quad \text{wfp}(m) \\
\text{(Equal-interfaces-src)} & \quad m_1 \times m_2 = \overline{m} \\
\text{(Exec-state-src)} & \quad \text{wfp\_params}(\text{mods}, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}) \\
\text{modIDs} = \{\text{mid} \mid (\text{mid}, \_\_\_) \in m_1\} = \{\text{mid} \mid (\text{mid}, \_\_\_) \in m_2\} \\
fDefs_1 = \{\text{fdef} \mid \text{fdef} \in fdefs \land (\_\_, \_\_\_, \text{fdefs}) \in m_1\} \\
fDefs_2 = \{\text{fdef} \mid \text{fdef} \in fdefs \land (\_\_, \_\_\_, \text{fdefs}) \in m_2\} \\
fSigs_1 = \{(\text{mid}, \text{fid}, n\text{Args}) \mid \text{fid} \in fDefs_1 \land \text{mid} = \text{moduleID}(\text{fid}) \land \text{fid} = \text{funID}(\text{fid}) \land n\text{Args} = |\text{args}(\text{fid})|\} \\
fSigs_2 = \{(\text{mid}, \text{fid}, n\text{Args}) \mid \text{fid} \in fDefs_2 \land \text{mid} = \text{moduleID}(\text{fid}) \land \text{fid} = \text{funID}(\text{fid}) \land n\text{Args} = |\text{args}(\text{fid})|\} \\
fSigs_1 = fSigs_2 \\
\overline{m_1} \sim \overline{m_2} \\
\text{frameSize}(Fd(\text{fid})) \times (\text{countIn}((\text{fid}, \_\_), \text{stk}) + (\text{pc} = (\text{fid}, \_\_) \land 0)) \\
\bigwedge_{\text{fid} \in \{\text{fid} \mid \text{moduleID}(\text{fid}) = \text{mid}\}} \forall \text{mid} \in \text{modIDs}. \Sigma(\text{mid}).1 + \Phi(\text{mid}) \leq \Sigma(\text{mid}).2 \\
\text{stk} = \text{nil} \implies \text{pc}.\text{fid} = \text{main} \\
\forall \text{mid}, a, \sigma, e. \Sigma(\text{mid}).(\_\_) \land \text{stk} = \text{nil} \implies \text{stk}(\emptyset).\text{fid} = \text{main} \\
\forall \text{mid}, a, \sigma, e. \text{stk} = (\delta, \_\_, \_\_) \\
\forall \text{mid}, a, \sigma, e. \text{stk}(\emptyset).\text{fid} = \text{main} \\
\text{nalloc} < 0 \\
\end{align*}
\]

\[
\begin{align*}
\text{K}_{\text{mod}}; \text{K}_{\text{fun}}; \overline{m}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{\text{exec}} (\text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc}) \\
\text{K}_{\text{mod}}; \text{K}_{\text{fun}}; \overline{m}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{\text{exec}} (\text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc}) \\
\text{nalloc} = -1 \\
\text{stk} = \text{nil} \\
\text{Mem} = \{a \rightarrow 0 \mid a \in \bigcup_{\text{mid} \in \Delta \cup \Sigma(\text{mid})} \text{dom}(\text{mid})\} \\
\text{pc} = (\text{main}, 0) \\
\Phi = \{\text{moduleID}(Fd(\text{main})) \mapsto \text{frameSize}(Fd(\text{main}))\} \cup \bigcup_{\text{mid} \in \Delta \cup \Sigma(\text{mid})} \{\text{mid} \mapsto 0\} \\
\end{align*}
\]

\[
\begin{align*}
\text{K}_{\text{mod}}; \text{K}_{\text{fun}}; \overline{m}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{\text{i}} (\text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc}) \\
\text{pc} = (\text{fid}, n) \\
\text{commands}(Fd(\text{fid}))(n) = \text{Exit} \\
\text{Fd} \vdash_{\text{i}} (\text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc}) \\
\end{align*}
\]
Definition 42 (Layout-ordered linking).
\[ C|_{\Delta; \Sigma} = m \iff C \prec m = [m] \land m \triangleright_{\Delta; \Sigma} C \]

Definition 43 (Linkability, loadability, and convergence of execution in the source language).
\[ \Sigma, \Delta, \beta, \nabla \vdash C|m| \downarrow \iff \exists m. C|_{\Delta; \Sigma} = m \land \exists s_t. \Sigma; \Delta; \beta; \text{mvar}(m); fd_{\text{map}}(m) \vdash \text{initial}_\text{state}(m, \Delta, \Sigma, \text{main}_\text{module}(m)) \rightarrow^* s_t \land \text{fd}_{\text{map}}(m) \vdash_t s_t \]

where \( \rightarrow^* \) is the reflexive transitive closure of the evaluation relation defined in fig. 6.

Definition 44 (Addition of an offset \( \omega \) to the data segment’s bounds).
\[ \Delta + \omega \overset{\text{def}}{=} \{ \text{mid} \mapsto \Delta(\text{mid}) + \omega \mid \text{mid} \in \text{dom}(\Delta) \} \]

where \( \Delta(\text{mid}) + \omega \) is the addition of a constant to an interval which is given by \([a, b] + c = [a + c, b + c] \).

Two programs \( m_1 \) and \( m_2 \) that have the same per-module data-segment size \( \Delta \) and that have respectively data segment layouts \( \beta_1 \) and \( \beta_2 \) are said to be contextually equivalent in the execution environment \( \Sigma, \nabla \) denoted \( \Delta, \beta_1, m_1 \overset{\Sigma, \nabla}{=} \Delta, \beta_2, m_2 \) when they are equi-linkable, equi-loadable, and equi-convergent in all contexts \( C \) with an arbitrary data segment size \( \Delta \), data segment layout \( \beta \), stack sizes \( \Sigma \).

Definition 45 (Source contextual equivalence).
\[ \Delta, \beta_1, m_1 \overset{\Sigma, \nabla}{=} \Delta, \beta_2, m_2 \overset{\text{def}}{=} \forall \Delta, \beta, \Sigma, C. \wfp(C) \implies \begin{array}{c}
(\Sigma \cup \Sigma, (\Delta \cup \Delta) + \omega, \beta \cup \beta_1, \nabla \vdash C|m| \downarrow \iff \\
\Sigma \cup \Sigma, (\Delta \cup \Delta) + \omega, \beta \cup \beta_2, \nabla \vdash C|m| \downarrow)
\end{array} \]

Lemma 56 (Preservation of \( \vdash_{\text{exec}} \)).
\[ \begin{array}{c}
K_{\text{mod}}; K_{\text{fun}}; \overline{\text{mods}}; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd} \vdash_{\text{exec}} \langle \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc} \rangle \\
\Sigma; \Delta; \beta; \text{MVar}; \text{Fd} \vdash \langle \text{Mem}', \text{stk}', \text{pc}', \Phi', \text{nalloc}' \rangle \\
\implies
\end{array} \]

Proof. By inversion using rules \text{Exec-state-src} and \text{Well-formed program and parameters}, we obtain the following assumptions:

Well formed program and parameters \text{wfp_params}(\overline{\text{mods}}, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}) \n
Module IDs \text{modIDs} = \{ \text{modID} \mid (\text{modID}, \_ , \_ ) \in \overline{\text{mods}} \} \n
Equal domains \text{dom}(K_{\text{mod}}) = \text{dom}(\text{MVar}) = \text{dom}(\Sigma) = \text{dom}(\Delta) = \text{modIDs} \n
Function definitions \text{funDefs} = \{ \text{modFunDef} \mid \text{modFunDef} \in \text{modFunDefs} \land (\_, \_, \_, \text{modFunDefs}) \in \overline{\text{mods}} \} \n
\text{Fd} = \{ \text{funID} \mapsto \text{funDef} \mid \text{funDef} \in \text{funDefs} \land \text{funDef} = (\_, \text{funID}, \_, \_, \_ ) \}$
MVar \( MVar = \{ modID \mapsto \text{varIDs} \mid (modID, \text{varIDs}, _) \in \text{mods} \} \)

pc points to an existing function \( pc = (\text{funID}, _) \land \text{funID} \in \text{dom}(Fd) \)

All pc’s on stack point to existing functions

\( \forall (fid, _) \in \text{elems}(stk). \; fid \in \text{dom}(Fd) \)

\( \text{dom}(\beta) = \{ (vid, fid, mid) \mid mid \in \text{modIDs} \land (vid \in MVar(mid) \land fid = \bot \land fid \in \text{dom}(Fd) \land vid \in \text{localIDs}(Fd(fid)) \cup \text{args}(Fd(fid)) \} \)

Arguments are non-arrays

\( \forall \; mid, fid, vid. \; vid \in \text{args}(Fd(fid)) \land \beta(vid, fid, mid) = (s, e) \implies |s - e| = 1 \)

Static addresses are mapped addresses

\( \text{static}\_\text{addresses}(\Sigma, \Delta, \text{modIDs}) \subseteq \text{dom}(\text{Mem}) \)

No address exists that is out-of-memory

\( \nabla \leq 0 \implies (\text{nalloc} > \nabla \land \forall a \in \text{dom}(\text{Mem}). \; a > \nabla \land \forall a, s, e. \; v \in \text{range}(\text{Mem}) \land v = (\delta, s, e) \land a \in [s, e] \implies a > \nabla) \)

No stack overflow

\( \forall \; mid \in \text{modIDs}. \; \Sigma(mid).1 + \Phi(mid) \leq \Sigma(mid).2 \)

Frame sizes are non-negative

\( \forall \; fid \in \text{dom}(Fd). \; \text{frameSize}(Fd(fid)) \geq 0 \)

Stack pointers are the sum of all frame sizes on stack

\( \forall \; mid \in \text{modIDs}. \; \Phi(mid) = \sum_{fid \in \{ fid \mid \text{moduleID}(Fd(fid)) = mid \}} \text{frameSize}(Fd(fid)) \times (\text{countIn}((fid, _), stk) + (pc = (fid, _) ? 1 : 0)) \)

Variables occupy exactly the frame

\( \forall \; mid \in \text{modIDs}, fid \in \text{dom}(Fd). \; \bigcup_{vid \in \text{localIDs}(Fd(fid)) \cup \text{args}(Fd(fid))} \beta(vid, fid, mid) = [\text{frameSize}(Fd(fid)), 0) \)

Static variables occupy exactly the data segment

\( \forall \; mid \in \text{modIDs}. \; \bigcup_{vid \in MVar(mid)} \beta(vid, \bot, mid) = [0, \Delta(mid).2 - \Delta(mid).1) \)

One address per command

\( \forall \; mid \in \text{modIDs}, fid \in \text{dom}(Fd). \; |K_{\text{fun}}(fid)| = |\text{commands}(Fd(fid))| \)

Module's code is a contiguous concatenation of its functions

\( \forall \; mid \in \text{modIDs}. \; \bigcup_{fid \in \{ fid \mid \text{moduleID}(Fd(fid)) = mid \}} K_{\text{fun}}(fid) = [0, |K_{\text{mod}}(mid)|) \)

Data segments are disjoint and code segments are disjoint

\( \forall \; mid, \; mid' \in \text{modIDs}. \; mid \neq mid' \implies \Delta(mid) \cap \Delta(mid') = \emptyset \land K_{\text{mod}}(mid) \cap K_{\text{mod}}(mid') = \emptyset \)

If no function has been called, then \text{main} is executing

\( stk = \text{nil} \implies pc.fid = \text{main} \)

The first function to start executing was \text{main}

\( stk \neq \text{nil} \implies stk(0).fid = \text{main} \)

Stack addresses (capabilities) only live on the stack

\( \forall \; mid, a, \sigma, e. \; \text{Mem}(a) = (\delta, \sigma, e, _) \land |\sigma, e) \cap \Sigma(mid) \neq \emptyset \implies a \in \Sigma(mid) \)
Dynamically-allocated addresses are negative

\[ \text{nalloc} < 0 \]

Our goal is \( \text{mods}; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd} \vdash_{\text{exec}} \langle \text{Mem}', \text{stk}', \text{pc}', \Phi', \text{nalloc}' \rangle \). We prove it using rule \( \text{Exec-state-src} \). We use the names that we gave to the assumptions above to also describe the subgoals about the state \( \langle \text{Mem}', \text{stk}', \text{pc}', \Phi', \text{nalloc}' \rangle \).

The following subgoals are immediate:

- **Well formed program and parameters** (This is a predicate of only the program text \( \text{mods} \), and the static parameters \( \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}} \).)
- Module IDs,
- Equal domains,
- Function definitions,
- \( \text{Fd} \), and
- \( \text{MVar} \)

It remains to prove the following subgoals:

- pc points to an existing function,
- All pc’s on stack point to existing functions,
- Static addresses are mapped addresses,
- No address exists that is out-of-memory,
- Stack pointers are the sum of all frame sizes on stack,
- No stack overflow,
- Stack addresses (capabilities) only live on the stack, and
- Dynamically-allocated addresses are negative.

We prove them by case distinction over the reduction relation \( \rightarrow \).

**Case Assign-to-var-or-arr:**

The goal “pc points to an existing function” is immediate from the corresponding assumption.

The goal “All pc’s on stack point to existing functions” is immediate from the corresponding assumption by substitution.

In this case, the goal “Static addresses are mapped addresses” about \( \text{Mem}' \) holds by transitivity of \( \subseteq \) after noticing that \( \text{dom(Mem)} \subseteq \text{dom(Mem')} \).

The goals “No stack overflow” and “Stack pointers are the sum of all frame sizes on stack” follow by substitution using \( \Phi' = \Phi \) and \( \text{stk} = \text{stk}' \).

The goal “No address exists that is out-of-memory” has three conjuncts:

Conjunct \( \text{nalloc}' > \nabla \) holds by substitution using the precondition \( \text{nalloc}' = \text{nalloc} \).
The second and third conjuncts follow from the corresponding assumption “**No address exists that is out-of-memory**” relying on Lemmas 57 and 81. (A detailed proof would be similar to the one in the next case. We skip it here for brevity.)

The goals “**If no function has been called, then main is executing**” and “**The first function to start executing was main**” are immediate from the corresponding assumptions after substitution using \( stk = stk' \) and \( pc.fid = pc'.fid. \)

To prove the goal “**Stack addresses (capabilities) only live on the stack**”, we obtain the precondition \( s'.Mem = s.Mem[\sigma + \text{off} \mapsto v] \).

Then, we fix an arbitrary memory address \( a \), and an arbitrary module ID \( mid \). We prove our goal for the following two cases:

- **Case \( a = \sigma + \text{off} \):**

  Here, we obtain the following preconditions of rule **Assign-to-var-or-arr**:

  (PRECOND-ASSN):
  \[
  \text{modID} = \text{moduleID}(\text{Fd}(s.pc.fid)), \quad \text{and} \quad v = (\delta, \sigma', e', \_) \implies ([\sigma', e'] \cap \Sigma(\text{modID}) = \emptyset \lor [\sigma, e] \subseteq \Sigma(\text{modID}))
  \]

  Assuming (STK-CAP-ASSM):
  \[
  v = (\delta, \sigma', e', \_) \land [\sigma', e'] \cap \Sigma(mid) \neq \emptyset,
  \]
  our goal is \( \sigma + \text{off} \in \Sigma(mid) \).

  We distinguish the following two cases:

  - **Case \( mid \neq \text{modID} \):**

    Here, we obtain a contradiction to the assumption \( [\sigma', e'] \cap \Sigma(mid) \neq \emptyset \). Here is how we show \( [\sigma', e'] \cap \Sigma(mid) = \emptyset \).

      - **First**, we show \( [\sigma', e'] \subseteq \text{reachable_addresses}(\Sigma, \Delta, \{ \text{modID} \}, s.Mem) \).

      - To prove this, we apply Lemma 81 choosing \( \text{modIDs} = \{ \text{modID} \} \) to obtain the following subgoals:

        - \( e_r, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, s.Mem, s.\Phi, s.pc \downarrow (\delta, \sigma', e', \_) \)
          This is immediate by the precondition of **Assign-to-var-or-arr** together with the assumption (STK-CAP-ASSM).

        - \( \_ \uparrow_{\text{exec}} s \)
          This is immediate by our lemma’s assumption.

        - \( \text{moduleID}(\text{Fd}(s.pc.fid)) \in \{ \text{modID} \} \)
          This is immediate by (PRECOND-ASSN).

      - **Second**, we show that \( \text{reachable_addresses}(\Sigma, \Delta, \{ \text{modID} \}, s.Mem) \cap \Sigma(mid) = \emptyset \)

        By unfolding Definitions 48 and 49, our goal is:
        \[
        (\Delta(\text{modID}) \cup \Sigma(\text{modID}) \cup \text{access}_{s,Mem}(\Delta(\text{modID}) \cup \Sigma(\text{modID}), s.Mem)) \cap \Sigma(mid) = \emptyset
        \]

        It suffices by easy set identities to show individually:

        - \( \Delta(\text{modID}) \cap \Sigma(mid) = \emptyset \)
          Immediate by **Well formed programs and parameters**.

        - \( \Sigma(\text{modID}) \cap \Sigma(mid) = \emptyset \)
          Immediate by **Well formed programs and parameters**.

        - \( \text{access}_{s,Mem}(\Delta(\text{modID}) \cup \Sigma(\text{modID}), s.Mem)) \cap \Sigma(mid) = \emptyset \)
          We prove it by induction on \( k \) with \( 0 \leq k \leq |s.Mem| \).

          - **Base case:**
            \( \text{access}_{0}(\Delta(\text{modID}) \cup \Sigma(\text{modID}), s.Mem)) \cap \Sigma(mid) = \emptyset \)
            By Definition 48, it suffices to prove \( (\Delta(\text{modID}) \cup \Sigma(\text{modID})) \cap \Sigma(mid) = \emptyset \).

            This is the same as the previous cases.
Inductive case:
The induction hypothesis is:
\[
\text{access}_k(\Delta(\text{modID}) \cup \Sigma(\text{modID}), s.Mem)) \cap \Sigma(\text{mid}) = \emptyset.
\]
And for convenience let:
\[
A = \text{access}_k(\Delta(\text{modID}) \cup \Sigma(\text{modID}), s.Mem))
\]
Our goal is:
\[
\text{access}_{k+1}(\Delta(\text{modID}) \cup \Sigma(\text{modID}), s.Mem)) \cap \Sigma(\text{mid}) = \emptyset.
\]
By Definitions 47 and 48 and after simplification using the induction hypothesis, it suffices for the remaining subgoal to prove:
\[
\forall a' \in A. s.Mem(a') = (\delta, \sigma', e', \_)) \implies [\sigma', e') \cap \Sigma(mid) = \emptyset.
\]
We prove it by contradiction. Assume the contrary, i.e., assume for an arbitrary address \(a' \in A\) that \(s.Mem(a') = (\delta, \sigma', e', \_) \land [\sigma', e') \cap \Sigma(mid) \neq \emptyset\).
Now by assumption “Stack addresses (capabilities) only live on the stack”, we have (*):
\[
a' \in \Sigma(mid)
\]
But we know \(a' \in A\), and by the induction hypothesis, we know \(A \cap \Sigma(mid) = \emptyset\).
Thus, we know that \(a' \notin \Sigma(mid)\) (contradiction to (*)).
This concludes our inductive proof that \(\text{access}_{s,Mem}(\Delta(\text{modID}) \cup \Sigma(\text{modID}), s.Mem)) \cap \Sigma(mid) = \emptyset\).
This concludes the proof of Second which concludes the proof of Case mid \(\neq\) modID.

- Case mid = modID:
  By instantiating (PRECOND-ASSN) using the assumptions above, we obtain the following two cases:
  * Case \([\sigma', e') \cap \Sigma(\text{modID}) = \emptyset\):
    Here, we obtain a contradiction to our assumptions. So, any goal is provable.
  * Case \([\sigma, e) \subseteq \Sigma(\text{modID})\):
    Here, our goal is immediate by compatibility of \(\in\) and \(\subseteq\) because of the precondition \(\sigma + off \in [\sigma, e)\) together with our case condition.
  * Case \(a \neq \sigma + off\):
    Here, our goal is immediate by the corresponding assumption.

The goal “Dynamically-allocated addresses are negative” is immediate by substitution using \(s.nalloc = s.nalloc\).

Case Allocate:
The goal “pc points to an existing function” is immediate from the corresponding assumption.

The goal “All pc’s on stack point to existing functions” is immediate from the corresponding assumption by substitution.

In this case, the goal “Static addresses are mapped addresses” about \(Mem'\) holds by transitivity of \(\subseteq\) after noticing that \(\text{dom}(Mem) \subseteq \text{dom}(Mem')\).

Next, we prove the goal “No address exists that is out-of-memory”.
In this case, we obtain the preconditions \(nalloc - v > \nabla\) and \(nalloc' = nalloc - v\) which by substitution in one another prove the first conjunct of the consequent of statement No address exists that is out-of-memory.
The second conjunct of No address exists that is out-of-memory is proved by fixing an arbitrary \(a \in \text{dom}(Mem')\) and distinguishing the cases that arise by the precondition \(Mem' = Mem[s + off \mapsto (\delta, nalloc', nalloc, 0)](a \mapsto 0 \mid a \in [nalloc', nalloc])\):
• Case \( a \notin \{s + \text{off}\} \cup [\text{nalloc}', \text{nalloc}) \):
Follows by the corresponding assumption, i.e., “No address exists that is out-of-memory”.

• Case \( a = s + \text{off} \):
In this case, we know by Lemma 81 that:
\( a \in \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \).
Thus, by Lemma 57, we know:
\( a \in \text{static_addresses}(\Sigma, \Delta, \text{modIDs}) \lor \exists s,e. (\delta, s, e, _) \in \text{range(Mem)} \land a \in [s,e) \)
Thus, we consider each case:
– Case \( a \in \text{static_addresses}(\Sigma, \Delta, \text{modIDs}) \):
Here, by transitivity of \( \subseteq \) from assumption “Static addresses are mapped addresses”, we have:
\( a \in \text{dom(Mem)} \).
So, our conclusion \( a > \nabla \) follows by assumption “No address exists that is out of memory”.
– Case \( \exists s,e. (\delta, s, e, _) \in \text{range(Mem)} \land a \in [s,e) \):
Here, our conclusion \( a > \nabla \) follows by assumption “No address exists that is out of memory”.

• Case \( a \in [\text{nalloc}', \text{nalloc}) \):
In this case, \( a \geq \text{nalloc}' \) and \( \text{nalloc}' > \nabla \) (which is a precondition of Allocate) give us our conclusion \( a > \nabla \).

The third conjunct of the goal “No address exists that is out-of-memory” is proved by fixing arbitrary \( s,e \) with \( (\delta, s, e, _) \in \text{range(Mem')} \land a \in [s,e) \) and proving that \( a > \nabla \).

We distinguish the following cases based on the definition of \( \text{Mem}' \) (similar to the cases above for \( a \in \text{dom(Mem')} \)):

• Case \( (\delta, s, e, _) \in \text{range(Mem)} \):
Here, our goal follows by the third conjunct of the corresponding assumption, i.e., “No address exists that is out-of-memory”.

• Case \( (\delta, s, e, _) = \text{Mem}'(s + \text{off}) \):
Here, the goal follows by the conclusion \( \text{nalloc}' > \nabla \) that we already argued.

• Case \( (\delta, s, e, _) = \text{Mem}'(a') \land a' \in [\text{nalloc}', \text{nalloc}) \):
This is an impossible case because \( \text{Mem}(a') = 0 \) in this case by the definition of \( \text{Mem}' \).

This concludes the proof of the goal “No address exists that is out-of-memory”.

The goals “No stack overflow” and “Stack pointers are the sum of all frame sizes on stack” follow by substitution using \( \Phi' = \Phi \) and \( \text{stk} = \text{stk}' \).

The goals “If no function has been called, then \text{main} is executing” and “The first function to start executing was \text{main}” are proved exactly as in the previous case.

We prove the goal “Stack addresses (capabilities) only live on the stack” by fixing an arbitrary address \( a \) where \( a \in \text{dom(Mem')} \) and distinguishing the cases that arise by the precondition \( \text{Mem}' = \text{Mem}[\sigma + \text{off} \mapsto (\delta, \text{nalloc}', \text{nalloc}, 0)][a \mapsto 0 \mid a \in [\text{nalloc}', \text{nalloc})] \):

• Case \( a \notin \{\sigma + \text{off}\} \cup [s'.\text{nalloc}, s.\text{nalloc}) \):
Here, our goal is immediate by the corresponding assumption.
• Case \(a = \sigma + \text{off}\):
  Here, we know \(s'.\text{Mem}(a) = (\delta, s'.\text{nalloc}, s.\text{nalloc}, 0)\).
  So, we prove our goal vacuously by proving that:
  \([s'.\text{nalloc}, s.\text{nalloc}] \cap \Sigma(mid) = \emptyset\).
  By inversion of rule **Well-formed program and parameters** in assumption **Well formed programs and parameters**, and by applying the obtained precondition:
  \(\Sigma(mid) \cap (-\infty, 0) = \emptyset\)
  to our goal, we obtain the following subgoal:
  \([s'.\text{nalloc}, s.\text{nalloc}] \subseteq (-\infty, 0)\)
  This is immediate by assumption “**Dynamically-allocated addresses are negative**”.

• Case \(a \in [s'.\text{nalloc}, s.\text{nalloc}]\):
  Here, our goal is vacuously true.

**Case Call:**

The goal “**pc points to an existing function**” follows from the precondition \(\text{modID} = \text{moduleID}(F_d(fid_{\text{call}}))\).

The goal “**All pc’s on stack point to existing functions**” follows from both the corresponding assumption and from the assumption **pc points to an existing function**.

In this case, the goal “**Static addresses are mapped addresses**” about \(\text{Mem}'\) holds by transitivity of \(\subseteq\) after noticing that \(\text{dom(\text{Mem})} \subseteq \text{dom(\text{Mem}')\)}.

The goal “**No address exists that is out-of-memory**” has three conjuncts:
Conjunct \(\text{nalloc}' > \nabla\) holds by substitution using the precondition \(\text{nalloc}' = \text{nalloc}\).

The second and third conjuncts follow from the corresponding assumption “**No address exists that is out-of-memory**” relying on Lemmas 57 and 81. (A detailed proof would be similar to the one in **case Allocate**. We skip it here for brevity.)

Next, we prove the goal “**No stack overflow**”, namely:
\(\forall mid \in \text{modIDs}, \Sigma(mid).1 + \Phi'(mid) \leq \Sigma(mid).2\).

We obtain from **Call** the preconditions:

- \(\Sigma(\text{modID}).1 + \Phi(\text{modID}) + \text{frameSize} \leq \Sigma(\text{modID}).2\)
- \(\Phi' = \Phi[\text{modID} \mapsto \Phi(\text{modID}) + \text{frameSize}]\)

These are sufficient to immediately prove our goal after case distinction on \(mid = \text{modID}\).

Next, we prove the goal “**Stack pointers are the sum of all frame sizes on stack**”.

Our goal is:
\(\forall mid \in \text{modIDs}, \Phi'(mid) = \sum_{fid \in \{fid \mid \text{moduleID}(F_d(fid)) = mid\}} \text{frameSize}(F_d(fid)) \times (\text{countIn}((fid, _), stk') + (pc' = (fid, _) ? 1 : 0))\)

We distinguish three cases:

- **Case mid = moduleID(F_d(fid_{\text{call}}))**: In this case, we further distinguish two cases:
  - **Case pc.fid = fid_{\text{call}}, and**
The goal “Stack pointers are the sum of all frame sizes on stack”.

The goal “If no function has been called, then main is executing” is vacuously true by noticing that stk’ ≠ nil.

To prove the goal “The first function to start executing was main”, i.e., stk’ ≠ nil ⇒ stk’(0).fid = main, we distinguish the following two cases:

- Case stk = nil:
  Here, by assumption “If no function has been called, then main is executing”, we know pc.fid = main. Thus, by the precondition stk = push(stk, pc), we have our goal.

- Case stk ≠ nil:
  Here, observe that stk(0) = stk’(0), so our goal is immediate by the corresponding assumption about stk.

We prove the goal “Stack addresses (capabilities) only live on the stack” by fixing an arbitrary address a where a ∈ dom(Mem’) and distinguishing the cases that arise by the precondition:

s’.Mem = s.Mem[a’ + s_i → v_i | β(argNames(i), fid_cal, modID) = [s_i, _] ∧ i ∈ [0, nArgs]]
[a’ + s_i → 0 | β(localIDs(i), fid_cal, modID) = [s_i, _] ∧ i ∈ [0, nLocal]]

- Case ∃i ∈ [0, nArgs), a ∈ a’ + β(argNames(i), fid_cal, modID):
  Here, we obtain the following precondition of rule Call:
  (PRECOND-CALL):
  curModID = moduleID(Fd(fid)), and
  ∀i ∈ [0, nArgs), s’, e’. v_i = (δ, s’, e’, _) → [s’, e’) ∩ Σ(curModID) = ∅
  We now obtain from our case condition i ∈ [0, nArgs), and we instantiate (PRECOND-CALL).
  The proof then proceeds exactly as in case Assign-to-var-or-arr replacing (PRECOND-ASSIGN) with (PRECOND-CALL). We avoid repetition.

- Case ∃i ∈ [0, nLocal). a ∈ a’ + β(localIDs(i), fid_cal, modID):
  Here, our goal holds vacuously.
Case \( \exists i. i \in [0, n\text{Args}) \land a \in \phi' + \beta(\text{argNames}(i), \text{fid}_{\text{call}}, \text{modID}) \lor i \in [0, n\text{Local}) \land a \in \phi' + \beta(\text{localIDs}(i), \text{fid}_{\text{call}}, \text{modID}) \):

Here, our goal is immediate by the corresponding assumption.

The goal \textit{“Dynamically-allocated addresses are negative”} is immediate by substitution using \( s'.\text{nalloc} = s.\text{nalloc} \).

Case Return:

The goal \textit{“pc points to an existing function”} follows from the assumption \textit{All pc’s on stack point to existing functions}.

The goal \textit{“All pc’s on stack point to existing functions”} follows from the corresponding assumption.

In this case, the goal \textit{“Static addresses are mapped addresses”} about \( \text{Mem}' \) holds by substitution using \( \text{Mem}' = \text{Mem} \).

The goal \textit{“No address exists that is out-of-memory”} holds by substitution using the preconditions \( \text{nalloc}' = \text{nalloc} \) and \( \text{Mem}' = \text{Mem} \).

Next, we prove the goal \textit{“Stack pointers are the sum of all frame sizes on stack”}.

Our goal is:

\[
\forall mid \in \text{modIDs}. \Phi'(mid) = \sum_{fid \in \{fid \mid \text{moduleID}(\text{Fd}(fid)) = mid\}} \text{frameSize}(\text{Fd}(fid)) \times (\text{countIn}((\text{fid}, \_), \text{stk}') + (\text{pc'} = (\text{fid}, \_)?1:0))
\]

We distinguish three cases:

- **Case \( mid = \text{moduleID}(\text{Fd}(pc.fid)) \):**
  In this case, we further distinguish two cases:
  - Case \( pc.fid = \text{pc'}.fid \), and
  - Case \( pc.fid \neq \text{pc'}.fid \):
    In both of these cases, we notice that the right-hand-side factor in the right side of the equality decreases by one for the term corresponding to \( pc.fid \).
    Thus, by the precondition \( \Phi'(mid) = \Phi(mid) - \text{frameSize}(\text{Fd}(pc.fid)) \), we can satisfy the equality.

- **Case \( mid \neq \text{moduleID}(\text{Fd}(pc.fid)) \land mid = \text{moduleID}(\text{Fd}(\text{pc'}.fid)) \):**
  In this case, we notice that all the terms on the right side of the equality remain the same. And in particular the term for \( \text{pc'}.fid \) remains the same because its right-hand-side factor remains the same because:
  \[
  (pc' = (\text{pc'}.fid)?1:0) - (pc = (\text{pc'}.fid)?1:0) = 1, \quad \text{and} \quad \text{countIn}((\text{pc'}.fid, \_), \text{stk}') - \text{countIn}((\text{pc'}.fid, \_), \text{stk}) = -1
  \]
  Thus, by substituting using the precondition \( \Phi'(mid) = \Phi(mid) \) in the left side of our goal equality, our goal holds by assumption.

- **Case \( mid \neq \text{moduleID}(\text{Fd}(pc.fid)) \land mid \neq \text{moduleID}(\text{Fd}(\text{pc'}.fid)) \):**
  In this case, our goal holds directly by the assumption.

This concludes the proof of the goal \textit{“Stack pointers are the sum of all frame sizes on stack”}.

Next, we prove the goal \textit{“No stack overflow”}, namely:

\[
\forall mid \in \text{modIDs}. \Sigma(mid).1 + \Phi'(mid) \leq \Sigma(mid).2
\]
Here, by case distinction on \( mid = \text{moduleID}(Fd(pc.fid)) \), our goal follows immediately by transitivity of \( \leq \) after obtaining the precondition
\[
\Phi'(mid) = \Phi(mid) - \text{frameSize}
\]
in one case, and immediately by assumption in the other case.
(The assumption “Frame sizes are non-negative” was used here.)

The goal “If no function has been called, then \text{main} is executing” follows from assumption “The first function to start executing was \text{main}” about \( stk \).
The goal “The first function to start executing was \text{main}” follows from the corresponding assumption about \( stk \).
The goal “Stack addresses (capabilities) only live on the stack” is immediate after substitution using \( s'.Mem = s.Mem \).
The goal “Dynamically-allocated addresses are negative” is immediate by substitution using \( s'.nalloc = s.nalloc \).

Case Jump-zero:
All remaining goals hold by substitution (using \( \Phi' = \Phi, stk = stk', nalloc' = nalloc, Mem' = Mem, \) and \( pc'.1 = pc.1 \))

Case Jump-non-zero:
All remaining goals hold by substitution (using \( \Phi' = \Phi, stk = stk', nalloc' = nalloc, Mem' = Mem, \) and \( pc'.1 = pc.1 \))

Case Exit:
Here, all goals hold by substitution (using \( \Phi' = \Phi, stk = stk', nalloc' = nalloc, Mem' = Mem, \) and \( pc' = pc \)).

This concludes the proof of Lemma 56.

Corollary 4 (Preservation of \( \vdash_{\text{exec}} \) by the reflexive transitive closure).
\[
\forall \text{mods}, s, s'. \text{mods} \vdash_{\text{exec}} s \land s \rightarrow^* s' \implies \text{mods} \vdash_{\text{exec}} s'
\]

Proof. Trivial by Lemma 56.

2.5 Memory Reachability

Given a memory context \( \Sigma; \Delta; \beta; MVar; Fd \) and a ImpMod program state \( \langle Mem, stk, pc, \Phi, nalloc \rangle \), we would like to characterize the set \( A \subseteq \mathbb{Z} \) of reachable memory addresses which informally captures all the addresses that an expression in the given state can evaluate to. In other words, the set \( A \) of reachable addresses should satisfy the condition that whenever an expression \( e \) evaluates to an address in the given state (i.e., \( e, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow a \) where \( a = (\delta, st, end, _) \)), then \( [st, end) \subseteq A \).

More formally, Lemma 81 captures the previous intuition.

Definition 46 (Static Addresses).
\[
\text{static_addresses}(\Sigma, \Delta, modIDs) \overset{\text{def}}{=} \\
\{ a \mid a \in \Delta(mid) \land mid \in modIDs \} \\
\cup \\
\{ a \mid a \in \Sigma(mid) \land mid \in modIDs \}
\]
Definition 47 (Memory accessibility).

\[
\text{access}(A, \text{Mem}) \overset{\text{def}}{=} A \cup \{a \mid a \in [s,e) \land \text{Mem}(a') = (\delta, s, e, _) \land a' \in A\}
\]

Definition 48 (Memory \(k\)-accessibility).

\[
\begin{align*}
\text{access}_0(A, _) &= A \\
\text{access}_{k+1}(A, \text{Mem}) &\overset{\text{def}}{=} \text{access}(\text{access}_k(A, \text{Mem}), \text{Mem})
\end{align*}
\]

Definition 49 (Reachable Addresses).

\[
\begin{align*}
\text{reachable Addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) &\overset{\text{def}}{=} \text{static Addresses}(\Sigma, \Delta, \text{modIDs}) \\
&\cup \text{access}[\text{Mem}](\text{static Addresses}(\Sigma, \Delta, \text{modIDs}), \text{Mem})
\end{align*}
\]

Lemma 57 (Reachable addresses are static addresses or are memory-stored).

\[
\forall a, \Sigma, \Delta, \text{modIDs}, \text{Mem}.
\begin{align*}
a \in &\text{reachable Addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \implies \\
&\text{a} \in \text{static Addresses}(\Sigma, \Delta, \text{modIDs}) \lor \exists s, e. (\delta, s, e, _) \in \text{range}(	ext{Mem}) \land a \in [s,e)
\end{align*}
\]

Proof. By Definitions 46 to 49.

Lemma 58 (access is expansive).

\[
\forall A, \text{Mem}. \text{access}(A, \text{Mem}) \supseteq A
\]

Proof. Similar to Lemma 7

Lemma 59 (access\(_n\) is expansive).

\[
\forall n, A, \text{Mem}. \text{access}_n(A, \text{Mem}) \supseteq A
\]

Proof. Similar to Lemma 8

Lemma 60 (Fixed points lead to convergence of access\(_k\)).

\[
\forall k, \text{Mem}, A. k > 0 \implies (\text{access}_k(A, \text{Mem}) = A \implies \text{access}_{k+1}(A, \text{Mem}) = A)
\]

Proof. Similar to Lemma 9

Lemma 61 (In an empty memory, only the starting addresses are reachable).

\[
\forall \Sigma, \Delta, \text{modIDs}, \text{Mem}.
\begin{align*}
(\forall v. v \in \text{range}(	ext{Mem}) \implies v \not= (\delta, _, _, _)) \\
\implies &\text{reachable Addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) = \text{static Addresses}(\Sigma, \Delta, \text{modIDs})
\end{align*}
\]

Proof. Similar to Lemma 10. Immediate by Definitions 47 to 49.

Lemma 62 (k-accessibility either adds a new memory address or a fixed point has been reached).

\[
\forall k, A, \text{Mem}. k > 0 \implies \\
\text{access}_k(A, \text{Mem}) \supseteq \text{access}_{k+1}(A, \text{Mem}) \implies \\
\exists a. a \in \text{dom}(	ext{Mem}) \land a \in \text{access}_k(A, \text{Mem}) \setminus \text{access}_{k-1}(A, \text{Mem})
\]

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Proof. Similar to Lemma 11

\textbf{Lemma 63} \textit{(k-accessibility set contains at least k mapped addresses).}

\begin{align*}
\forall k, A, Mem. \\
\text{access}_{k+1}(A, Mem) \supseteq \text{access}_k(A, Mem) \implies \\
|\{a \mid a \in \text{access}_k(A, Mem) \land a \in \text{dom}(Mem)\}| > k
\end{align*}

Proof. Similar to Lemma 12

\textbf{Lemma 64} \textit{(|Mem|-accessibility suffices).}

\begin{align*}
\forall A, Mem, k. k \geq 0 \implies \\
\text{access}_{|Mem|+k}(A, Mem) = \text{access}_{|Mem|}(A, Mem)
\end{align*}

Proof. Similar to Lemma 13

\textbf{Lemma 65} \textit{(Safe allocation adds only allocated addresses to k-accessibility).}

\begin{align*}
\forall A, Mem, \hat{a}, a_\sigma, \sigma_\epsilon, k. \\
\forall a \in [\sigma, e). \ Mem[\hat{a} \mapsto (\delta, \sigma, e, _)](a) = v \implies v \notin \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \land \\
a_\sigma \in \text{access}_k(A, Mem[\hat{a} \mapsto (\delta, \sigma, e, _)]) \\
\implies a_\sigma \in \text{access}_k(A, Mem) \lor a_\sigma \in [\sigma, e)
\end{align*}

Proof. Similar to Lemma 39.

\textbf{Lemma 66} \textit{(Safe allocation adds only allocated addresses to reachability).}

\begin{align*}
\forall \Sigma, \Delta, \text{modIDs}, Mem, \hat{a}, a_\sigma, \sigma_\epsilon. \\
\forall a \in [\sigma, e). \ Mem[\hat{a} \mapsto (\delta, \sigma, e, _)](a) = v \implies v \notin \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \land \\
a_\sigma \in \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, Mem[\hat{a} \mapsto (\delta, \sigma, e, _)]) \\
\implies a_\sigma \in \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, Mem) \lor a_\sigma \in [\sigma, e)
\end{align*}

Proof. Similar to Lemma 40.

\textbf{Lemma 67} \textit{(Safe allocation causes reduction of k-accessibility to \(\chi_k\) and addition of exactly the allocated addresses).}

\begin{align*}
\forall A, Mem, \hat{a}, a_\sigma, \sigma_\epsilon, k. \\
\forall a \in [\sigma, e). \ Mem[\hat{a} \mapsto (\delta, \sigma, e, _)](a) = v \implies v \notin \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \land \\
\hat{a} \in \text{access}_k(A, Mem) \\
\implies \\
\text{access}_k(A, Mem[\hat{a} \mapsto (\delta, \sigma, e, _)]) = \chi_k(A, Mem, \hat{a}) \cup [\sigma, e)
\end{align*}

Proof. Similar to Lemma 41. Should follow by induction on \(k\), and should be similar to the proof of Lemma 65.

\textbf{Lemma 68} \textit{(Invariance to unreachable memory updates).}

\begin{align*}
\forall \Sigma, \Delta, \text{modIDs}, Mem, a, v. a \notin \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, Mem) \implies \\
\text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, Mem) = \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, Mem[a \mapsto v])
\end{align*}

Proof.

Similar to Lemma 21 using Lemmas 59, 63 and 69.
Lemma 69 (Updating k-inaccessible locations does not affect the k-accessibility set).

\[ \forall a, k, Mem, A, v. \ a \notin \text{access}_k(A, Mem) \implies \text{access}_k(A, Mem) = \text{access}_k(A, Mem[a \mapsto v]) \]

Proof.
Similar to Lemma 22 using Definitions 47 and 48.

Lemma 70 (Updating a location does not affect its own k-accessibility).

\[ \forall a, k_a, Mem, v. \ a \in \text{access}_{k_a}(A, Mem) \implies a \in \text{access}_{k_a}(A, Mem[a \mapsto v]) \]

Proof.
Similar to Lemma 23 using Lemma 69.

Lemma 71 (Updating a location does not affect its own reachability).

\[ \forall \Sigma, \Delta, \text{modIDs}, a, v, Mem. \ a \in \text{reachable\_addresses}(\Sigma, \Delta, \text{modIDs}, Mem) \implies a \in \text{reachable\_addresses}(\Sigma, \Delta, \text{modIDs}, Mem[a \mapsto v]) \]

Proof.
Similar to Lemma 24 using Lemma 70 and definition 49.

Lemma 72 (\(\chi_k\) is upper-bounded by k-accessibility).

\[ \forall k, Mem, A, a. \ \chi_k(A, Mem, a) \subseteq \text{access}_k(A, Mem) \]

Proof.
Immediate by Definitions 25 and 48.

Lemma 73 (One capability is potentially lost from accessible addresses as a result of a non-capability update).

\[ \forall A, a, Mem, v. \ v \neq (\delta, \_, \_, \_) \implies \text{access}(A, Mem[a \mapsto v]) = \chi(A, Mem, a) \]

Proof.
Similar to Lemma 32. Follows from Definitions 24 and 47 by observing that \(Mem[a \mapsto v](a) \neq (\delta, \_, \_, \_)\) and that \(Mem[a \mapsto v](a') = Mem(a')\) for \(a' \neq a\).

Lemma 74 (\(\chi_k\) captures k-accessibility after potential deletion of a capability).

\[ \forall A, a, Mem, v. \ v \neq (\delta, \_, \_, \_) \implies \text{access}_k(A, Mem[a \mapsto v]) = \chi_k(A, Mem, a) \]

Proof.
Similar to Lemma 33. Follows by induction on \(k\) from Definitions 25 and 48 using Lemma 73.

Lemma 75 (Reachability is captured by union over \(\chi_k\) after potential deletion of a capability).

\[ \forall \Sigma, \Delta, \text{modIDs}, Mem, a, v. \ v \neq (\delta, \_, \_, \_) \implies \text{reachable\_addresses}(\Sigma, \Delta, \text{modIDs}, Mem[a \mapsto v]) = \bigcup_k (\chi_k(\text{static\_addresses}(\Sigma, \Delta, \text{modIDs}, Mem, Mem, a))) \]

Proof.
Similar to Lemma 34. Immediate by Definition 49 and lemma 74.

Definition 50 (Derivable capability). A capability \(c^* = (x, \sigma, e, \_\_)\) is derivable from reachability parameters \(\Sigma, \Delta, \text{modIDs}\) on memory \(Mem\), written \(\Sigma, \Delta, \text{modIDs} \models c^*\) iff

\[ \forall a \in [\sigma, e). \ a \in \text{reachable\_addresses}(\Sigma, \Delta, \text{modIDs}, Mem) \]
Lemma 76 (Reachability traverses all derivable capabilities).

\[ \forall \Sigma, \Delta, \text{modIDs}, \text{Mem}, c. \]
\[ \Sigma, \Delta, \text{modIDs}, \text{Mem} \vdash c \implies \]
\[ \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \supseteq [c.\sigma, c.e) \cup \text{access}_{\text{Mem}}([c.\sigma, c.e), \text{Mem}) \]

Proof. Similar to Lemma 28.

Lemma 77 (Additivity of access).

\[ \forall A_1, A_2, M_d. \text{access}(A_1 \cup A_2, \text{Mem}) = \text{access}(A_1, \text{Mem}) \cup \text{access}(A_2, \text{Mem}) \]

Proof. Similar to Lemma 16.

Lemma 78 (Additivity of access\(_k\)).

\[ \forall k, A_1, A_2, M_d. \text{access}_k(A_1 \cup A_2, \text{Mem}) = \text{access}_k(A_1, \text{Mem}) \cup \text{access}_k(A_2, \text{Mem}) \]

Proof. Similar to Lemma 17. Follows by induction on \(k\) using Lemma 77.

Lemma 79 (Effect of assigning a derivable capability).

\[ \forall \Sigma, \Delta, \text{modIDs}, \text{Mem}, a, c. \]
\[ \Sigma, \Delta, \text{modIDs}, \text{Mem} \vdash c \land a \in \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \implies \]
\[ \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}[a \mapsto c]) = \]
\[ \bigcup_k (\text{static_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \cup [c.\sigma, c.e), \text{Mem}, a) \]

Proof. Follows from Lemmas 30, 75 and 78.

Lemma 80 (Assigning a derivable capability does not enlarge reachability).

\[ \forall \Sigma, \Delta, \text{modIDs}, \text{Mem}, a, c. \]
\[ \Sigma, \Delta, \text{modIDs}, \text{Mem} \vdash c \land a \in \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \implies \]
\[ \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}[a \mapsto c]) \subseteq \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \]

Proof. After substitution using Lemma 79, we apply Lemma 30 to get two subgoals that are provable using Lemma 72 and Lemma 76 respectively.

Lemma 81 (Completeness of reachable_addresses).

\[ \forall \text{st}, \text{end}, e, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc}, \text{modIDs}. \]
\[ e, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow (\delta, \text{st}, \text{end}, \_ ) \land \]
\[ \exists \text{mods}, \text{nalloc}, \text{stk}. \overline{\text{mods}}; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd} \vdash_{\text{exec}} \langle \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc} \rangle \land \]
\[ \text{moduleID}(\text{Fd}(\text{pc}.\text{fid})) \in \text{modIDs} \]
\[ \implies \]
\[ \{\text{st}, \text{end}\} \subseteq \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \]

Proof.
• We fix arbitrary \( st, end, e, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \), and assume the antecedent
\( e, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \Downarrow (\delta, st, end, _) \).

• We prove the consequent by induction on the evaluation of \( e \).
  
  - Case \textbf{Evaluate-expr-const}:
  
  - Case \textbf{Evaluate-expr-cast-to-integer-start}:
  
  - Case \textbf{Evaluate-expr-cast-to-integer-end}:
  
  - Case \textbf{Evaluate-expr-cast-to-integer-offset}:
  
  - Case \textbf{Evaluate-expr-cap-type}:
  
  - Case \textbf{Evaluate-expr-binop}:
    
    All of these cases are vacuous because in all, the antecedent does not hold because
\( e, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \Downarrow z \) with \( z \notin \{ \delta \} \times \mathbb{Z} \times \mathbb{Z} \).

  - Case \textbf{Evaluate-expr-addr-local}:
    
    In this case, we obtain the preconditions:
\[ (fid, _) = pc, vid \in \text{localIds}(Fd(fid)) \cup \text{args}(Fd(fid)), \mid \beta(vid, fid, mid) = (s, e) \mid \text{and } \phi = \Sigma(mid).1 + \Phi(mid). \]

    Our goal is to show that:
\[ [\phi + s, \phi + e] \subseteq \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, Mem). \]

    We instead show the following goal:
\[ [\phi - \text{frameSize}, \phi] \subseteq \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, Mem) \]
where \text{frameSize} = \text{frameSize}(Fd(fid)).

    The latter follows immediately by Definitions 46 and 49.

    And it suffices for our goal by transitivity of \( \subseteq \) assuming:
\[ [\phi + s, \phi + e] \subseteq [\phi - \text{frameSize}, \phi]. \]

    This latter assumption follows by interval arithmetic identities from:
\[ [s, e] \subseteq [-\text{frameSize}, 0]. \]

    This last statement follows from:
\[ \bigcup_{vid \in \text{localIds}(Fd(fid)) \cup \text{args}(Fd(fid))} \beta(vid, fid, mid) = [-\text{frameSize}, 0] \]

    which in turn can be obtained from the assumption
\[ \text{mods}; \Sigma, \Delta, \beta, MVar; Fd \vdash \text{exec} \langle Mem, stk, pc, \Phi, nalloc \rangle \]

    of our lemma by inversion using rule \text{Exec-state-src} then inversion using rule \text{Well-formed}
program and parameters.

    This concludes case \textbf{Evaluate-expr-addr-local}.

  - Case \textbf{Evaluate-expr-addr-module}:
    
    This case is similar to the previous one, but not identical.

    We obtain the preconditions:
\[ (fid, _) = pc, vid \notin \text{localIds}(Fd(fid)) \cup \text{args}(Fd(fid)), \]
\[ \mid \beta(vid, \bot, mid) = (s, e) \mid \text{and } vid \in MVar(mid). \]

    Our goal is to show that:
\[ [\Delta(mid).1 + s, \Delta(mid).1 + e] \subseteq \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, Mem). \]

    We instead show the following goal:
\[ [\Delta(mid).1, \Delta(mid).2] \subseteq \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, Mem). \]

    The latter follows immediately by Definitions 46 and 49.

    And it suffices for our goal by transitivity of \( \subseteq \) assuming:
\[ [\Delta(mid).1 + s, \Delta(mid).1 + e] \subseteq [\Delta(mid).1, \Delta(mid).2]. \]

    This last statement follows from:
\[ \bigcup_{vid \in MVar(mid)} \beta(vid, \bot, mid) = \Delta(mid) \]
which in turn can be obtained from the assumption
\[ \text{mods}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{\text{exec}} \langle \text{Mem}, stk, pc, \Phi, nalloc \rangle \]
of our lemma by inversion using rule \text{Exec-state-src} then inversion using rule \text{Well-formed program and parameters}.
This concludes case \text{Evaluate-expr-addr-module}.

– Case \text{Evaluate-expr-var}:
We obtain the preconditions \( \text{addr}(vid), \Sigma, \Delta, \beta, MVar, Fd, \text{Mem}, \Phi, pc \vdash a' \)
and \( \text{Mem}(a') = v \).
We distinguish the following two cases:
* Case \( v \not\in \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \):
  This case is vacuous.
* Case \( v \in \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \):
  Here, we know \( v = (\delta, st, end, _) \) and our goal is to show that:
  \( [st, end) \subseteq \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \).
  We first show that \( a' \in \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \) by distinguishing
  the following two cases:
  - Case \( \text{vid} \in \text{localIDs}(Fd(fid)) \cup \text{args}(Fd(fid)) \):
    This case is then identical to case \text{Evaluate-expr-addr-local} of our current lemma.
  - Case \( \text{id} \not\in \text{localIDs}(Fd(fid)) \cup \text{args}(Fd(fid)) \):
    This case is then identical to case \text{Evaluate-expr-addr-module} of our current
    lemma.

Now, having proved that \( a' \in \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \),
we distinguish by unfolding Definition 49 the following cases:
  - Case \( a' \in \text{static_addresses}(\Sigma, \Delta, \text{modIDs}) \):
    In this case, we know by \( a' \in \text{dom}(	ext{Mem}) \) which was obtained above that \(|\text{Mem}| \geq 1\)
    and thus by unfolding Definitions 47 to 49 of our goal, we’re done.
  - Case \( a' \in \text{access}_{\text{Mem}}(\text{static_addresses}(\Sigma, \Delta, \text{modIDs}), \text{Mem}) \):
    Here, by unfolding Definitions 47 and 48, we know that:
    \( [st, end) \subseteq \text{access}_{\text{Mem}}(\text{static_addresses}(\Sigma, \Delta, \text{modIDs}), \text{Mem}) \)
    But then by Lemma 64, we conclude:
    \( [st, end) \subseteq \text{access}_{\text{Mem}}(\text{static_addresses}(\Sigma, \Delta, \text{modIDs}), \text{Mem}) \).
    The last statement by Definition 49 gives us our goal.

– Case \text{Evaluate-expr-addr-arr}:
  Immediate by the induction hypothesis.

– Case \text{Evaluate-expr-arr}:
  Similar to case \text{Evaluate-expr-var}.

– Case \text{Evaluate-expr-deref}:
  Similar to cases \text{Evaluate-expr-var} and \text{Evaluate-expr-arr}.

– Case \text{Evaluate-expr-limrange}:
  Immediate by the induction hypothesis and transitivity of \( \subseteq \).

This concludes the proof of Lemma 81. \( \square \)

**Definition 51** (Data segment capability of a module).

\[
data_{\text{segment}}_{\text{capability}}(\Delta, \text{modID}) \overset{\text{def}}{=} (\delta, \Delta(\text{modID}).1, \Delta(\text{modID}).2, 0)
\]

**Definition 52** (Stack capability of a module).

\[
\text{stack}_{\text{capability}}(\Sigma, \text{modID}) \overset{\text{def}}{=} (\delta, \Sigma(\text{modID}).1, \Sigma(\text{modID}).2, 0)
\]
Definition 53 (Capabilities of a module).

\[
\text{module\_caps}(\Delta, \Sigma, \text{modID}) \overset{\text{def}}{=} \{\text{data\_segment\_capability}(\Delta, \text{modID}), \text{stack\_capability}(\Sigma, \text{modID})\}
\]

Definition 54 (Static capabilities).

\[
\text{static\_capabilities}(\Sigma, \Delta, \text{modIDs}) \overset{\text{def}}{=} \bigcup_{\text{modID} \in \text{modIDs}} \text{module\_caps}(\Delta, \Sigma, \text{modID})
\]

Lemma 82 (Static addresses are precisely those of static capabilities).

\[
\text{static\_addresses}(\Sigma, \Delta, \text{modIDs}) = \text{addr}(\text{static\_capabilities}(\Sigma, \Delta, \text{modIDs}))
\]

Proof. Immediate by unfolding \text{addr}, Definition 54, Definition 53, Definition 52, Definition 51, and Definition 46.

Definition 55 (Access to capabilities).

\[
\text{access\_cap}(C, \text{Mem}) \overset{\text{def}}{=} C \cup \{(\delta, \sigma, e, 0) \mid \text{Mem}(a') = (\delta, \sigma, e, _) \land a' \in \text{addr}(C)\}
\]

Lemma 83 (Accessed addresses are precisely the addresses of accessed capabilities).

\[
\text{access}(\text{addr}(C), \text{Mem}) = \text{addr}(\text{access\_cap}(C, \text{Mem}))
\]

Proof. Straightforward by unfolding \text{addr}, Definition 55, and Definition 47.

Definition 56 (k-access to capabilities).

\[
\text{access\_cap}_0(C, \text{Mem}) \overset{\text{def}}{=} C
\]

\[
\text{access\_cap}_{k+1}(C, \text{Mem}) \overset{\text{def}}{=} \text{access\_cap}(\text{access\_cap}_k(C, \text{Mem}), \text{Mem})
\]

Lemma 84 (k-accessed addresses are precisely the addresses of k-accessed capabilities).

\[
\text{access}_k(\text{addr}(C), \text{Mem}) = \text{addr}(\text{access\_cap}_k(C, \text{Mem}))
\]

Proof. Straightforward by induction on \(k\); the base case is immediate then we apply Lemma 83 in the inductive case, after unfolding the goal using \text{addr}, Definition 56, and Definition 48.

Definition 57 (Reachable capabilities).

\[
\text{reachable\_caps}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \overset{\text{def}}{=} \text{static\_capabilities}(\Sigma, \Delta, \text{modIDs}) \cup \text{access\_cap}_{\text{Mem}}(\text{static\_capabilities}(\Sigma, \Delta, \text{modIDs}), \text{Mem})
\]

Lemma 85 (Reachable addresses are precisely the addresses of the reachable capabilities).

\[
\text{reachable\_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) = \text{addr}(\text{reachable\_caps}(\Sigma, \Delta, \text{modIDs}, \text{Mem}))
\]

Proof. By unfolding Definition 57 and Definition 49, and by applying the linearity of \text{addr}, our goal follows from Lemma 84 and Lemma 82.
3 Compiling pointers as capabilities (ImpMod to CHERIExpress)

Definition 58 (Expression Translation).

- $\lfloor z \rfloor_{\cdot \cdot, \cdot} \overset{\text{def}}{=} z$ for $z \in \mathbb{Z}$
- $\lfloor \text{addr}(\text{vid}) \rfloor_{\cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{lim}(\text{ddc}, \text{capStart}(\text{ddc}) + s, \text{capStart}(\text{ddc}) + e)$ with $\beta(\text{vid}, \perp, \text{mid}) = (s, e)$
- $\lfloor \text{addr}(\text{vid}) \rfloor_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{lim}(\text{stc}, \text{capStart}(\text{stc}) + \text{capOff}(\text{stc}) + s, \text{capStart}(\text{stc}) + \text{capOff}(\text{stc}) + e)$ with $\text{fid} \neq \perp$, $\beta(\text{vid}, \text{fid}, \text{mid}) = (s, e)$
- $\lfloor \text{vid} \rfloor_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{deref} (\lfloor \text{addr}(\text{vid}) \rfloor_{\cdot, \cdot, \cdot, \cdot})$
- $\lfloor e_1 \oplus e_2 \rfloor_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \lfloor e_1 \rfloor_{\cdot, \cdot, \cdot, \cdot} \oplus \lfloor e_2 \rfloor_{\cdot, \cdot, \cdot, \cdot}$
- $\lfloor \text{deref} (e) \rfloor_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{deref} (\lfloor e \rfloor_{\cdot, \cdot, \cdot, \cdot})$
- $\lfloor \text{addr}(e_{\text{arr}[e_{\text{off}}])} \rfloor_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{inc} (\lfloor \text{addr}(e_{\text{arr}}) \rfloor_{\cdot, \cdot, \cdot, \cdot}, \lfloor e_{\text{off}} \rfloor_{\cdot, \cdot, \cdot, \cdot})$
- $\lfloor e_{\text{arr}[e_{\text{off}}]} \rfloor_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{deref} (\lfloor \text{addr}(e_{\text{arr}}[e_{\text{off}}]) \rfloor_{\cdot, \cdot, \cdot, \cdot})$
- $\lfloor \text{start}(e) \rfloor_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{capStart}(\lfloor e \rfloor_{\cdot, \cdot, \cdot, \cdot})$
- $\lfloor \text{end}(e) \rfloor_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{capEnd}(\lfloor e \rfloor_{\cdot, \cdot, \cdot, \cdot})$
- $\lfloor \text{offset}(e) \rfloor_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{capOff}(\lfloor e \rfloor_{\cdot, \cdot, \cdot, \cdot})$
- $\lfloor \text{capType}(e) \rfloor_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{capType}(\lfloor e \rfloor_{\cdot, \cdot, \cdot, \cdot})$
- $\lfloor \text{limRange}(e, c, e) \rfloor_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{lim}(\lfloor e \rfloor_{\cdot, \cdot, \cdot, \cdot}, \lfloor c \rfloor_{\cdot, \cdot, \cdot, \cdot}, \lfloor e \rfloor_{\cdot, \cdot, \cdot, \cdot})$

We also define expression translation for a list of expressions as $\lfloor \overline{e} \rfloor_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \lfloor e_0 \rfloor_{\cdot, \cdot, \cdot, \cdot} \ldots \lfloor e_{n-1} \rfloor_{\cdot, \cdot, \cdot, \cdot}$ where $\overline{e} \equiv e_0 \ldots e_{n-1}$.

Definition 59 (Command Translation).

- $\langle \text{Assign} \; e_1, e_2 \rangle_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{Assign} (\lfloor e_1 \rfloor_{\cdot, \cdot, \cdot, \cdot}, \lfloor e_2 \rfloor_{\cdot, \cdot, \cdot, \cdot})$
- $\langle \text{Alloc} \; e_1, e_{\text{size}} \rangle_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{Alloc} (\lfloor e_1 \rfloor_{\cdot, \cdot, \cdot, \cdot}, \lfloor e_{\text{size}} \rfloor_{\cdot, \cdot, \cdot, \cdot})$
- $\langle \text{Call} \; \text{fid}_\text{call}, \overline{c} \rangle_{\cdot, \cdot, \cdot, \cdot} \overset{\text{def}}{=} \text{Cinvoke moduleID}(\text{Fd}(\text{fid}_\text{call})) \; \text{fid}_\text{call} \; \lfloor \overline{c} \rfloor_{\cdot, \cdot, \cdot, \cdot}$
- $\langle \text{Return} \rangle_{\ldots, \ldots, \ldots} \overset{\text{def}}{=} \text{Creturn}$
- $\langle \text{JumpIfZero} \; e_c, e_{\text{off}} \rangle_{\ldots, \ldots, \cdot, \cdot} \overset{\text{def}}{=} \text{JumpIfZero} (\lfloor e_c \rfloor_{\cdot, \cdot, \cdot, \cdot}, \lfloor e_{\text{off}} \rfloor_{\cdot, \cdot, \cdot, \cdot})$
- $\langle \text{Exit} \rangle_{\ldots, \ldots, \ldots} \overset{\text{def}}{=} \text{Exit}$

Lemma 86 (Code and data segment capabilities are precise with respect to the code and data memory initializations).

\[ \forall M_c, M_d, \text{imp}. \; (M_c, M_d, \text{imp}, \cdot, \cdot) \in \text{range}(\cdot) \]
\[ \implies \]
\[ \forall a. \; a \in \text{dom}(M_c) \iff \exists c \in \text{range}(\text{imp}). \; c.1 = (\kappa, s, e, \cdot) \land a \in [s, e) \land \]
\[ \forall a. \; a \in \text{dom}(M_d) \iff \exists c \in \text{range}(\text{imp}). \; c.2 = (\delta, s, e, \cdot) \land a \in [s, e) \]

Proof. Follows from rules Module-list-translation, Module-translation and Function-translation. \[\square\]
3.1 Whole-program compiler correctness

Compiler correctness is given by Theorem 1 (backward simulation).

**Definition 60** (Source-target value relatedness).
Value relatedness $\cong \subseteq V \times V$ is syntactic equality:

$$\forall v. \ v \cong v$$

**Lemma 87** (Expression translation forward simulation - case $\text{addr}(vid)$).

$$\exists \modss, \Sigma, \Delta, \beta; \text{MVar}, \text{Fd}, \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc}, \text{mid}, \text{fid}, \text{vid}, \mathcal{M}_d, \text{stc}, \text{ddc}.$$  

$$\text{pc} = (\text{fid}, \text{vid}) \land \Delta(\text{mid}) = (\text{ddc}.\sigma, \text{ddc}.e) \land$$  

$$\Sigma(\text{mid}) = (\text{stc}.\sigma, \text{stc}.e) \land \Phi(\text{mid}) = \text{stc}.\sigma \land$$  

$$\exists \text{vid}, \text{addr}(\text{vid}), \Sigma, \Delta, \beta; \text{MVar}, \text{Fd} \vdash_{\text{exec}} (\text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc}) \land$$  

$$\exists \text{vid}, \text{addr}(\text{vid}), \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \Downarrow v \implies \exists \text{vid}. \ \text{addr}(\text{vid}) \in (\text{localIDS}(\text{Fd}(\text{vid}))) \cup \text{args}(\text{Fd}(\text{vid}))$$

\begin{itemize}
  \item From the assumption $\exists \text{vid}, \text{addr}(\text{vid}), \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \Downarrow v$, there are two cases for inversion:

  \textbf{Case Evaluate-expr-addr-local:}

  Here, we have by inversion:

  1. $\text{vid} \in \text{localIDS}(\text{Fd}(\text{fid})) \cup \text{args}(\text{Fd}(\text{fid}))$
2. \( mid = \text{moduleId}(Fd(fid)) \)
3. \( \beta(vid, fid, mid) = [s, e] \)
4. \( \phi = \Sigma(mid).1 + \Phi(mid) \)
5. \( v = (\delta, \phi + s, \phi + e, 0) \)

Thus, by value relatedness, we would like to show that:
\[
\{ \text{addr}(vid) \}_{fid, mid, \beta, M_d, ddc, stc, _} \downarrow (\delta, \phi + s, \phi + e, 0).
\]

We know that \( \bot \notin \text{dom}(Fd) \). Thus, we conclude \( fid \neq \bot \), which by Definition 58 gives us:
\[
\{ \text{addr}(vid) \}_{fid, mid, \beta} = \lim(stc, \text{capStart}(stc) + \text{capOff}(stc) + s, \text{capStart}(stc) + \text{capOff}(stc) + e).
\]

By substitution, our goal becomes:
\[
\lim(stc, \text{capStart}(stc) + \text{capOff}(stc) + s, \text{capStart}(stc) + \text{capOff}(stc) + e), M_d, ddc, stc, _ \downarrow (\delta, \phi + s, \phi + e, 0).
\]

By applying \text{evalLim} \text{, (and evalstc} to some of the subgoals), we obtain three subgoals:
\[
\begin{align*}
&- \text{capStart}(stc) + \text{capOff}(stc) + s, M_d, ddc, stc, _ \downarrow \phi + s \\
&- \text{capStart}(stc) + \text{capOff}(stc) + e, M_d, ddc, stc, _ \downarrow \phi + e \\
&- (\phi + s, \phi + e) \subseteq [stc, \sigma, stc.e]
\end{align*}
\]

These subgoals become (by further applying evalBinOp, evalCapStart, evalCapOff and evalstc and substitution using the definition of \( \phi, s \text{ and } e \) given above):
\[
\begin{align*}
&- \text{stc}, \sigma + \text{stc off} = \Sigma(mid).1 + \Phi(mid) \\
&- \Sigma(mid).1 + \Phi(mid) + \beta(vid, fid, mid).1, \Sigma(mid).1 + \Phi(mid) + \beta(vid, fid, mid).2 \\
&\subseteq [stc, \sigma, stc.e]
\end{align*}
\]

The first subgoal holds immediately by reflexivity after substitution from the assumptions of our lemma.

The second subgoal after substitution becomes:
\[
\Sigma(mid).1 + \Phi(mid) \subseteq \Sigma(mid).2.
\]

In order to prove this goal, we invert the assumption
\[
\_ : _; \_ : \text{mods}; \_ : \Sigma; \_ : MVar; Fd \vdash \_ : (Mem, stk, pc, \Phi, nalloc)
\]

using rule Exec-state-src then we invert wfp_params(\_ : \_ : \_ : \Sigma, \_ : \_ : K_mod, K_fun) using rule Well-formed program and parameters to obtain the following:

No stack overflow
\[
\forall mid \in \text{mods}. \Sigma(mid).1 + \Phi(mid) \leq \Sigma(mid).2
\]

Frame sizes are non-negative
\[
\forall fid \in \text{dom}(Fd). \text{frameSize}(Fd(fid)) \geq 0
\]

Stack pointers are the sum of all frame sizes on stack
\[
\forall mid \in \text{mods}. \Phi(mid) = \sum_{fid \in \{fid | \text{moduleId}(Fd(fid)) = mid\}} \text{frameSize}(Fd(fid)) \times (\text{countIn}(fid, \_), stk) + (pc = (fid, \_)? 1 : 0)
\]

Variables occupy exactly the frame
\[
\forall mid \in \text{mods}, fid \in \text{dom}(Fd).
\]
\[
\_{fid, \_} \beta(vid, fid, mid) = [\neg \text{frameSize}(Fd(fid)), 0)
\]

Now, by substituting the assumption:
\[
\text{pc} = (\text{fid}, \_)
\]

of our lemma into statement:

\textbf{Stack pointers are the sum of all frame sizes on stack}

instantiated with the assumption (obtained above by inversion):
\[
mid = \text{moduleId}(Fd(fid)),
\]

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together with the constraint:

**Frame sizes are non-negative,**
we can conclude that:
\[ \Phi(mid) \geq \text{frameSize}(Fd(fid)). \]
The latter statement, together with:

**Variables occupy exactly the frame**
suffice to show that:
\[ \Sigma(mid).1 + \Phi(mid) + \beta(vid, fid, mid).1 \geq \Sigma(mid).1. \]

Thus, it remains to show that:
\[ \Sigma(mid).1 + \Phi(mid) + \beta(vid, fid, mid).2 \leq \Sigma(mid).2. \]
We already know:
\[ \Sigma(mid).1 + \Phi(mid) \leq \Sigma(mid).2 \]
by “No stack overflow”.
And we know:
\[ \beta(vid, fid, mid).2 < 0 \]
by “Variables occupy exactly the frame”.
So, we immediately have the desired inequality by arithmetic identities.
This proves the second subgoal, and concludes case **Evaluate-expr-addr-local.**

**Case Evaluate-expr-addr-module:**
Here, we have by inversion:
1. \( vid \notin \text{localIDs}(Fd(fid)) \cup \text{args}(Fd(fid)) \)
2. \( mid = \text{moduleID}(Fd(fid)) \)
3. \( vid \in MVar(mid) \)
4. \( \beta(vid, \bot, mid) = (s, e) \)
5. \( v = (\delta, \Delta(mid).1 + s, \Delta(mid).1 + e, 0) \)
Thus, by value relatedness, we would like to show that:
\[ \{ \text{addr}(vid) \}_\text{fid,mid,\beta}, M_d, ddc, stc, \_ \Downarrow (\delta, \Delta(mid).1 + s, \Delta(mid).1 + e, 0). \]
Here, by the precondition \( \beta(vid, \bot, mid) = (s, e) \), we know by Definition 58 that:
\[ \{ \text{addr}(vid) \}_\text{fid,mid,\beta} = \lim(ddc, \text{capStart}(ddc) + s, \text{capStart}(ddc) + e) \]
Thus, substituting this into our goal, our goal becomes:
\[ \lim(ddc, \text{capStart}(ddc) + s, \text{capStart}(ddc) + e), M_d, ddc, stc, \_ \Downarrow (\delta, \Delta(mid).1 + s, \Delta(mid).1 + e, 0) \]
By applying evalLim, we obtain three subgoals:
- \( \text{capStart}(ddc) + s, M_d, ddc, stc, \_ \Downarrow \Delta(mid).1 + s \)
- \( \text{capStart}(ddc) + e, M_d, ddc, stc, \_ \Downarrow \Delta(mid).1 + e \)
- \( \Delta(mid).1 + s, \Delta(mid).1 + e \subseteq [ddc.\sigma, ddc.e] \)

For each of the first two subgoals, we apply evalBinOp and evalCapStart to end up with the following subgoal instead:
\( ddc.\sigma = \Delta(mid).1 \)
which is immediate by our lemma’s assumptions.

For the third subgoal, by substitution from the assumptions, we obtain the following subgoal instead:
\( \Delta(mid).1 + s, \Delta(mid).1 + e \subseteq [\Delta(mid).1, \Delta(mid).2] \)
To prove this subgoal, we invert the assumption:
\( \text{mods}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{\text{exec}} \langle \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc} \rangle \)
using rule \( \text{Exec-state-src} \) then by inversion using rule \( \text{Well-formed program and parameters} \), we obtain:

**Static variables occupy exactly the data segment**
\[ \forall mid \in \text{modIDs}. \bigcup_{vid \in \text{MVar}(mid)} \beta(vid, \bot, mid) = [0, \Delta(mid).2 - \Delta(mid).1) \]
from which we conclude:
\[ [s, e) \subseteq [0, \Delta(mid).2 - \Delta(mid).1) \].

In this last statement, by adding \( \Delta(mid).1 \) to both components of the intervals on each side, we immediately obtain our goal.

This concludes case \( \text{Evaluate-expr-addr-module} \).

This concludes the proof of Lemma 87. \( \square \)

**Lemma 88** (Expression translation forward simulation).

\[ \forall \text{mods}, \Sigma, \Delta, \beta, MVar, Fd, \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc}, mid, vid, M_d, \text{stc}, \text{ddc}. \]
\[ pc = (fid, \_) \land \Delta(mid) = (\text{ddc}.$\sigma$, \text{ddc}.$e$) \land \]
\[ \Sigma(mid) = (\text{stc}.$\sigma$, \text{stc}.$e$) \land \Phi(mid) = \text{stc}.\text{off} \land \]
\[ \text{imp}(mid).\text{ddc} \equiv \text{ddc} \land \text{mstc}(mid) \equiv \text{stc} \land \]
\[ \_; \_; \text{mods}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{\text{exec}} \langle \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc} \rangle \land \]
\[ \text{moduleID}(Fd(fid)) \in \text{modIDs} \land \]
\[ A_s = \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \land \]
\[ A_t = \text{reachable_addresses}(\bigcup_{mid \in \text{modIDs}} \{ \text{imp}(mid).\text{ddc}, \text{mstc}(mid) \}, M_d) \land \]
\[ A_s = A_t \land \text{Mem}|_{A_s} = M_d|_{A_t} \land \]
\[ \exists v. e, \Sigma, \Delta, \beta, MVar, Fd, \text{Mem}, \Phi, pc \downarrow v \]
\[ \implies \exists v. \downarrow e \downarrow_{\text{fid}, mid, \beta, M_d, ddc, \text{stc}, \_} v \land v \equiv v \]

**Proof.**

Our goal by Definition 60 is:
\[ \downarrow e \downarrow_{\text{fid}, mid, \beta, M_d, ddc, \text{stc}, \_} v \]

We assume the antecedents and prove it by induction on the evaluation of the source expression \( e \).

**Case Evaluate-expr-const:**
By substitution in Definition 58, our goal becomes:
\[ z, M_d, ddc, \text{stc}, \_ \downarrow z \]
which is immediate by evalconst.

**Case Evaluate-expr-cast-to-integer-start:**
By substitution in Definition 58, our goal becomes:
\[ \text{capStart}(\downarrow e \downarrow_{\text{fid}, mid, \beta, M_d, ddc, \text{stc}, \_} z) \]
where we have the induction hypothesis:
\[ \downarrow e \downarrow_{\text{fid}, mid, \beta, M_d, ddc, \text{stc}, \_} (\_, z, \_, \_). \]
Thus, by applying rule evalCapStart to the goal, we can immediately show our goal using the induction hypothesis on \( e' \).
Case **Evaluate-expr-cast-to-integer-end:**

By substitution in Definition 58, our goal becomes:
\[
\text{capEnd}(\{e'; fid, mid, \beta\}, M_d, ddc, stc, \downarrow z)
\]
where we have the induction hypothesis:
\[
\{e'; fid, mid, \beta\}, M_d, ddc, stc, \downarrow (\_, \_, z, \_).
\]
Thus, by applying rule evalCapEnd to the goal, we can immediately show our goal using the induction hypothesis on \(e'\).

Case **Evaluate-expr-cast-to-integer-offset:**

By substitution in Definition 58, our goal becomes:
\[
\text{capOff}(\{e'; fid, mid, \beta\}, M_d, ddc, stc, \downarrow z)
\]
where we have the induction hypothesis:
\[
\{e'; fid, mid, \beta\}, M_d, ddc, stc, \downarrow (\_, \_, \_, z).
\]
Thus, by applying rule evalCapOff to the goal, we can immediately show our goal using the induction hypothesis on \(e'\).

Case **Evaluate-expr-cap-type:**

By substitution in Definition 58, our goal becomes:
\[
\text{capType}(\{e'; fid, mid, \beta\}, M_d, ddc, stc, \downarrow v)
\]
where we have the induction hypothesis:
\[
\{e'; fid, mid, \beta\}, M_d, ddc, stc, \downarrow (x, \_, \_, \_).
\]
and the assumptions:
\[
x = \kappa \implies v = 0 \quad \text{and} \quad x = \delta \implies v = 1.
\]
Thus, by applying rule evalCapType to the goal, we can immediately show our goal using the induction hypothesis on \(e'\) and the assumptions on \(x\) and \(v\).

Case **Evaluate-expr-binop:**

By substitution in Definition 58, our goal becomes:
\[
\{e_1 \oplus e_2; fid, mid, \beta\}, M_d, ddc, stc, \downarrow z
\]
where we have the induction hypotheses:
\[
\{e_1; fid, mid, \beta\}, M_d, ddc, stc, \downarrow z_1 \quad \text{and} \quad \{e_2; fid, mid, \beta\}, M_d, ddc, stc, \downarrow z_2
\]
and the assumption:
\[
z = z_1[\oplus]z_2.
\]
Thus, by applying rule evalBinOp to the goal, we can immediately show our generated subgoals using the induction hypotheses on \(e_1\) and \(e_2\) and the assumption on \(z, z_1\) and \(z_2\).

Case **Evaluate-expr-addr-local** and **Evaluate-expr-addr-module:**

These two cases are proved by Lemma 87.

Case **Evaluate-expr-var:**

By substitution in Definition 58, our goal becomes:
\[
deref(\{addr(vid); fid, mid, \beta\}, M_d, ddc, stc, \downarrow v)
\]
And we have the assumptions:
- \(addr(vid), \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow (\delta, s, e, \text{off})\)
- \(s \leq s + \text{off} < e\)
- \(Mem(s + \text{off}) = v\)
By Lemma 87, we have that:
\[ \text{addr}(\text{vid}, \text{M}_d, \text{ddc}, \text{stc}) \downarrow \delta, s, e, \text{off} \]
Thus, from the assumption \( s + \text{off} < e \), it follows by substitution that \( s \leq s + \text{off} < e \).
Applying rule evalDeref to our goal, we get the following goals:

1. \( s \leq s + \text{off} < e \) which is immediate.
2. \( \text{M}_d(s + \text{off}) = v \)

For the latter goal, we notice first from assumptions:
\[ \text{addr}(\text{vid}, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc}) \downarrow \delta, s, e, \text{off} \]

and by Lemma 81 that:
\[ s + \text{off} \in \text{reachable\_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) = A_s \]
And, also by the definition of the value \( (\delta, s, e, \text{off}) \), we have using Lemma 25 that:
\[ s + \text{off} \in \text{reachable\_addresses}(\{\text{ddc}, \text{stc}\}, \text{M}_d) \]
Applying Lemma 18, and using the assumptions \( \text{imp}(\text{mid}).\text{ddc} = \text{ddc} \), and \( \text{mstc}(\text{mid}) = \text{stc} \), we hence conclude that:
\( s + \text{off} \in A_s \). (Here, we used Lemmas 6 and 18, and a little hand-waving to prove that the offsets of both \( \text{ddc} \) and \( \text{stc} \) do not affect the function \text{reachable\_addresses}.)

So, by assumptions
\( A_s = A_t \) and \( \text{Mem}|_{A_s} = \text{M}_d|_{A_t} \),
we conclude:
\( \text{M}_d(s + \text{off}) = \text{Mem}(s + \text{off}) \)
This last statement together with assumption \( \text{Mem}(s + \text{off}) = v \) immediately prove our remaining goal.

Case Evaluate-expr-addr-arr:
By substitution in Definition 58, our goal becomes:
\[ \text{inc}(\{ \text{addr}(e_{arr}) \}_{\text{fid}, \text{mid}, \beta}, \text{M}_d, \text{ddc}, \text{stc}) \downarrow \delta, s, e, \text{off} + \text{off}' \]
with the induction hypotheses and assumptions:
1. \( \text{off}' \in \mathbb{Z} \)
2. \( \{ \text{addr}(e_{arr}) \}_{\text{fid}, \text{mid}, \beta}, \text{M}_d, \text{ddc}, \text{stc} \downarrow \delta, s, e, \text{off} \)
3. \( \{ e_{off} \}_{\text{fid}, \text{mid}, \beta}, \text{M}_d, \text{ddc}, \text{stc} \downarrow \text{off}' \)
Thus, by applying rule evalIncCap to our goal, we get five subgoals which are immediately satisfiable by our induction hypotheses and assumptions.

Case Evaluate-expr-arr:
By substitution in Definition 58, our goal becomes:
\[ \text{deref}(\{ \text{addr}(e_{arr}[e_{off}]) \}_{\text{fid}, \text{mid}, \beta}, \text{M}_d, \text{ddc}, \text{stc}) \downarrow v \]
with the assumptions:
1. \( \text{addr}(e_{arr}[e_{off}]), \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow \delta, s, e, \text{off} \)
2. \( s \leq s + \text{off} < e \)
3. \( \text{Mem}(s + \text{off}) = v \)
From the first assumption using an argument exactly the same as case Evaluate-expr-addr-arr, we conclude that:
\[ \{ \text{addr}(e_{arr}[e_{off}]) \}_{\text{fid}, \text{mid}, \beta}, \text{M}_d, \text{ddc}, \text{stc} \downarrow \delta, s, e, \text{off} \]
Thus, by applying rule evalDeref to our goal, we obtain three subgoals.
Two of them are immediate by our conclusions so far (after unfolding $\vdash_\delta (\delta, s, e, \text{off})$ using Definition 2).
The subgoal $\mathcal{M}_d(s + \text{off}) = v$ is proved by using the assumptions:
$A_s = A_t$ and $\text{Mem}|_{A_s} = \mathcal{M}_d|_{A_t}$,
and Lemma 81 as in case Evaluate-expr-var.

**Case Evaluate-expr-deref:**
By substitution in Definition 58, our goal becomes:
$$\text{deref}(\underbrace{e}_{\text{fid}, \text{mid}, \beta}, \mathcal{M}_d, \text{ddc}, \text{stc}, _{-} \downarrow v)$$
with the induction hypothesis and assumptions:
- $\underbrace{e}_{\text{fid}, \text{mid}, \beta}, \mathcal{M}_d, \text{ddc}, \text{stc}, _{-} \downarrow (\delta, s, e, \text{off})$
- $s \leq s + \text{off} < e$
- $\text{Mem}(s + \text{off}) = v$

Thus, by applying rule evalDeref to our goal, we obtain three subgoals.
Two of them are immediate by our conclusions so far (after unfolding $\vdash_\delta (\delta, s, e, \text{off})$ using Definition 2).
The subgoal $\mathcal{M}_d(s + \text{off}) = v$ is proved by using the assumptions:
$A_s = A_t$ and $\text{Mem}|_{A_s} = \mathcal{M}_d|_{A_t}$,
and Lemmas 18, 25 and 81 as in case Evaluate-expr-var.

**Case Evaluate-expr-limrange:**
By substitution in Definition 58, our goal becomes:
$$\lim(\underbrace{e}_{\text{fid}, \text{mid}, \beta}, \underbrace{e_s}_{\text{fid}, \text{mid}, \beta}, \underbrace{e_e}_{\text{fid}, \text{mid}, \beta}, \mathcal{M}_d, \text{ddc}, \text{stc}, _{-} \downarrow (x, s', e', \text{off}))$$
with the induction hypotheses and assumptions:
- $\underbrace{e}_{\text{fid}, \text{mid}, \beta}, \mathcal{M}_d, \text{ddc}, \text{stc}, _{-} \downarrow (x, s, e, \text{off})$
- $\underbrace{e_s}_{\text{fid}, \text{mid}, \beta}, \mathcal{M}_d, \text{ddc}, \text{stc}, _{-} \downarrow s'$
- $\underbrace{e_e}_{\text{fid}, \text{mid}, \beta}, \mathcal{M}_d, \text{ddc}, \text{stc}, _{-} \downarrow e'$
- $[s', e') \subseteq [s, e)$

Thus, by applying rule evalLim to our goal, we obtain four subgoals which are immediate by our four assumptions/hypotheses above.

This concludes the proof of Lemma 88. 

Lemma 89 (Expression translation backward simulation - case addr(vi).

$$\sqrt{\text{mods}}, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc}, \text{mid}, \text{fid}, \text{vid}, \mathcal{M}_d, \text{stc}, \text{ddc}.$$ 
$$\text{pc} = (\text{fid}, _) \land \Delta(\text{mid}) = (\text{ddc}, \sigma, \text{ddc}, e) \land$$
$$\Sigma(\text{mid}) = (\text{stc}, \sigma, \text{stc}, e) \land \Phi(\text{mid}) = \text{stc}, \text{off} \land$$
$$\vdash_\Delta \vdash_\beta \vdash_{\text{exec}} \langle \text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc} \rangle \land$$
$$\exists v. \underbrace{\text{addr}(\text{vid})}_{\text{fid}, \text{mid}, \beta}, \mathcal{M}_d, \text{ddc}, \text{stc}, _{-} \downarrow v \implies \exists v. \text{addr}(\text{vid}), \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \downarrow v \land v \equiv v$$

Proof.
We assume the antecedents, and by Definition 58, we consider the following two cases:
• Case $\beta(vid, fid, mid) = (s,e)$:

In this case, we know, by Definition 58 and by assumption that:
$$\lim(stc, capStart(stc) + capOff(stc) + s, capStart(stc) + capOff(stc) + e), M_d, ddc, stc, \_ \downarrow v$$

Thus, by rule evalLim, we have (ANTECS-evalLim):
$$stc, M_d, ddc, stc, pcc \downarrow v',$$
$$\text{capStart}(stc) + \text{capOff}(stc) + s, M_d, ddc, stc, pcc \downarrow s',$$
$$\text{capStart}(stc) + \text{capOff}(stc) + e, M_d, ddc, stc, pcc \downarrow e',$$
$$s' \in \mathbb{Z},$$
$$e' \in \mathbb{Z},$$
$$v' = (x, s, e, \_ ) \in \text{Cap},$$
$$[s', e'] \subseteq [s, e],$$
$$v = (x, s', e', 0)$$

Thus, by applying rules evalCapStart, evalCapOff, and evalStc to the first three statements of (ANTECS-evalLim), we conclude by substitution from the assumption that:
$$v = (\delta, \Sigma(mid).1 + \Phi(mid) + s, \Sigma(mid).1 + \Phi(mid) + e, 0)$$

Thus, our goal is to show that:
$$\text{addr}(vid), \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow (\delta, \Sigma(mid).1 + \Phi(mid) + s, \Sigma(mid).1 + \Phi(mid) + e, 0)$$

By rule Evaluate-expr-addr-local, it suffices to show that:
$$\text{vid} \in \text{localIDs}(Fd(fid)) \cup \text{args}(Fd(fid))$$

This follows from the case condition $\beta(vid, fid, mid) = (s,e)$ together with assumption $\_; \_; \text{mod}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{exec} (Mem, stk, pc, \Phi, nalloc)$ after inversion using rule Exec-state-src then rule Well-formed program and parameters and Well-formed program.

• Case $\beta(vid, \bot, mid) = (s,e)$:

In this case, we know, by Definition 58 and by assumption that:
$$\lim(ddc, capStart(ddc) + s, capStart(ddc) + e), M_d, ddc, stc, \_ \downarrow v$$

Thus, by rule evalLim, we have (ANTECS-evalLim-2):
$$stc, M_d, ddc, stc, pcc \downarrow v',$$
$$\text{capStart}(ddc) + s, M_d, ddc, stc, pcc \downarrow s',$$
$$\text{capStart}(ddc) + e, M_d, ddc, stc, pcc \downarrow e',$$
$$s' \in \mathbb{Z},$$
$$e' \in \mathbb{Z},$$
$$v' = (x, s, e, \_ ) \in \text{Cap},$$
$$[s', e'] \subseteq [s, e],$$
$$v = (x, s', e', 0)$$

Thus, by applying rules evalCapStart, and evalddc to the first three statements of (ANTECS-evalLim-2), we conclude by substitution from the assumption that:
$$v = (\delta, \Delta(mid).1 + s, \Delta(mid).1 + e, 0)$$

Thus, our goal is to show that:
$$\text{addr}(vid), \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow (\delta, \Delta(mid).1 + s, \Delta(mid).1 + e, 0)$$

By rule Evaluate-expr-addr-module, it suffices to show that:
$$\text{vid} \notin \text{localIDs}(Fd(fid)) \cup \text{args}(Fd(fid))$$

This follows from the case condition $\beta(vid, \bot, mid) = (s,e)$ together with assumption $\_; \_; \text{mod}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{exec} (Mem, stk, pc, \Phi, nalloc)$ after inversion using rule Exec-state-src then rule Well-formed program and parameters and Well-formed program.

This concludes the proof of Lemma 89.
Lemma 90 (Expression translation backward simulation).

∀mods, Σ, ∆, β, MVar, Fd, Mem, stk, pc, Φ, nalloc, mid, vid, M_d, stc, ddc.

\[ pc = ( fid, _ ) \land \Delta( mid ) = ( ddc.σ, ddc.e ) \land \]

\[ \Sigma( mid ) = ( stc.σ, stc.e ) \land \Phi( mid ) = stc.off \land \]

\[ imp( mid ). ddc \equiv ddc \land mstc( mid ) \equiv stc \land \]

_ : _ ⊨ \text{exec} ( Mem, stk, pc, Φ, nalloc ) \land \)

moduleID( Fd( fid ) ) ∈ modIDs \land \)

\[ A_s = \text{reachable_addresses}( Σ, ∆, modIDs, Mem ) \land \]

\[ A_t = \text{reachable_addresses}( \bigcup_{ mid ∈ modIDs } \{ imp( mid ). ddc, mstc( mid ) \} , M_d ) \land \]

\[ \exists v. \_ \downarrow v \]

⇒ \exists v. e, Σ, ∆, β, MVar, Fd, Mem, Φ, pc \downarrow v \land v ≜ v

Proof.
We assume the antecedents and prove our goal by induction on the expression evaluation \{ e \}_{ fid, mid, β } . M_d, ddc, stc, _ \downarrow v.

Case evalconst:
Here, \{ e \}_{ fid, mid, β } = z.

By Definition 58, we thus know e = z.

Thus, by rule Evaluate-expr-const, we have our goal.

Case evalddc:
Here, \{ e \}_{ fid, mid, β } = ddc.

By Definition 58, we thus know this case is impossible.

Case evalstc:
Here, \{ e \}_{ fid, mid, β } = stc.

By Definition 58, we thus know this case is impossible.

Case evalCapType:
Here, \{ e \}_{ fid, mid, β } = capType( E' ) ,

with E', M_d, ddc, stc, _ \downarrow v',

and by Definition 58, we know:

\[ \exists e'. e = \text{capType}( e' ) \land E' = \{ e' \}_{ fid, mid, β } . \]

Thus, by the induction hypothesis, we know (IH):

\[ e', Σ, ∆, β, MVar, Fd, Mem, Φ, pc \downarrow v' . \]

Now, we consider the following cases:

- Case v' ∈ Z:
  In this case, our goal is: e, Σ, ∆, β, MVar, Fd, Mem, Φ, pc \downarrow 0.
  But this is immediate by (IH), and rule Evaluate-expr-cap-type.

- Case v' ∈ { κ } × Z × Z × Z:
  In this case, our goal is: e, Σ, ∆, β, MVar, Fd, Mem, Φ, pc \downarrow 1.
  But this is immediate by (IH), and rule Evaluate-expr-cap-type.
• Case $v' \in \{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$:
  In this case, our goal is: $e, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow 2$.
  But this is immediate by (IH), and rule Evaluate-expr-cast-to-integer-end.

Case evalCapStart:
Here, $[e]_{fid, mid, \beta} = \text{capStart}(E')$,
with $E', \mathcal{M}_d, ddc, stc, \_ \downarrow v'$,
and by Definition 58, we know:
$\exists e'. e = \text{start}(e') \land E' = [e']_{fid, mid, \beta}$.
Thus, by the induction hypothesis, we know (IH):
$e', \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v'$.
Our goal is thus immediate by (IH) and rule Evaluate-expr-cast-to-integer-start.

Case evalCapEnd:
Here, $[e]_{fid, mid, \beta} = \text{capEnd}(E')$,
with $E', \mathcal{M}_d, ddc, stc, \_ \downarrow v'$,
and by Definition 58, we know:
$\exists e'. e = \text{end}(e') \land E' = [e']_{fid, mid, \beta}$.
Thus, by the induction hypothesis, we know (IH):
$e', \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v'$.
Our goal is thus immediate by (IH) and rule Evaluate-expr-cast-to-integer-end.

Case evalCapOff:
Here, $[e]_{fid, mid, \beta} = \text{capOff}(E')$,
with $E', \mathcal{M}_d, ddc, stc, \_ \downarrow v'$,
and by Definition 58, we know:
$\exists e'. e = \text{offset}(e') \land E' = [e']_{fid, mid, \beta}$.
Thus, by the induction hypothesis, we know (IH):
$e', \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v'$.
Our goal is thus immediate by (IH) and rule Evaluate-expr-cast-to-integer-offset.

Case evalBinOp:
By rule evalBinOp and by Definition 58, we know $e = e_1 \oplus e_2$, so we know:
$[e]_{fid, mid, \beta} = [e_1 \oplus e_2]_{fid, mid, \beta} = [e_1]_{fid, mid, \beta} \oplus [e_2]_{fid, mid, \beta}$,
$[e_1]_{fid, mid, \beta} = \mathcal{M}_d, ddc, stc, \_ \downarrow v_1$, and
$[e_2]_{fid, mid, \beta} = \mathcal{M}_d, ddc, stc, \_ \downarrow v_2$.
Thus, by the induction hypothesis, we know (IH1):
$e_1, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v_1$,
and (IH2):
$e_2, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v_2$.
Thus, our goal is immediate by (IH1), (IH2), and rule Evaluate-expr-binop.

Case evalIncCap:
Here, by Definition 58, we know:
$[e]_{fid, mid, \beta} = [\text{addr}(e_{arr} \downarrow e_{off})]_{fid, mid, \beta} = \text{inc}([e_{arr}]_{fid, mid, \beta}, [e_{off}]_{fid, mid, \beta})$
And by rule evalIncCap, we know:
$[e_{arr}]_{fid, mid, \beta} = \mathcal{M}_d, ddc, stc, pcc \downarrow v \in \text{Cap}$, and
$[e_{off}]_{fid, mid, \beta} = \mathcal{M}_d, ddc, stc, pcc \downarrow v_2 \in \mathbb{Z}$
By the induction hypothesis, we thus know (IH1):
\[ e_{\text{arr}}, \Sigma, \Delta, \beta, \text{MVar}, \Phi, \text{M}, \Phi, \text{pc} \Downarrow v, \]
and (IH2):
\[ e_{\text{off}}, \Sigma, \Delta, \beta, \text{MVar}, \Phi, \text{M}, \Phi, \text{pc} \Downarrow v_z \]
Our goal is to show that:
\[ \text{addr}(e_{\text{arr}}, e_{\text{off}}), \Sigma, \Delta, \beta, \text{MVar}, \Phi, \text{M}, \Phi, \text{pc} \Downarrow (\delta, v.\sigma, v.e, v.\text{off} + v_z), \]
This is immediate by rule Evaluate-addr-arr.

Case evalDeref:

By rule evalDeref, we know (DEREF-ASSMS):
\[ E, \Sigma, \Delta, \beta, \text{MVar}, Fd, \text{Mem}, \Phi, \text{pc} \Downarrow v \]
Our goal is to show that:
\[ e, \Sigma, \Delta, \beta, \text{MVar}, Fd, \text{Mem}, \Phi, \text{pc} \Downarrow v \]
By Definition 58, we distinguish the following cases:

- **Case** \( e = \text{deref}(e') \):
  
  Here, by Definition 58, we also know:
  \[ \{e'\} \Downarrow \text{fid}, \text{mid}, \beta = E' \]
  
  Thus, together, with the assumption above, we have by the induction hypothesis that:
  \[ e', \Sigma, \Delta, \beta, \text{MVar}, Fd, \text{Mem}, \Phi, \text{pc} \Downarrow v' \]
  
  By Rule Evaluate-expr-deref, we thus have the following two subgoals:
  
  - \[ v'.\sigma \leq v'.\sigma + v'.\text{off} < v'.e \]
    
    This is immediate by (DEREF-ASSMS)’s \( \Downarrow \text{off} \) (unfolding Definition 2).
  
  - \[ \text{Mem}(v'.\sigma + v'.\text{off}) = v \]
    
    Here, by (DEREF-ASSMS)’s \( v = \text{M}_d(v'.\sigma + v'.\text{off}) \), and \( \Downarrow \text{off} \), and the antecedents, it suffices to show that:
    
    \[ v'.\sigma + v'.\text{off} \in A_e \]
    
    This is immediate by Lemma 81.

- **Case** \( e = \text{vid} \):
  
  By inverting our goal using rule Evaluate-expr-var, we obtain the following subgoals:
  
  - \[ \text{addr}(\text{vid}), \Sigma, \Delta, \beta, \text{MVar}, Fd, \text{Mem}, \Phi, \text{pc} \Downarrow (\delta, s, e, \text{off}) \]
    
    By Definition 58, we know:
    \[ E' = \{\text{addr}(\text{vid})\} \Downarrow \text{fid}, \text{mid}, \beta \]
    
    Thus, by Lemma 89, we know (ADDR-EVAL):
    \[ \text{addr}(\text{vid}), \Sigma, \Delta, \beta, \text{MVar}, Fd, \text{Mem}, \Phi, \text{pc} \Downarrow v' \]
    
    which together with the knowledge of \( \Downarrow \text{off} \) (DEREF-ASSMS) immediately satisfy our subgoal.
  
  - \[ v'.\sigma \leq v'.\sigma + v'.\text{off} < v'.e \]
    
    Immediate by \( \Downarrow \text{off} \) (unfolding Definition 2).
  
  - \[ \text{Mem}(v'.\sigma + v'.\text{off}) = v \]
    
    Here, by (DEREF-ASSMS)’s \( v = \text{M}_d(v'.\sigma + v'.\text{off}) \), and \( \Downarrow \text{off} \), and the antecedents, it suffices to show that:
    
    \[ v'.\sigma + v'.\text{off} \in A_e \]
    
    This is immediate by Lemma 81.

- **Case** \( e = e_{\text{arr}}[e_{\text{off}}] \):
  
  Here, by Definition 58, we have:
  \[ E' = \{\text{addr}(e_{\text{arr}}[e_{\text{off}}])\} \Downarrow \text{fid}, \text{mid}, \beta = \text{inc}(\{e_{\text{arr}}\} \Downarrow \text{fid}, \text{mid}, \beta, \{e_{\text{off}}\} \Downarrow \text{fid}, \text{mid}, \beta) \]
  
  Thus, by (DEREF-ASSMS), and inversion using rule evalIncCap, we obtain (INC-ASSMS):
  \[ \{e_{\text{arr}}\} \Downarrow \text{fid}, \text{mid}, \beta, \text{M}_d, \text{dc}, \text{stc}, \Downarrow (\delta, \sigma_a, e_a, \text{off}_a), \]
  \[ \{e_{\text{off}}\} \Downarrow \text{fid}, \text{mid}, \beta, \text{M}_d, \text{dc}, \text{stc}, \Downarrow v_z \in \mathbb{Z}, \]
  
  and
  \[ v'.\text{off} = \text{off}_a + v_z \]

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By the induction hypothesis (instantiated with (INC-ASSMS)), we thus have (IH-E-ARR):
\[ e_{arr}, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow (\delta, \sigma_a, e_a, off_a) \]
and (IH-E-OFF):
\[ e_{off}, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v_z \in Z \]
By inverting our goal using rule Evaluate-expr-arr, we obtain the following subgoals:
- \( \text{addr}(e_{arr}[e_{addr}]), \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v' \)
  By inversion using rule Evaluate-expr-addr, we obtain the following subgoals:
  * \( e_{arr}, MVar, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow (\delta, \sigma_a, e_a, off_a) \)
    Immediate by (IH-E-ARR).
  * \( e_{off}, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v_z, \) and
    \* \( v_z \in Z \)
    Immediate by (IH-E-OFF).
- \( v'.\sigma \leq v'.\sigma + v'.\text{off} < v'.e \)
  Immediate by \( \vdash_\delta v' \) of (DEREF-ASSMS).
- \( \text{Mem}(v'.\sigma + v'.\text{off}) = v \)
  Here, by (DEREF-ASSMS)’s \( v = M_d(v'.\sigma + v'.\text{off}) \), and \( \vdash_\delta v' \), and the antecedents, it suffices to show that:
  \( v'.\sigma + v'.\text{off} \in A_4 \).
  This is immediate by Lemma 81.

**Case evalLim:**
Here, \( \{ e \}_{\text{fid, mid, } \beta} = \text{lim}(E, E_s, E_e) \)

By rule evalLim, we know (LIM-ASSMS):
\[ E, M_d, ddc, stc, pcc \downarrow v, \]
\[ E_s, M_d, ddc, stc, pcc \downarrow s', \]
\[ E_e, M_d, ddc, stc, pcc \downarrow e', \]
\[ s' \in Z, \]
\[ e' \in Z, \]
\[ v = (x, s, e, _) \in \text{Cap}, \]
\[ [s', e'] \subseteq [s, e], \) and
\[ v' = (x, s', e', 0) \]

By Definition 58, we distinguish the following cases:

- **Case** \( e = \text{limRange}(e_{cap}, e_s, e_e) \):
  Here, \( E = \{ e_{cap}\}_{\text{fid, mid, } \beta}, E_s = \{ e_s\}_{\text{fid, mid, } \beta}, \) and \( E_e = \{ e_e\}_{\text{fid, mid, } \beta} \)
  We thus get the following induction hypotheses (IH-limRange):
  \[ e_{cap}, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v, \]
  \[ e_s, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow s', \) and
  \[ e_e, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow e' \]
  By inverting our goal using rule Evaluate-expr-limrange, we get the following subgoals instead:
  \[ e_{cap}, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow (x, s, e, _), \]
  \[ e_s, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow s', \]
  \[ s' \in Z, \]
  \[ e_e, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow e', \]
  \[ e' \in Z, \]
  \[ [s', e'] \subseteq [s, e], \) and
  \[ v' = (x, s', e', 0) \]
  which are all immediate by (IH-limRange) and (LIM-ASSMS).
• Case \( e = \text{addr}(vid) \land \beta(vid, \perp, mid) = (st, end) \):

Here, \( \mathcal{E} = \text{ddc}, \mathcal{E}_s = \text{capStart}(\text{ddc}) + st \), and \( \mathcal{E}_e = \text{capStart}(\text{ddc}) + end \)

Thus, by (LIM-ASSMS), inversion using rules evalddc and evalCapStart, and by our lemma assumptions, we conclude:

\[
v = (x, s, e, \perp) = (\delta, \Delta(mid).1, \Delta(mid).2, \perp),
\]

\[
s' = \Delta(mid).1 + st, \text{ and}
\]

\[
e' = \Delta(mid).1 + end
\]

Thus, \( \nu' = (\delta, \Delta(mid).1 + st, \Delta(mid).1 + end, 0) \)

Thus, by inverting our goal using rule Evaluate-expr-addr-local, only the following subgoals are not immediate:

\[
- \text{vid} \notin \text{localIDs}(Fd(fid)) \cup \text{args}(Fd(fid)), \text{ and}
\]

\[
- \text{vid} \in \text{MVar}(mid)
\]

They both follow by assumption

\[
\perp; \perp; \mod; \Sigma; \delta; \beta; \text{MVar}; Fd \vdash_{\text{exec}} \langle \text{Mem}, stk, pc, \Phi, \text{nalloc} \rangle \text{ after inversion using rule Exec-state-src then rule Well-formed program and parameters and Well-formed program.}
\]

• Case \( e = \text{addr}(vid) \land \beta(vid, fid, mid) = (st, end) \):

Here, \( \mathcal{E} = \text{stc}, \mathcal{E}_s = \text{capStart}(\text{stc}) + \text{capOff}(\text{stc}) + st \), and

\( \mathcal{E}_e = \text{capStart}(\text{stc}) + \text{capOff}(\text{stc}) + end \).

Thus, by (LIM-ASSMS), inversion using rules evalstc, evalCapStart, and evalCapOff, and by our lemma assumptions, we conclude:

\[
v = (x, s, e, \perp) = (\delta, \Sigma(mid).1, \Sigma(mid).2, \Phi(mid)),
\]

\[
s' = \Sigma(mid).1 + \Phi(mid) + st, \text{ and}
\]

\[
e' = \Delta(mid).1 + \Phi(mid) + end
\]

Thus, \( \nu' = (\delta, \Sigma(mid).1 + \Phi(mid) + st, \Sigma(mid).1 + \Phi(mid) + end, 0) \)

Thus, by inverting our goal using rule Evaluate-expr-addr-local, only the following subgoal is not immediate: \( \text{vid} \in \text{localIDs}(Fd(fid)) \cup \text{args}(Fd(fid)) \)

This follows by assumption

\[
\perp; \perp; \mod; \Sigma; \delta; \beta; \text{MVar}; Fd \vdash_{\text{exec}} \langle \text{Mem}, stk, pc, \Phi, \text{nalloc} \rangle \text{ after inversion using rule Exec-state-src then rule Well-formed program and parameters and Well-formed program.}
\]

This concludes case evalLim.

This concludes the proof of Lemma 90. \( \Box \)

**Lemma 91** (Memory bounds are preserved by compilation).

\[
\Phi \models \forall \text{mods}, mid, fid, \Sigma, \delta, \beta, K_{\text{mod}}, K_{\text{fun}}, \mathcal{M}_c, \text{imp}, \text{mstc}, \phi.
\]

\[
\Phi \models \delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}} = (\mathcal{M}_c, \mathcal{M}_d, \text{imp}, \text{mstc}, \phi) \land
\]

\[
\mathcal{M}_c \models \text{modDefs} \equiv \{ \text{modFunDef} \mid \text{modFunDef} \in \text{modFunDefs} \land (\perp, \perp, \text{modFunDefs}) \in \text{mods} \} \land
\]

\[
\mathcal{M}_d \models \text{funDefs} \equiv \{ \text{funDef} \mid \text{funDef} \in \text{funDefs} \land \text{funDef} = (\perp, \text{funDef}, \perp, \perp, \text{funDef}) \} \land
\]

\[
(mid, \perp, \perp) \in \text{mods}
\]

\[
\mathcal{M}_d(a) = 0 \land
\]

\[
\text{offs} = \{ \text{funId} \mapsto K_{\text{fun}}(\text{fid}).1 \mid \text{funId} \in \text{dom}(Fd) \} \land
\]

\[
\text{imp}(mid) = ((\kappa, K_{\text{mod}}(mid).1, K_{\text{mod}}(mid).2, 0), (\delta, \Delta(mid).1, \Delta(mid).2, 0), \text{offs}) \land
\]

\[
\text{mstc}(mid) = (\delta, \Sigma(mid).1, \Sigma(mid).2, 0) \land
\]

\[
\forall \text{fid}. \mid \text{moduleID}(Fd(fid)) \implies \phi(mid, fid) = (\text{length(args}(Fd(fid))), \text{length(localIDs}(Fd(fid))))
\]

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Lemma 92 (No additional code/data is added by the compiler).

\[
\forall \text{mods}, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}, M_c, \text{imp}, \text{mstc}, \phi. \\
\models \text{mods} \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}} = (M_c, M_d, \text{imp}, \text{mstc}, \phi) \land \\
\text{funDefs} = \{ \text{modFunDef} \mid \text{modFunDef} \in \text{modFunDefs} \land (\_ , \_ , \text{modFunDefs}) \in \text{mods} \} \land \\
F_d = \{ \text{funID} \rightarrow \text{funDef} \mid \text{funDef} \in \text{funDefs} \land \text{funDef} = (\_ , \text{funID}, \_ , \_ , \_ ) \} \\
\implies \\
(\text{mid} \in \text{dom} (\text{imp}) \implies \exists \text{fid}. \text{mid} = \text{moduleId}(F_d(\text{fid})) \land \\
\text{mid} \in \text{dom} (\text{mstc}) \implies \exists \text{fid}. \text{mid} = \text{moduleId}(F_d(\text{fid})) \land \\
(\text{mid}, \text{fid}) \in \text{dom} (\phi) \implies \text{mid} = \text{moduleId}(F_d(\text{fid})) \land \\
\text{a} \in \text{dom} (M_d) \implies \exists \text{fid}. \text{mid} = \text{moduleId}(F_d(\text{fid})) \land \text{a} \in \Delta (\text{mid}) \land \\
\text{a} \in \text{dom} (M_c) \implies \exists \text{mid}, \text{fid}, n. \text{mid} = \text{moduleId}(F_d(\text{fid})) \land \text{n} \in [0, |\text{commands}(F_d(\text{fid}))|] \land \\
\text{a} = K_{\text{mod}}(\text{mid}).1 + K_{\text{fun}}(\text{fid}).1 + n)
\]

Proof. Immediate from the assumptions after inversion using rules Module-list-translation and Module-translation.

Lemma 93 (Code memory is the translation of the commands arranged according to \( K_{\text{mod}} \) and \( K_{\text{fun}} \)).

\[
\forall \text{mods}, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}, \text{mid}, \text{fid}, n, M_c. \\
\models \text{mods} \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}} = (M_c, \_ , \_ , \_ ) \land \\
\text{funDefs} = \{ \text{modFunDef} \mid \text{modFunDef} \in \text{modFunDefs} \land (\_ , \_ , \text{modFunDefs}) \in \text{mods} \} \land \\
F_d = \{ \text{funID} \rightarrow \text{funDef} \mid \text{funDef} \in \text{funDefs} \land \text{funDef} = (\_ , \text{funID}, \_ , \_ , \_ ) \} \land \\
\text{mid} = \text{moduleId}(F_d(\text{fid})) \land \text{n} \in [0, |\text{commands}(F_d(\text{fid}))|) \land \\
M_c(\text{K}_{\text{mod}}(\text{mid}).1 + \text{K}_{\text{fun}}(\text{fid}).1 + n) = (\text{commands}(F_d(\text{fid}))(\text{n}))^{F_d, \text{K}_{\text{fun}}(\text{fid}), \text{mid}, \beta}
\]


Definition 61 (Related program counters).

\[
\begin{align*}
K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; M\text{Var}; F_d; \text{pc} & \equiv \text{pcc} \overset{\text{def}}{=} \\
K_{\text{mod}}(\text{moduleID}(F_d(\text{pc}, \text{fid}))).1 + K_{\text{fun}}(\text{pc}, \text{fid}).1 + \text{pc} \cdot \text{n} = \text{pcc} \cdot \sigma + \text{pcc} \cdot \text{off} \land \\
K_{\text{mod}}(\text{moduleID}(F_d(\text{pc}, \text{fid}))) = [\text{pcc} \cdot \sigma, \text{pcc} \cdot e]
\end{align*}
\]

Definition 62 (Related stacks).

\[
K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; M\text{Var}; F_d; \text{stk}_s \equiv \text{stk}_t \overset{\text{def}}{=} \\
\text{length}(\text{stk}_s) = \text{length}(\text{stk}_t) \land \\
\forall i \in \text{dom}(\text{stk}_s) K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; M\text{Var}; F_d; \text{stk}_s(i) \equiv \text{stk}_t(j). \text{pcc}
\]

Definition 63 (Related local stack usage).

The usage of local stacks is related between a candidate pair of source and target states when 1. the stack usage \( \Phi(\text{mid}) \) in the source state is equal to that given by the capability offset \( \text{mstc} \cdot \text{mid} \).off
of the stack capability of the target state, and 2. for all functions \( \text{fid} \), \( \text{fid} \) is callable (i.e., there is enough stack space to call it according to \( \Phi \)) in the source state iff it is callable in the target state (according to \( \text{mstc} \)). Additionally, the number of arguments specified in the source interface by the function definitions map \( Fd \) matches the number of arguments given by the implementation of the target functions specified by the map \( \phi \) of call frame sizes.

\[
K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; Fd; \Phi \equiv \text{mstc}, \phi \\
def \forall \text{mid} \in \text{dom}(\Phi). \Phi(\text{mid}) = \text{mstc}(\text{mid}).\text{off} \\
\land \\
\forall \text{fid} \in \text{dom}(Fd), \text{mid}.\text{moduleID}(Fd(\text{fid})) = \text{mid} \implies \\
(\text{frameSize}(Fd(\text{fid})) + \Sigma(\text{mid}).1 + \Phi(\text{mid}) < \Sigma(\text{mid}).2 \iff \\
\phi(\text{mid}, \text{fid}).1 + \phi(\text{mid}, \text{fid}).2 + \text{mstc}(\text{mid}).\sigma + \text{mstc}(\text{mid}).\text{off} < \text{mstc}(\text{mid}).e) \\
\land \\
\forall \text{fid} \in \text{dom}(Fd), \text{mid}.\text{moduleID}(Fd(\text{fid})) = \text{mid} \implies \\
\text{length}(\text{args}(Fd(\text{fid}))) = \phi(\text{mid}, \text{fid}).1 \\
\land \\
\forall (\text{mid}, \text{fid}) \in \text{dom}(\phi). \text{fid} \in \text{dom}(Fd) \land \text{mid} = \text{moduleID}(Fd(\text{fid}))
\]

Definition 64 (Cross-language compiled-program state similarity).

\[
K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; Fd; \{\text{Mem}, \text{stk}, \text{pc}, \Phi, \text{nalloc}\} \equiv_{\text{modIDs}} \{\mathcal{M}_c, \mathcal{M}_d, \text{stk}, \text{imp}, \phi, \text{ddc}, \text{stc}, \text{pcc}, \text{mstc}, \text{nalloc}\} \\
def \equiv \\
n\text{alloc} = \text{nalloc} \land \\
A_s = \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}) \land \\
A_t = \text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{\text{imp}(\text{mid}).\text{ddc}, \text{mstc}(\text{mid}).\text{stc}\}, \mathcal{M}_d) \land \\
A_s = A_t \land \text{Mem}|A_s = \mathcal{M}_d|A_t \land \\
\Delta(\text{moduleID}(Fd(\text{pc}.\text{fid}))) = \{\text{ddc.}\sigma, \text{ddc.}e\} \land \\
\Sigma(\text{moduleID}(Fd(\text{pc}.\text{fid}))) = \{\text{stc.}\sigma, \text{stc.}e\} \land \\
\Phi(\text{moduleID}(Fd(\text{pc}.\text{fid}))) = \text{stc.}\text{off} \land \\
K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; Fd; \text{pc} \equiv \text{pcc} \land \\
K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; Fd; \text{stk} \equiv \text{stk} \land \\
K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; Fd; \Phi \equiv \text{mstc}, \phi \\
\lor \\
(\text{pc} = \bot \land \mathcal{M}_c(\text{pcc}) = \bot)
\]

Lemma 94 (Cross-language equi-k-accessibility and memory equality is preserved by deleting as-
signments and safe allocation).

∀A, a, v, Mem, Md.
∀k, ∃A′. A′ = access_k(A, Mem) = access_k,Md,A ∧
Mem|A′ = Md|A′ ∧
(v ≠ (δ, _, _, _)) ∨
(v = (δ, σ, e, _) ∧ ∀a* ∈ [σ, e). Md[a → v](a*) ≠ (δ, _, _, _) ∧ Mem[a → v](a*) ≠ (δ, _, _, _))
⇒
∀k, ∃A′. A′ = access_k(A, Mem[a → v]) = access_k,Md[a→v],A ∧
Mem[a→v]|A′ = Md[a→v]|A′.

Proof.
We fix arbitrary A, a, v, Mem, Md and consider the following two cases from the disjunctive assumption:

- **Case v ≠ (δ, _, _, _)**:  
  In this case, by Lemma 33, we know access_k,Md[a→v],A = χ_k(A, Md, a).
  Also, by Lemma 74, we know access_k(A, Mem[a → v]) = χ_k(A, Mem, a).
  Then, our first subgoal becomes:

  ∀k. χ_k(A, Mem, a) = χ_k(A, Md, a).
  
  This can be shown by an easy induction on k with the help of Lemmas 31 and 72, Definitions 24 and 25 and the assumptions:

  ∀k, ∃A′. A′ = access_k(A, Mem) = access_k,Md,A ∧ Mem|A′ = Md|A′.
  
  Our next subgoal ∀k. Mem[a → v]|A′ = Md[a → v]|A′ (now with A′ = χ_k(A, Mem, a) = χ_k(A, Md, a)) follows again immediately from Lemmas 31 and 72, and the assumptions.

- **Case v = (δ, σ, e, _) ∧ ∀a* ∈ [σ, e). Md[a → v](a*) ≠ (δ, _, _, _) ∧ Mem[a → v](a*) ≠ (δ, _, _, _)**:  
  Here, we distinguish two cases:

  - **Case a ∈ access_k(A, Mem) = access_k,Md,A**:  
    In this case, our goals follow by Lemmas 41 and 67 together with Lemmas 31 and 72 and the assumptions.

  - **Case a /∈ access_k(A, Mem) = access_k,Md,A**:  
    In this case, our goals follow immediately from the assumptions after applying Lemmas 22 and 69.

Lemma 95 (Cross-language equi-reachability and memory equality is preserved by deleting assign-
ments, safe allocation, and assigning derivable capabilities).

∀a, v, Σ, Δ, modIDs, Mem, C, M_d.
A = \text{static\_addresses}(\Sigma, Δ, modIDs) = \bigcup_{c \in C} \{c.\sigma, c.e\} \land

∃A_r. A_r = \text{reachable\_addresses}(\Sigma, Δ, modIDs, Mem) = \text{reachable\_addresses}(C, M_d) \land

\text{Mem}|_{A_r} = M_d|_{A_r} \land

a \in A_r \land

(v \neq (\delta, _, _, _)) \lor

(v = (\delta, \sigma, e, _) \land \exists a^* \in [\sigma, e]. M_d[a \mapsto v](a^*) \neq (\delta, _, _, _)) \lor

(v = (\delta, \sigma, e, _) \land \Sigma, Δ, modIDs, Mem \models v \land C, M_d \models v))

\implies

∃A'_r, A'_r = \text{reachable\_addresses}(\Sigma, Δ, modIDs, Mem|_{a \mapsto v}) = \text{reachable\_addresses}(C, M_d|_{a \mapsto v})

\text{Mem}[a \mapsto v]|_{A'_r} = M_d[a \mapsto v]|_{A'_r}

\text{Proof.}
Here, we can use Lemma 13, and by an easy argument using assumptions Mem|_{A_r} = M_d|_{A_r} and A_r = \text{reachable\_addresses}(\Sigma, Δ, modIDs, Mem) = \text{reachable\_addresses}(C, M_d), we obtain the antecedent of Lemma 94, which proves two cases of our goal (again after applying Lemma 13 to pick a finite k).

The remaining case of our goal is proved by applying Lemmas 42 and 79 which give the first subgoal, and then applying Lemmas 43 and 80 to get the second subgoal from the assumptions. □

Lemma 96 (Compiled-program state similarity implies equi-reachability).

∀K_mod, K_fun, Σ; Δ; β; MVar; Fd: ⟨Mem, stk, pc, Φ, nalloc⟩, ⟨M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc⟩.
K_mod, K_fun, Σ; Δ; β; MVar; Fd; ⟨Mem, stk, pc, Φ, nalloc⟩ \equiv_{\text{modIDs}} ⟨M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc⟩

\implies

\text{reachable\_addresses}(\Sigma, Δ, modIDs, Mem) = \text{reachable\_addresses}(\bigcup_{mid \in \text{modIDs}} \{imp(mid).ddc, mstc(mid).stc\}, M_d)

\text{Proof.}
Immediate by Definition 64. □

Lemma 97 (Compiler forward simulation).

∀K_mod, K_fun, Σ; Δ; β; MVar; Fd: ⟨Mem, stk, pc, Φ, nalloc⟩, \overline{\text{mods}_1}.
M_c, M_d, imp, mstc, φ.
\[\|\text{mods}_1\|_{\Delta, \Sigma, \beta, K_mod, K_fun} = t \land \]
K_mod; K_fun; \overline{\text{mods}_1}; Σ; Δ; β; MVar; Fd \vdash_{\text{exec}} ⟨Mem, stk, pc, Φ, nalloc⟩ \land

t \vdash_{\text{exec}} ⟨M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc⟩ \land

\text{modIDs} = \{modID \mid (modID, _, _) \in \text{mods}_1\} \land

K_mod; K_fun; Σ; Δ; β; MVar; Fd; ⟨Mem, stk, pc, Φ, nalloc⟩ \equiv_{\text{modIDs}} ⟨M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc⟩ \land

Σ; Δ; β; MVar; Fd \vdash ⟨Mem, stk, pc, Φ, nalloc⟩ \rightarrow ⟨Mem', stk', pc', Φ', nalloc'⟩

\implies

⟨M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc⟩ \rightarrow ⟨M_c', M_d', stk', imp, φ, ddc', stk', pcc', mstc', nalloc'⟩ \land

K_mod; K_fun; Σ; Δ; β; MVar; Fd; ⟨Mem', stk', pc', Φ', nalloc'⟩ \equiv_{\text{modIDs}} ⟨M_c, M_d', stk', imp, φ, ddc', stk', pcc', mstc', nalloc'⟩
Proof.
We assume the antecedents, and we unfold assumption

\[K_{mod}; K_{fun}; \Sigma; \Delta; \beta; MVar; Fd; \langle Mem, stk, pc, \Phi, nalloc \rangle \cong_{modIDs} \langle M_c, M_d, stk, imp, \phi, ddc, stc, pcc, mstc, nalloc \rangle\]

using Definition \text{64} to obtain:

**Equal allocation**
\[nalloc = nalloc\]

**Equal reachable memories**
\[A_s = \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, Mem) \land A_t = \text{reachable_addresses}(\bigcup_{mid \in \text{modIDs}} \{\text{imp}(mid).ddc, \text{mstc}(mid).stc\}, \mathcal{M_d}) \land A_s = A_t \land \text{Mem}|A_s = \mathcal{M_d}|A_t\]

**Equal data segments**
\[\Delta(\text{moduleID}(Fd(pc.fid))) = (\text{ddc}., ddc.e)\]

**Equal stack regions**
\[\Sigma(\text{moduleID}(Fd(pc.fid))) = (\text{stc}.\sigma, \text{stc}.e)\]

**Equal stack pointers**
\[\Phi(\text{moduleID}(Fd(pc.fid))) = \text{stc}.off\]

**Related program counters**
\[K_{mod}; K_{fun}; \Sigma; \Delta; \beta; MVar; Fd; pc \cong pcc\]

**Related trusted stacks**
\[K_{mod}; K_{fun}; \Sigma; \Delta; \beta; MVar; Fd; stk \cong stk\]

**Related local stack usage**
\[K_{mod}; K_{fun}; \Sigma; \Delta; \beta; MVar; Fd; \Phi \cong \text{mstc, } \phi\]

**Static addresses are the same as module’s capabilities**
We let \[C = \bigcup_{mid \in \text{modIDs}} \{\text{imp}(mid).ddc, \text{mstc}(mid).stc\}.\]

Then, using assumption \[\llbracket\text{mods}\rrbracket_{\Delta, \Sigma, \beta, K_{mod}, K_{fun}} = \langle M_{c1}, M_{d1}, \text{imp}_{c1}, \text{mstc}_{c1}, \phi_1 \rangle\] and by Lemmas \text{91} and \text{92}, we have: static_addresses(\Sigma, \Delta, \text{modIDs}) = \bigcup_{e \in C} [\epsilon.e, \epsilon.e] \]

Then, we prove our goal by case distinction on the source reduction
\[(\Sigma, \Delta; \beta; MVar; Fd \vdash \langle Mem, stk, pc, \Phi, nalloc \rangle \rightarrow \langle Mem', stk', pc', \Phi', nalloc' \rangle)\]

**Case Assign-to-var-or-arr:**
In this case, by inversion, we have the following assumptions:

1. \[(fid, n) = pc\]
2. \[\text{commands}(Fd(fid))(n) = \text{Assign} \ e_l, e_r\]
3. \[\text{frameSize} = \text{frameSize}(Fd(fid))\]
4. \[e_l, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \Downarrow (\delta, s, e, off)\]
5. \[e_r, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \Downarrow v\]
6. \[\text{modID} = \text{moduleID}(Fd(fid))\]
7. \[\phi = \Sigma(\text{modID}).1 + \Phi(\text{modID})\]
8. \[\forall s', e'. v = (\delta, s', e', \_ \Rightarrow (s', e') \cap \Sigma(\text{modID}) = \emptyset \lor [s, e] \subseteq \Sigma(\text{modID}))\]
9. \[s \leq s + \text{off} < e\]
10. \[\text{Mem}' = \text{Mem}[s + \text{off} \rightarrow v]\]
11. $pc' = \text{inc}(pc)$

And we would like to prove the first subgoal:

$$\langle M_c, M_d, stk', imp, \phi, ddc, stc, pcc, mstc, nalloc \rangle \rightarrow \langle M_c, M' d, stk', imp, \phi, ddc', stc', pcc', nalloc' \rangle$$

By inversion using rule assign, we obtain the following subgoals:

(a) $\vdash_{\kappa} pcc$

By unfolding Definition 2, the condition on the capability type follows from assumption $t \vdash_{\text{exec}} \langle M_c, M_d, stk', imp, \phi, ddc, stc, pcc, mstc, nalloc \rangle$ by inversion using exec-state.

It remains to show the condition on the bounds:

$$pcc.\sigma \leq pcc.\sigma + pcc.\text{off} < pcc.e$$

By substitution using assumption Related program counters after unfolding Definition 61, our goal is:

$$pcc.\sigma \leq K_{\text{mod}}(\text{moduleID}(Fd(pc.fid))).1 + K_{\text{fun}}(pc.fid).1 + pc.\text{off} < pcc.e$$

By assumption Related program counters after unfolding Definition 61, we know uniquely the values of $pcc.\sigma$ and $pcc.e$:

$$|pcc.\sigma, pcc.e| = K_{\text{mod}}(\text{moduleID}(Fd(pc.fid)))$$

Thus, by substitution and a simple rewriting into interval notation, our goal becomes:

$$K_{\text{mod}}(\text{moduleID}(Fd(pc.fid))).1 + K_{\text{fun}}(pc.fid).1 + pc.\text{off} \in K_{\text{mod}}(\text{moduleID}(Fd(pc.fid)))$$

This goal can now be proved by substitution and interval arithmetic:

first by obtaining the condition on $K_{\text{fun}}(pc.fid)$ and $K_{\text{mod}}(\text{moduleID}(Fd(pc.fid)))$ from Exec-state-src,

then by noticing that $pc.\text{off} \in |\text{commands}(Fd(fid))|$ which we have from assumption (2.) obtained above.

The argument above proves $\vdash_{\kappa} pcc$.

(b) $M_c.(pcc) = \text{Assign } E_L, E_R$

This follows immediately by Lemma 93 and definition 59 after replacing $pcc.\sigma + pcc.\text{off}$ as in the previous goal.

By unrolling Definition 59, we immediately get the following substitutions which we use in the coming goals:

$$E_R = \{e_r \mid pc.fid, \text{moduleID}(Fd(fid)), \beta \}$$

and

$$E_L = \{e_l \mid pc.fid, \text{moduleID}(Fd(fid)), \beta \}$$

By assumption Equal reachable memories, we can apply Lemma 88 for the next two goals (we have all the assumptions).

(c) $E_R, M_d, ddc, stc, pcc \Downarrow v$ and

(d) $E_L, M_d, ddc, stc, pcc \Downarrow c$

are proved by Lemma 88.

(e) $\vdash_{\delta} c$

This follows by Lemma 88, then by assumptions (4.) and (9.).

(f) $\vdash_{\delta} v \implies (v \cap \text{stk} = \emptyset \lor c \subseteq \text{stc})$

After substitution using the assumption [Equal stack regions]:

$$\Sigma(\text{moduleID}(Fd(pc.fid))) = (\text{stc.}\sigma, \text{stc.}c),$$

this goal is immediately satisfied by using assumption (8.).

(g) $pcc' = \text{inc}(pcc, 1)$, and

(h) $M'_ d = M_d[c \mapsto v]$

These are inevitable by noticing that only rule assign applies after having proved the precondition
\( \mathcal{M}_c(pcc) = \text{Assign } E_L, E_R. \)

We also have to prove:
\[
K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; MVar; Fd; (\text{Mem}', \text{stk}', \text{pc}', \Phi', \text{nalloc}') \equiv_{\text{modIDs}}
(\mathcal{M}_c, \mathcal{M}'_d, \text{stk}', \text{imp}', \Phi', \text{ddc}', \text{stc}', pcc', \text{nalloc}').
\]

By unfolding Definition 64, we obtain the following subgoals:

(i) \( nalloc' = nalloc \)
   Immediate by assumption after substitution using the preconditions \( nalloc' = nalloc \) and \( nalloc' = nalloc \).

(j) \( A'_s = \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}') \land \)
   \( A'_t = \text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{ \text{imp}'(\text{mid}).ddc, \text{mstc}'(\text{mid}) \}, \mathcal{M}'_d) \land \)
   \( A'_s = A'_t \land \text{Mem}'|_{A'_s} = \mathcal{M}'_d|_{A'_t} \)

First, we obtain the following statement (*):
\( \text{imp}(\text{moduleID}(Fd(pc'.fid))).ddc \equiv \text{ddc} \) and \( \text{mstc}(\text{moduleID}(Fd(pc'.fid))) \equiv \text{stc} \)
which follows from rule exec-state together with Lemmas 91 and 93.

Then, we distinguish two cases:

- **Case** \( v \neq (\delta, \_\_\_\_, \_\_\_) \):
   In this case, we apply Lemma 95 to obtain the following subgoals:
   - \( c = (\delta, s, e, \text{off}), \) and
   - \( v = v \)
     These two follow from the successful application of Lemma 88 in the proof of subgoals (c) and (d) above.
   - The remaining subgoals follow immediately from the assumptions **Equal reachable memories** and **Static addresses are the same as module’s capabilities**.

- **Case** \( v = (\delta, \sigma, e, \_\_\_) \):
   In this case, by Lemmas 18, 25 and 81 (using assumption \( \text{moduleID}(Fd(pc'.fid)) \in \text{modIDs} \) for Lemma 81 and statement (*) for Lemmas 18 and 25), we know:
   \( [\sigma, e] \subseteq A_s = A_t \)
   which by folding Definitions 23 and 50, gives us (**):
\( \Sigma, \Delta, \text{modIDs}, \text{Mem} \models v, \) and
\( \bigcup_{\text{mid} \in \text{modIDs}} \{ \text{imp}(\text{mid}).ddc, \text{mstc}(\text{mid}) \}, \mathcal{M}d \models v \)

Now, we apply Lemma 95 to obtain the following subgoals:
   - \( c = (\delta, s, e, \text{off}), \) and
   - \( v = v \)
     These two follow from the successful application of Lemma 88 in the proof of subgoals (c) and (d) above.
   - The remaining subgoals follow immediately from (**) and the assumptions **Equal reachable memories** and **Static addresses are the same as module’s capabilities**.

(k) \( \Delta(\text{moduleID}(Fd(pc'.fid))) = (\text{ddc}'.\sigma, \text{ddc}'.e) \)
   Immediate by assumptions after rewriting using \( \text{ddc}' = \text{ddc} \) and \( \text{pc}'.\text{fid} = \text{pc}..\text{fid} \).

(l) \( \Sigma(\text{moduleID}(Fd(pc'.fid))) = (\text{stc}'.\sigma, \text{stc}'.e) \)
   Immediate by assumptions after rewriting using \( \text{stc}' = \text{stc} \) and \( \text{pc}'.\text{fid} = \text{pc}..\text{fid} \).
(m) \( \Phi(\text{moduleID}(\text{Fd}(\text{pc}.\text{fid}))) = \text{stc}.\text{off} \)
   Immediate by assumptions after rewriting using \( \text{stc} = \text{stc} \) and \( \text{pc}.\text{fid} = \text{pc}.\text{fid} \).

(n) \( K_{\text{mod}}(\text{moduleID}(\text{Fd}(\text{pc}.\text{fid}))).1 + K_{\text{fun}}(\text{pc}.\text{fid}).1 + \text{pc}.n = \text{pcc}.\sigma + \text{pcc}.\text{off} \)
   \( \wedge K_{\text{mod}}(\text{moduleID}(\text{Fd}(\text{pc}.\text{fid}))) = [\text{pcc}.\sigma, \text{pcc}.\epsilon] \)
   This is immediate after substitution using the assumptions on \( \text{pcc} \) and \( \text{pc} \) and after having proved 
   \( \text{pcc} = \text{inc}(\text{pcc}, 1) \).

(o) \( K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd}; \text{stc}' \cong \text{stk}' \)
   Immediate by assumption after rewriting using \( \text{stc}' = \text{stc} \) and \( \text{stk}' = \text{stk} \).

(p) \( K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd}; \Phi' \cong \text{mstc}', \phi \)
   Immediate by assumption after rewriting using \( \Phi' = \Phi \) and \( \text{mstc}' = \text{mstc} \).

Case Allocate:
In this case, by inversion, we have the following assumptions:

1. \( (\text{fid}, n) = \text{pc} \)
2. \( \text{commands}(\text{Fd}(\text{fid}))(n) = \text{Alloc} \ e_l \ e_{\text{size}} \)
3. \( e_l, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \Downarrow (\delta, s, e, \text{off}) \)
4. \( e_{\text{size}}, \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{Mem}, \Phi, \text{pc} \Downarrow v \)
5. \( s \leq s + \text{off} < e \)
6. \( v \in \mathbb{Z}^+ \)
7. \( \text{nalloc} - v > \nabla \)
8. \( \text{nalloc}' = \text{nalloc} - v \)
9. \( \text{Mem}' = \text{Mem}[s + \text{off} \mapsto (\delta, \text{nalloc}', \text{nalloc}, 0)](a \mapsto 0 | a \in [\text{nalloc}', \text{nalloc}]) \)

And we would like to prove the first subgoal:
\( (\mathcal{M}_c, \mathcal{M}_d, \text{stk}. \text{imp}, \phi, \text{ddc}, \text{stc}, \text{pcc}, \text{mstc}, \text{nalloc}) \rightarrow (\mathcal{M}_c', \mathcal{M}_d', \text{stk}', \text{imp}, \phi, \text{ddc}', \text{stc}', \text{pcc}', \text{nalloc}') \)

By inversion using rule allocate, we obtain the following subgoals:

(a) \( \vdash_{\alpha} \text{pcc} \)
   Same as in the previous case.

(b) \( \text{pcc}' = \text{inc}(\text{pcc}, 1) \)
   Same as in the previous case.

(c) \( \mathcal{M}_c(\text{pcc}) = \text{Alloc} \ E_{\text{size}} \ E_{\text{size}} \)
   This follows immediately by Lemma 93 and definition 59 after replacing \( \text{pcc}.\sigma + \text{pcc}.\text{off} \).
   By unrolling Definition 59, we immediately get the following substitutions which we use in the coming goals:
   \( E_{\text{size}} = [e_{\text{size}}]_{\text{pc}.\text{fid}.\text{moduleID}(\text{Fd}(\text{fid})), \beta} \)
   and \( E_{\text{size}} = [e_l]_{\text{pc}.\text{fid}.\text{moduleID}(\text{Fd}(\text{fid})), \beta} \).

By assumption equal reachable memories, we can apply Lemma 88 for the next two goals (we have all the assumptions).

(d) \( E_{\text{size}}, \mathcal{M}_d, \text{ddc}, \text{stc}, \text{pcc} \Downarrow v \) and

(e) \( E_{\text{size}}, \mathcal{M}_d, \text{ddc}, \text{stc}, \text{pcc} \Downarrow c \)
   are proved by Lemma 88.
We also have to prove:

This follows by Lemma 88 and the corresponding check of precondition (7.).

The check follows from Lemma 88 and the corresponding check of precondition (7.).

We prove (*) by applying Lemma 21, so we must prove:

We obtain (**1) and (**2) by inverting assumption t ⊩_{exec} \langle M_c, M_d, stk, imp, φ, ddc, stc, pcc, mstc, nalloc \rangle using rule exec-state.

Thus, having (*), we can now apply Lemma 95 to our goal which immediately proves it.

This is immediate after substitution using the assumptions on pcc and pc and after having proved pcc' = inc(pcc, 1).

(f) \( v \in \mathbb{Z}^+ \)

This follows by Lemma 88, then by assumption (6).

(g) \( \vdash δ \in \mathcal{C} \)

This follows by Lemma 88, then by assumptions (3.) and (5.).

(h) \( M'_d = M_d[c \mapsto (\delta, \text{nalloc} - v, \text{nalloc}, 0), i \mapsto 0 \forall i \in [\text{nalloc} - v, \text{nalloc}]] \)

Same as in the previous case (i.e., inevitable after proving that only rule allocate applies).

(i) \( \text{nalloc}' = \text{nalloc} - v \)

(j) \( \text{nalloc}' > \nabla \)

The definition of \( \text{nalloc}' \) is inevitable by rule allocate.

The check follows from Lemma 88 and the corresponding check of precondition (7.).
(q) \( K_{\text{mod}'} : K_{\text{fun}} : \Sigma ; \Delta ; \beta ; \text{MVar} ; Fd ; stk' \equiv stk' \)

Immediate by assumption after rewriting using \( stk' = stk \) and \( stk' = stk \).

(r) \( K_{\text{mod}'} : K_{\text{fun}} : \Sigma ; \Delta ; \beta ; \text{MVar} ; Fd ; \Phi' \equiv \text{mstc}' , \phi \)

Immediate by assumption after rewriting using \( \Phi' = \Phi \) and \( \text{mstc}' = \text{mstc} \).

Case Call:
In this case, by inversion, we have the following assumptions:

1. \((fid, n) = pc\)
2. \(\text{commands}(Fd(fid))(n) = \text{Call} \ fid_{\text{call}} \) \( \overline{\tau} \)
3. \(modID = \text{moduleID}(Fd(fid_{\text{call}}))\)
4. \(\text{argNames} = \text{args}(Fd(fid_{\text{call}}))\)
5. \(\text{localIDs} = \text{localIDs}(Fd(fid_{\text{call}}))\)
6. \(n\text{Ar}gs = \text{length}(\text{argNames}) = \text{length}(\overline{\tau})\)
7. \(n\text{Local} = \text{length}(\text{localIDs})\)
8. \(\text{frameSize} = \text{frameSize}(Fd(fid_{\text{call}}))\)
9. \(\text{curFrameSize} = \text{frameSize}(Fd(fid))\)
10. \(\text{curModID} = \text{moduleID}(Fd(fid))\)
11. \(\Sigma(\text{modID}).1 + \Phi(\text{modID}) + \text{frameSize} < \Sigma(\text{modID}).2\)
12. \(\Phi' = \Phi[\text{modID} \mapsto \Phi(\text{modID}) + \text{frameSize}]\)
13. \(\phi = \Sigma(\text{curModID}).1 + \Phi(\text{curModID})\)
14. \(\phi' = \Sigma(\text{modID}).1 + \Phi'(\text{modID})\)
15. \(\overline{\tau}(i), \Sigma, \Delta, \beta, \text{MVar}, Fd, \text{Mem}, \Phi, \text{pc} \downarrow v_i \quad \forall i \in [0, n\text{Ar}gs)\)
16. \(\forall i \in [0, n\text{Ar}gs), s', e'. \quad v_i = (s', e', \_ \_ \_ ) \implies [s', e'] \cap \Sigma(\text{modID}) = \emptyset\)
17. \(stk' = \text{push}(stk, \text{pc})\)
18. \(pc' = (fid_{\text{call}}, 0)\)
19. \(\text{Mem}' = \text{Mem}[\phi' + s_i \mapsto v_i | \beta(\text{argNames}(i)) = [s_i, \_ \_ \_ ] \wedge i \in [0, n\text{Ar}gs)]\)
   \([\phi' + s_i \mapsto 0 | \beta(\text{localIDs}(i)) = [s_i, \_ \_ \_ ] \wedge i \in [0, n\text{Local}]\]

And we would like to prove the first subgoal:
\(\langle \text{M}_{c}, \text{M}_{d}, \text{stk}, \text{imp}, \phi, \text{ddc}, \text{stc}, \text{pc}, \text{mstc}, \text{ndloc} \rangle \rightarrow \langle \text{M}_{c'}, \text{M}_{d}', \text{stk}', \text{imp}, \phi, \text{ddc}', \text{stc}', \text{pc}', \text{mstc}', \text{ndloc}' \rangle\)

By inversion using rule cinvoke then cinvoke-aux, we obtain the following subgoals:

(a) \(\vdash_{\kappa} \text{pcc}\)

Same as in the previous cases.

(b) \(\text{M}_{c}.(\text{pcc}) = \text{Cinvoke} \ \text{modID} \ fid_{\text{call}} \ \overline{\tau}\)

This follows immediately by Lemma 93 and definition 59
after replacing \(\text{pcc}.\sigma + \text{pcc}.\text{off}\).

By unrolling Definition 59, we immediately get the following substitutions which we use in the coming goals:

(EXPR-TRANS):
\(\overline{\tau} = [\overline{\tau}]_{\text{pc}, \text{fid}, \text{moduleID}(Fd(fid)))}, \beta\)

and
\(\text{modID} = \text{moduleID}(Fd(fid_{\text{call}}))\).

By assumption Equal reachable memories, we can apply Lemma 88 for the next goal (we have all the assumptions).
(c) \(\varphi(i), M_d, ddc, stc, pcc \downarrow v_i \forall i \in [0, nArg] \)
- First, we need to prove that \(*\) \(nArg = nArg\).
  This follows from assumption Related local stack usage after unfolding Definition 63 and obtaining conjunct
  \(\forall fid \in \text{dom}(Fd), \text{mid. moduleID}(Fd(fid)) = \text{mid} \implies \text{length(args}(Fd(fid)) = \phi(\text{mid. fid}).1\)
  which we instantiate using \(fid_{cal}\) from assumption (2.) and the substitution (EXPR-TRANS) from the previous subgoal’s proof.
- Then, for an arbitrary \(i \in [0, nArg]\), we apply Lemma 88 to the \(i\)-th goal (namely, \(\varphi(i), M_d, ddc, stc, pcc \downarrow v_i\)) obtaining subgoals that are immediate by assumptions (including crucially assumption (15.) and the substitutions (EXPR-TRANS) from the previous subgoal’s proof).

(d) \(\phi(modID, fid_{cal}) = (nArg, nLocal)\)
- Here, we just need to prove that \(\phi(modID, fid_{cal})\) is defined and that \(\phi(modID, fid_{cal}).1 = nArg\).
  This argument was given in the previous subgoal’s proof.

(e) \((\delta, \sigma, e, off) = mstc(modID)\)
- That the entry \(modID\) exists in the domain of \(mstc\) follows by inversion of the antecedent using rule exec-state from the fact that \(\phi(modID, fid_{cal})\) is defined which is proven in previous subgoals.

(f) \(\forall i \in [0, nArg]. \models_\delta v_i \implies v_i \cap stc = \emptyset\)
- Here, we need to prove that \(nArg = nArg\). This fact is proven in previous subgoals.
  Then, after substituting using that equality, the stated goal follows by assumption (16.) and subgoal (c) after substituting using assumption Equal stack regions.

(g) \((c, d, offs) = imp(modID)\)
- That the entry \(modID\) exists in the domain of \(imp\) follows by Lemma 91 and by assumption \(\text{moduleID}(Fd(pc'.fid)) \in \text{modIDs}\).

(h) \(off' = off + nArg + nLocal\),
(i) \(stc' = (\delta, s, e, off')\),
(j) \(stk' = \text{push}(stk, (ddc, pcc, modID, fid_{cal}))\),
(k) \(M'_d = M_d[s + off + i \mapsto v_i \forall i \in [0, nArg]][s + off + nArg + i \mapsto 0 \forall i \in [0, nLocal]}\),
(l) \(mstc = mstc[modID \mapsto stc']\),
(m) \(ddc' = d\), and
(n) \(pcc' = \text{inc}(c, offs(fid_{cal})))\)
- Nothing to prove. (Immediate by cinvoke-aux after knowing that only rule cinvoke possibly applies).

(o) \(\models_\delta stc'\)
- By Definition 2, we have to prove that:
  \(mstc(modID).\sigma + off + nArg + nLocal \in [mstc(modID).\sigma, mstc(modID).e]\).
  By unfolding assumption Related local stack usage using Definition 63, we obtain \(*\):

\[
\forall fid \in \text{dom}(Fd), \text{mid. moduleID}(Fd(fid)) = \text{mid} \implies
\text{frameSize}(Fd(fid)) + \Sigma(\text{mid}).1 + \Phi(\text{mid}) < \Sigma(\text{mid}).2 \iff
\phi(\text{mid. fid}).1 + \phi(\text{mid. fid}).2 + mstc(\text{mid}).\sigma + mstc(\text{mid}).off < mstc(\text{mid}).e
\]

which we instantiate using \(fid_{cal}\) and assumptions (3.) and (11.) respectively to immediately obtain our goal (after simple interval arithmetic).
We also have to prove:

\[ K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; Fd; (\text{Mem}', \text{stk}', \text{pc}', \Phi', \text{nalloc}') \cong_{\text{modIDs}} (\mathcal{M}_c, \mathcal{M}'_d, \text{imp}, \phi, \text{ddc}', \text{stc}', \text{pcc}', \text{nalloc}') \]

By unfolding Definition 64, we obtain the following subgoals:

\((p)\) \text{nalloc}' = \text{nalloc}'

Immediate from the assumption \textbf{Equal allocation} after substitution.

\((q)\) \[ A'_s = \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}') \land \]
\[ A'_t = \text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{ \text{imp}'(\text{mid}).\text{ddc}, \text{mstc}'(\text{mid}) \}, \mathcal{M}'_d) \land \]
\[ A'_s = A'_t \land \text{Mem'}|A'_t = \mathcal{M}'_d|A'_t \]

This is similar to the corresponding subgoal (i.e., (j)) of case Assign-to-var-or-arr.

We sketch the differences:

- First, we prove that \(\phi(\text{modID}, \text{fid}_{\text{call}}) = (\text{nArgs}, \text{nLocal})\) (i.e., we prove that \(\text{nLocal} = \text{nLocal}'\))

  After unfolding the definitions of argNames and localIDs, we can apply Lemma 91 to our goal to obtain subgoals that are provable using:

  assumption (6.), and

  \[ \llbracket \text{mods}_1 \rrbracket_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}} = t. \]

- We then prove our goal by induction on \(\text{nArgs} + \text{nLocal}\).

- In the \(k\)-th induction step, we distinguish two cases:
  - Case \(k \in \{0, \text{nArgs}\}\):
    Here, we know from subgoal (c) about \(v_i\) that we can apply Lemma 95 obtaining subgoals that are provable similarly to subgoal (j) of case Assign-to-var-or-arr.
  - Case \(k \in \{\text{nArgs}, \text{nArgs} + \text{nLocal}\}\):
    Here, we know from subgoal (k) that we can apply Lemma 95 obtaining subgoals that are provable similarly to subgoal (j) of case Assign-to-var-or-arr.

\((r)\) \Delta(\text{moduleID}(Fd(\text{pc}', \text{fid}))) = (\text{ddc}'.\sigma, \text{ddc}'.e)

This is immediate by Lemma 91.

\((s)\) \Sigma(\text{moduleID}(Fd(\text{pc}', \text{fid}))) = (\text{stc}'.\sigma, \text{stc}'.e)

This is also immediate by Lemma 91.

\((t)\) \Phi(\text{moduleID}(Fd(\text{pc}', \text{fid}))) = \text{stc}'.\text{off}

This is provable using assumption \textbf{Related local stack usage}.

\((u)\) \[ K_{\text{mod}}(\text{moduleID}(Fd(\text{pc}', \text{fid}))).1 + K_{\text{fun}}(\text{pc}', \text{fid}).1 + \text{pc}'.n = \text{pcc}'.\sigma + \text{pcc}'.\text{off} \land \]
\[ K_{\text{mod}}(\text{moduleID}(Fd(\text{pc}', \text{fid}))) = \lbrack \text{pcc}'.\sigma, \text{pcc}'.e \rbrack \]

Immediate by the already-established subgoals ((n) and (g)), and Lemma 91.

\((v)\) \[ K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; Fd; \text{stk}' \cong \text{stk}' \]

By unfolding Definition 62, our goal follows easily from assumption \textbf{Related program counters}.

\((w)\) \[ K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; Fd; \Phi' \cong \text{mstc}', \phi \]
By Definition 63, our goal is:

\[
\forall mid \in \text{dom}(\Phi'). \quad \Phi'(mid) = \text{mstc}'(mid).\text{off} \\
\land \\
\forall fid \in \text{dom}(Fd), mid. \quad \text{moduleID}(Fd(fid)) = mid \implies \\
(frameSize(Fd(fid)) + \Sigma(mid).1 + \Phi'(mid) < \Sigma(mid).2 \iff \\
\phi(mid, fid).1 + \phi(mid, fid).2 + \text{mstc}'(mid).\sigma + \text{mstc}'(mid).\text{off} < \text{mstc}'(mid).e \\
\land \\
\forall fid \in \text{dom}(Fd), mid. \quad \text{moduleID}(Fd(fid)) = mid \implies \\
\text{length(args}(Fd(fid))) = \phi(mid, fid).1 \\
\land \\
\forall(mid, fid) \in \text{dom}(\phi). \quad fid \in \text{dom}(Fd) \land mid = \text{moduleID}(Fd(fid))
\]

- The first conjunct is immediate by assumption Related local stack usage (after unfolding Definition 63) together with assumption (12.) and subgoals (l), (i) and (h).
- For the second conjunct, we fix arbitrary fid and mid, then we distinguish two cases:
  - Case mid = moduleID(Fd(fid, call)):
    Here, the “⇒” direction of our goal follows from subgoal (o) after substitution using subgoal (l).
    And the “⇐” direction follows from assumptions (11.) and (12.).
  - Case mid ≠ moduleID(Fd(fid, call)):
    Here, our goal is immediate by assumption Related local stack usage after substitution using mstc'(mid) = mstc(mid) of subgoal (l), and Φ'(mid) = Φ(mid) of assumption (12.).
- The remaining subgoals are immediate by assumption Related local stack usage.

Case Return:
In this case, by inversion, we have the following assumptions:

1. \((fid, n) = pc\)
2. \(\text{commands}(Fd(fid))(n) = \text{Return}\)
3. \((pc', stk') = \text{pop}(stk)\)
4. \(pc' = (fid', \_2)\)
5. \(\text{curFrameSize} = \text{frameSize}(Fd(fid))\)
6. \(\text{curModID} = \text{moduleID}(Fd(fid))\)
7. \(\Phi' = \Phi[\text{curModID} \mapsto \Phi(\text{curModID}) - \text{curFrameSize}]\)

And we would like to prove the first subgoal:

\((\mathcal{M}_c, \mathcal{M}_d, stk, imp, \phi, ddc, stc, pcc, mstc, \text{nalloc}) \rightarrow (\mathcal{M}_c', \mathcal{M}_d', stk', imp, \phi, ddc', stc', pcc', mstc', \text{nalloc}')\)

By inversion using rule creturn, we obtain the following subgoals:

(a) ⊢ pc
   Same as in the previous cases.
(b) \(\mathcal{M}_c(pcc) = \text{Creturn}\)
   This follows immediately by Lemma 93 and definition 59 after replacing pcc.σ + pcc.\text{off}.
\[\exists \phi (d) \mid \Delta (g) mstc (e) \equiv \text{related trusted stacks} \]

Furthermore, by the definition of \( \text{exec-state} \), we prove:
\[\exists \phi (d) \mid \Delta (g) mstc (e) \equiv \text{related trusted stacks} \]

Again, from Lemma 91, we know that \( \text{related trusted stacks} \)

Thus, assume for the sake of contradiction that (STK-NIL):
\[\exists \phi (d) \mid \Delta (g) mstc (e) \equiv \text{related trusted stacks} \]

Thus, assumption (STK-NIL) must be false which is our goal.

\[\phi (mid, fid) = (n\text{Args}, n\text{Local})\]

Using assumption \( \text{Execution in compile code} \), and from Lemma 91, we know that \( \phi (\text{moduleID}(Fd(pc.fid)), pc.fid) \) exists.

Furthermore, by the definition of \( \text{frameSize} \), we can conclude that (##):
\[n\text{Args} + n\text{Local} = \text{curFrameSize} \]

Again, from Lemma 91, we know that \( \text{mstc}(mid) \) exists.

\[\text{mstc} = \text{mstc}[mid \mapsto (\delta, s, e, \text{off}')]\]

Nothing to prove.

\[\exists \text{mid}', pcc' \equiv \text{imp}(\text{mid}'), pcc \land \text{stc}' = \text{mstc}(\text{mid}')\]

For the first conjunct, it suffices by rule \( \text{exec-state} \) to prove:
\[t \vdash_{\text{exec}} \langle M_c, M_d', stk', imp, \phi, ddc', stc', pcc', \text{mstc}', \text{nalloc}' \rangle\]

The latter follows from the assumption \[t \vdash_{\text{exec}} \langle M_c, M_d, stk, imp, \phi, ddc, stc, pcc, \text{mstc}, \text{nalloc} \rangle\]

by Lemma 52.

For second conjunct, all we need is to prove that \( \text{mid}' \in \text{dom(mstc)} \).
This follows from the precondition \( \text{dom(imp)} = \text{dom(mstc)} \) of also rule \( \text{exec-state} \).

We also have to prove:
\[R_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; M\text{Var}; Fd; \langle M_c, stk', pcc', \text{phi}', \text{nalloc}' \rangle \cong_{\text{modIDs}} \]
\[\langle M_c, M_d', stk', imp, \phi, ddc', stc', pcc', \text{mstc}', \text{nalloc}' \rangle\]

By unfolding Definition 64, we obtain the following subgoals:

\[\text{nalloc}' = \text{nalloc}'\]

This is immediate by assumption \( \text{Equal allocation} \) after substitution.

\[A'_i = \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem}') \land A'_i = \text{reachable_addresses}(\text{mid} \in \text{modIDs}) \{ \text{imp}'(\text{mid}).\text{ddc}, \text{mstc}'(\text{mid}) \}, M_d' \land A'_i = A'_i' \land \text{Mem}'|_{A'_i} = M_d'|_{A'_i'}\]

This is immediate (after substitution) by assumption \( \text{Equal reachable memories} \).

\[\Delta(\text{moduleID}(Fd(pc.fid))) = (ddc'.\sigma, ddc'.e)\]
By assumption Related trusted stacks (unfolding Definition 62), we know that:
$$K_{mod}(moduleID(Fd(pc',fid))) = [pcc'.σ, pcc'.e]$$
Thus, immediately, by exec-state, and the disjointness constraints of valid-linking, we know that:
$$imp(moduleID(Fd(pc'.fid))).ddc = ddc'$$
This (after unfolding Definition 6) suffices for our goal by Lemma 91.

(l) $$Σ(moduleID(Fd(pc'.fid))) = (stc'.σ, stc'.e)$$
Again, by assumption Related trusted stacks (unfolding Definition 62), we know that:
$$K_{mod}(moduleID(Fd(pc'.fid))) = [pcc'.σ, pcc'.e]$$
Thus, immediately, by exec-state, and the disjointness constraints of valid-linking, we know that:
$$mstc(moduleID(Fd(pc'.fid))) = stc'$$
This (after unfolding Definition 6) suffices for our goal by Lemma 91.

(m) $$Φ(moduleID(Fd(pc'.fid))) = stc'.off$$
This follows from the assumption Related local stack usage.

(n) $$K_{mod}(moduleID(Fd(pc'.fid))).1 + K_{fun}(pc'.fid).1 + pc'.n = pcc'.σ + pcc'.off \land K_{mod}(moduleID(Fd(pc'.fid))) = [pcc'.σ, pcc'.e]$$
This follows from assumption Related trusted stacks (unfolding Definition 62). Here is how:
Using assumption 3 and subgoal (c), together with folding Definition 61, it suffices to show that:
$$K_{mod}: K_{fun}; Σ; Δ; β; MVar; Fd; stk(length(stk) - 1) \equiv stk(length(stk) - 1).pcc$$
The latter is immediate by unfolding assumption Related trusted stacks using Definition 62.

(o) $$K_{mod}: K_{fun}; Σ; Δ; β; MVar; Fd; stk' \equiv stk'$$
Follows easily from assumption Related trusted stacks (unfolding Definition 62), assumption 3, and subgoal (c).

(p) $$K_{mod}: K_{fun}; Σ; Δ; β; MVar; Fd; Φ' ≡_{modIds} mstc', φ$$
By Definition 63, our goal is:
$$∀mid \in dom(Φ'). Φ'(mid) = mstc'(mid).off$$
$$\land ∀fid \in dom(Fd), mid. moduleID(Fd(fid)) = mid \implies$$
$$(frameSize(Fd(fid)) + Σ(mid).1 + Φ'(mid) < Σ(mid).2 \iff$$
$$φ(mid, fid).1 + φ(mid, fid).2 + mstc'(mid).σ + mstc'(mid).off < mstc'(mid).e)$$
$$\land ∀fid \in dom(Fd), mid. moduleID(Fd(fid)) = mid \implies$$
$$length(args(Fd(fid))) = φ(mid, fid).1$$
$$\land ∀(mid, fid) \in dom(φ). fid \in dom(Fd) \land mid = moduleID(Fd(fid))$$

- For the first conjunct, we fix an arbitrary mid and distinguish the following two cases:
  - Case mid = moduleID(Fd(pc, fid)):
    Here, after substitution using assumptions (5.), and (7.), and subgoals (e) and (h), our goal follows from assumption Related local stack usage.
– Case \( mid \neq \text{moduleID}(Fd(pc.fid)) \):
Here, our goal follows after substitution using assumption (7.) and subgoal (h)
from assumption Related local stack usage.

• For the second conjunct, we fix arbitrary \( fid \) and \( mid \) and again distinguish the
following two cases:

– Case \( mid = \text{moduleID}(Fd(pc.fid)) \):
Here, both the " \( \implies \) " and " \( \iff \) " directions follow by substitution using
Lemma 91.

– Case \( mid \neq \text{moduleID}(Fd(pc.fid)) \):
Here, our goal follows after substitution using assumption (7.) and subgoal (h)
from assumption Related local stack usage.

• The remaining conjuncts are immediate by assumption Related local stack usage.

Case Jump-non-zero:
In this case, by inversion, we have the following assumptions:

1. \((fid, n) = pc\)
2. \(\text{commands}(Fd(fid))(n) = \text{JumpIfZero} e_c, e_{off}\)
3. \(e_c, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v\)
4. \(v \neq 0\)
5. \(pc' = \text{inc}(pc)\)

And we would like to prove the first subgoal:
\(\langle M_c, M_d, stk, imp, \phi, ddc, stc, pcc, \Phi, nalloc \rangle \rightarrow \langle M'_c, M'_d, stk', imp, \phi, ddc', stc', pcc', \Phi, nalloc' \rangle\)

By inversion using rule jump1, we obtain the following subgoals:

(a) \(\vdash_K pcc\)
   Same as in the previous cases.

(b) \(M_c(pcc) = \text{JumpIfZero} \ E_{\text{cond}} E_{\text{off}}\)
   This follows immediately by Lemma 93 and definition 59
   after replacing \(pcc.\sigma + pcc.\text{off}\).
   By Definition 59, we have the following substitution which we use in the coming goals:
   \(E_{\text{cond}} = \{e_c\}_{\text{fid}, \text{mid}, \beta}\)
(c) \(E_{\text{cond}}, M_d, ddc, stc, pcc \downarrow v\), and
(d) \(v \neq 0\)
   After the substitution, and by assumption Equal reachable memories, we can apply
   Lemma 88 for these two subgoals (we have all the assumptions).
   From assumption \(v \neq 0\), we thus conclude \(v \neq 0\).

(e) \(pcc' = \text{inc}(pcc, 1)\)
   Immediate by rule jump1.

We also have to prove:
\(K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; MVar; Fd; \langle Mem', stk', pc', \Phi', nalloc' \rangle \cong_{\text{modIDs}}\)
\(\langle M'_c, M'_d, stk', imp, \phi, ddc', stc', pcc', nalloc' \rangle\).

By unfolding Definition 64, we obtain the following subgoals:
(f) \textit{nalloc}' = \textit{nalloc}c

Immediate by assumption after substitution using the preconditions \textit{nalloc}' = \textit{nalloc} and \textit{nalloc}' = \textit{nalloc} (of rule \texttt{jump1}).

(g) \textit{A}' = \textit{reachableAddresses}(\textit{Σ}, \Delta, \textit{modIDs}, \textit{Mem}') \land
\textit{A}' = \textit{reachableAddresses}(\bigcup_{\textit{mid} \in \textit{modIDs}} \{\textit{imp}'(\textit{mid}).\textit{ddc}, \textit{mstc}'(\textit{mid})\}, \textit{M}_d') \land
\textit{A}' = \textit{A}' \land \textit{Mem}'|_{\textit{A}'} = \textit{M}_d'|_{\textit{A}'}

Immediate by assumptions after rewriting using \textit{M}_d = \textit{M}_d and \textit{Mem}' = \textit{Mem}.

(h) \textit{Δ}(\textit{moduleID}(\textit{Fd}(\textit{pc}', \textit{fid}))) = (\textit{ddc}'.\textit{σ}, \textit{ddc}'.\textit{e})

Immediate by assumptions after rewriting using \textit{ddc}' = \textit{ddc} and \textit{pc}'.\textit{fid} = \textit{pc}.\textit{fid}.

(i) \textit{Σ}(\textit{moduleID}(\textit{Fd}(\textit{pc}', \textit{fid}))) = (\textit{stc}'.\textit{σ}, \textit{stc}'.\textit{e})

Immediate by assumptions after rewriting using \textit{stc}' = \textit{stc} and \textit{pc}'.\textit{fid} = \textit{pc}.\textit{fid}.

(j) \textit{Φ}(\textit{moduleID}(\textit{Fd}(\textit{pc}', \textit{fid}))) = \textit{stc}'.\textit{off}

Immediate by assumptions after rewriting using \textit{stc}' = \textit{stc} and \textit{pc}'.\textit{fid} = \textit{pc}.\textit{fid}.

(k) \textit{K}_{\textit{mod}}(\textit{moduleID}(\textit{Fd}(\textit{pc}', \textit{fid}))).1 + \textit{K}_{\textit{fun}}(\textit{pc}', \textit{fid}).1 + \textit{pc}'.\textit{n} = \textit{pcc}'.\textit{σ} + \textit{pcc}'.\textit{off} \land
\textit{K}_{\textit{mod}}(\textit{moduleID}(\textit{Fd}(\textit{pc}', \textit{fid}))) = [\textit{pcc}'.\textit{σ}, \textit{pcc}'.\textit{e}]

This is immediate after substitution using the assumptions on \textit{pcc} and \textit{pc} and after having proved
\textit{pcc}' = \textit{inc}(\textit{pcc}, 1).

(l) \textit{K}_{\textit{mod}}; \textit{K}_{\textit{fun}}; \textit{Σ}; \Delta; \beta; \textit{MVar}; \textit{Fd}; \textit{stk}' \cong \textit{stk}'

Immediate by assumption after rewriting using \textit{stk}' = \textit{stk} and \textit{stk}' = \textit{stk}.

(m) \textit{K}_{\textit{mod}}; \textit{K}_{\textit{fun}}; \textit{Σ}; \Delta; \beta; \textit{MVar}; \textit{Fd}; \textit{Φ}' \cong \textit{mstc}', \textit{φ}

Immediate by assumption after rewriting using \textit{Φ}' = \textit{Φ} and \textit{mstc}' = \textit{mstc}.

Case \texttt{Jump-zero}:

In this case, by inversion, we have the following assumptions:

1. (\textit{fid}, \textit{n}) = \textit{pc}
2. \textit{commands}(\textit{Fd}(\textit{fid}))(\textit{n}) = \texttt{JumpIfZero} \textit{e}_c \textit{e}_off
3. \textit{e}_c, \textit{Σ}, \Delta, \beta, \textit{MVar}, \textit{Fd}, \textit{Mem}, \textit{Φ}, \textit{pc} \Downarrow \textit{v}
4. \textit{e}_off, \textit{Σ}, \Delta, \beta, \textit{MVar}, \textit{Fd}, \textit{Mem}, \textit{Φ}, \textit{pc} \Downarrow \textit{off}
5. \textit{v} = 0
6. \textit{off} \in \mathbb{Z}
7. \textit{pc}' = (\textit{fid}, \textit{n} + \textit{off})

And we would like to prove the first subgoal:

\langle \textit{M}_c, \textit{M}_d, \textit{stk}, \textit{imp}, \textit{φ}, \textit{ddc}, \textit{stc}, \textit{pcc}, \textit{mstc}, \textit{nalloc} \rangle \rightarrow \langle \textit{M}_c, \textit{M}_d', \textit{stk}', \textit{imp}, \textit{φ}, \textit{ddc}', \textit{stc}', \textit{pcc}', \textit{nalloc}' \rangle

By inversion using rule \texttt{jump0}, we obtain the following subgoals:

(a) \textit{Γ}_\textit{pcc}

Same as in the previous cases.

(b) \textit{M}_c(\textit{pcc}) = \texttt{JumpIfZero} \textit{E}_\textit{cond} \textit{E}_\textit{off}

This follows immediately by Lemma 93 and definition 59
after replacing \textit{pcc}.\textit{σ} + \textit{pcc}.\textit{off}.

By Definition 59, we have the following substitutions which we use in the coming goals:

\textit{E}_\textit{cond} = \{\textit{e}_c\}_\textit{fid, mid, β}, and
\textit{E}_\textit{off} = \{\textit{e}_off\}_\textit{fid, mid, β}
In this case, by inversion, we have the following assumptions:

(c) $E_{\text{cond}}, M_d, ddc, stc, pcc \downarrow v,$

(d) $E_{\text{off}}, M_d, ddc, stc, pcc \downarrow \text{off},$ and

(e) $v = 0$

After the substitution, and by assumption Equal reachable memories, we can apply Lemma 88 for each of these subgoals (we have all the assumptions).

From assumption $v = 0,$ we thus conclude $v = 0.$

From assumption $\text{off} \in \mathbb{Z},$ we conclude $\text{off} \in \mathbb{Z}.$

(f) $pcc' = \text{inc}(pcc, \text{off})$

Immediate by rule jump0.

We also have to prove:

$K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; Fd; (\text{Mem'}, stk', pe', \Phi', \text{nalloc'}) \cong_{\text{modIDs}}$

$\langle M_c, M'_{d}, stk', imp, \phi, ddc', stc', pcc', \text{nalloc'} \rangle.$

By unfolding Definition 64, we obtain the following subgoals:

(g) $\text{nalloc'} = \text{nalloc'}$

Immediate by assumption after substitution using the preconditions $\text{nalloc'} = \text{nalloc}$ and $\text{nalloc'} = \text{nalloc}$ (of rule jump0).

(h) $A_s' = \text{reachable_addresses}(\Sigma, \Delta, \text{modIDs}, \text{Mem'}) \land$

$A'_i = \text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{\text{imp}(\text{mid}).ddc, \text{mstc}(\text{mid})\}, M'_d) \land$

$A'_s = A'_i \land \text{Mem'}|_{A'_i} = M'_{d}|_{A'_i}$

Immediate by assumptions after rewriting using $M'_{d} = M_{d}$ and $\text{Mem'} = \text{Mem}.$

(i) $\Delta(\text{moduleID}(Fd(pc'.fid))) = (\text{ddc}'.\sigma, \text{ddc}'.e)$

Immediate by assumptions after rewriting using $\text{ddc'} = \text{ddc}$ and $\text{pc}'.\text{fid} = \text{pc}.\text{fid}.$

(j) $\Sigma(\text{moduleID}(Fd(pc'.fid))) = (stc'.\sigma, stc'.e)$

Immediate by assumptions after rewriting using $\text{stc'} = \text{stc}$ and $\text{pc}'.\text{fid} = \text{pc}.\text{fid}.$

(k) $\Phi(\text{moduleID}(Fd(pc'.fid))) = \text{stc'.off}$

Immediate by assumptions after rewriting using $\text{stc'} = \text{stc}$ and $\text{pc}'.\text{fid} = \text{pc}.\text{fid}.$

(l) $K_{\text{mod}}(\text{moduleID}(Fd(pc'.fid))_1 + K_{\text{fun}}(pc'.\text{fid}).1 + pc'.n = pcc'.\sigma + pcc'.\text{off} \land$

$K_{\text{mod}}(\text{moduleID}(Fd(pc'.\text{fid}))) = [pcc'.\sigma, pcc'.e])$

This is immediate after substitution using the assumptions on $pcc$ and $pc$ and after having proved

$pcc' = \text{inc}(pcc, \text{off}).$

(m) $K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; Fd; \text{stk'} \cong \text{stk'}$

Immediate by assumption after rewriting using $\text{stk'} = \text{stk}$ and $\text{stk'} = \text{stk}.$

(n) $K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; Fd; \Phi' \cong \text{mstc'}, \phi$

Immediate by assumption after rewriting using $\Phi' = \Phi$ and $\text{mstc'} = \text{mstc}.$

Case Exit:

In this case, by inversion, we have the following assumptions:

1. $(\text{fid}, n) = pc$

2. $\text{commands}(Fd(\text{fid}))(n) = \text{Exit}$
And we would like to prove the first subgoal:
\( \langle M_c, M_d, stk, imp, \phi, ddc, stk, pcc, stc, nalloc \rangle \rightarrow \langle M_c', M_d', stk', imp, \phi, ddc', stc', pcc', nalloc' \rangle \)

By inversion using rule \texttt{cexit}, we obtain the following subgoals:

(a) \( \vdash K \)

Same as in the previous cases.

(b) \( M_c(\text{pcc}) = \text{Exit} \)

This follows immediately by Lemma 93 and definition 59

(All the remaining subgoals are immediate from the assumptions after substitution.)

This concludes the proof of Lemma 97.

\[ \square \]

\textbf{Lemma 98} (Compiler backward simulation).

\( \forall K_{mod}, K_{fun}, \Sigma; \Delta; \beta; MVar; Fd, \langle Mem, stk, pc, \Phi, nalloc \rangle, \text{mods}_1, t \)

\( M_c, M_d, \text{imp}, \text{mstc}, \phi. \)

\( \vdash \begin{array}{l}
\langle \text{mods}_1 \rangle_{\Delta, \Sigma, \beta, K_{mod}, K_{fun}, } = t \\
K_{mod}, K_{fun}; \langle \text{mods}_1 \rangle_{\Sigma; \Delta; \beta; MVar; Fd} \vdash_{exec} \langle Mem, stk, pc, \Phi, nalloc \rangle \\
t \vdash_{exec} \langle M_c, M_d, stk, \text{imp}, \phi, ddc, stc, pcc, stc, nalloc \rangle \\
\text{modIDs} = \{\text{modID} | (\text{modID}, \_ , \_ ) \in \text{mods}_1 \} \wedge \\
K_{mod}; K_{fun}; \Sigma; \Delta; \beta; MVar; Fd; \langle Mem, stk, pc, \Phi, nalloc \rangle \equiv_{\text{modIDs}} \langle M_c, M_d, stk, \text{imp}, \phi, ddc, stc, pcc, stc, nalloc \rangle \\
\langle M_c, M_d, stk, \text{imp}, \phi, ddc, stc, pcc, stc, nalloc \rangle \rightarrow \langle M_c', M_d', stk', \text{imp}, \phi, ddc', stc', pcc', nalloc' \rangle \\
\rightarrow \\
\Sigma; \Delta; \beta; MVar; Fd \vdash \langle Mem, stk, pc, \Phi, nalloc \rangle \rightarrow \langle Mem', stk', pc', \Phi', nalloc' \rangle \\
K_{mod}; K_{fun}; \Sigma; \Delta; \beta; MVar; Fd; \langle Mem', stk', pc', \Phi', nalloc' \rangle \equiv_{\text{modIDs}} \langle M_c, M_d', stk', \text{imp}, \phi, ddc', stc', pcc', nalloc' \rangle \\
\end{array} \)

\textit{Proof.}

- We assume the antecedents, and we assume for the sake of contradiction that

(\text{ASSM-NO-SRC-STEP}):

\( \#\text{Mem}', \text{stk}', \text{pc}', \text{nalloc}'. \Sigma; \Delta; \beta; MVar; Fd \vdash \langle Mem, stk, pc, \Phi, nalloc \rangle \rightarrow \langle Mem', stk', pc', \Phi', nalloc' \rangle \)

- Using assumptions

\( \vdash \begin{array}{l}
\text{mods}_1 \end{array} \)

\( \text{modIDs} = \{\text{modID} | (\text{modID}, \_ , \_ ) \in \text{mods}_1 \} \), and

\( K_{mod}; K_{fun}; \Sigma; \Delta; \beta; MVar; Fd; \langle Mem, stk, pc, \Phi, nalloc \rangle \equiv_{\text{modIDs}} \langle M_c, M_d, stk, \text{imp}, \phi, ddc, stc, pcc, stc, nalloc \rangle \)

we know by Lemma 93, and Definitions 61 and 64 that

(\text{CURR-COM-COMPILED}):

\( M_c(\text{pcc}) = \langle \text{commands}(Fd(pc, fid))(pc, n) \rangle_{Fd, K_{fun}, pc, fid, moduleID(pc, fid), \beta} \)

- We consider the following possible cases of the assumption

(\text{TRG-STEMS}):

\( \langle M_c, M_d, stk, \text{imp}, \phi, ddc, stc, pcc, stc, nalloc \rangle \rightarrow \langle M_c', M_d', stk', \text{imp}, \phi, ddc', stc', pcc', nalloc' \rangle \)

and derive a contradiction to (\text{ASSM-NO-SRC-STEP}) for each case.

\textbf{Case assign:}

In this case, by inversion, we have the following assumptions:
1. \(\vdash \kappa\ pcc\)
2. \(pcc' = inc(pcc, 1)\)
3. \(\mathcal{M}_c(pcc) = Assign E_L E_R\)
4. \(E_R, M_d, ddc, stc, pcc \downarrow v\)
5. \(E_L, M_d, ddc, stc, pcc \downarrow c\)
6. \(\vdash c\)
7. \(\vdash v \implies (v \cap stc = \emptyset \lor c \subseteq stc)\)
8. \(\mathcal{M}'_d = M_d[c \mapsto v]\)

By unfolding assumption (CURR-COM-COMPILED) using Definition 59, we conclude:

\[
\text{commands}(Fd(pc.fid))(pc.n) = \text{Assign } e_l e_r
\]

with \(E_L = \{e_l\}_{pc.fid \text{ moduleID}(pc.fid), \beta}\), and \(E_R = \{e_r\}_{pc.fid \text{ moduleID}(pc.fid), \beta}\)

To contradict (ASSM-NO-SRC-STEP), we have the following subgoals using rule Assign-to-var-or-arr:

- \((fid, n) = pc, \text{ and}\)
- \(\text{commands}(Fd(fid))(n) = \text{Assign } e_l e_r\)
  - Proved above.
- \(\text{frameSize} = \text{frameSize}(Fd(fid))\)
  - Nothing to prove.
- \(e_l, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow (\delta, s, e, \text{off}), \text{ and}\)
- \(e_r, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v\)
  - Follow from Lemma 90, and we obtain \(v = v\) and \((\delta, s, e, \text{off}) = c\).
- \(\text{modID} = \text{moduleID}(Fd(fid))\)
  - Existence of \(Fd(fid)\) is immediate by assumption.
- \(\phi = \Sigma(\text{modID}), 1 + \Phi(\text{modID})\)
  - Nothing to prove.
- \(\forall s', e'. v = (\delta, s', e', _) \implies ((s', e') \cap \Sigma(\text{modID}) = \emptyset \lor [s, e] \subseteq \Sigma(\text{modID}))\)
  - Follows from assumption (7), after substitution using assumption \(K_{mod}; K_{fun}; \Sigma; \Delta; \beta; MVar; Fd; (Mem, stk, pc, \Phi, nalloc) \equiv_{modIDs}\)
  - \(\langle \mathcal{M}_c, M_d, stk, \text{imp}, \phi, ddc, stc, pcc, mstc, nalloc\rangle\) (unfolding Definition 64).
- \(s \leq s + \text{off} < e\)
  - Immediate by assumption (6) after substitution using \((\delta, s, e, \text{off}) = c\) (obtained above).
- \(\text{Mem}' = \text{Mem}[s + \text{off} \mapsto v]\)
  - Nothing to prove.

**Case allocate:**

In this case, by inversion, we have the following assumptions:

1. \(\vdash \kappa\ pcc\)
2. \(pcc' = inc(pcc, 1)\)
3. \(\mathcal{M}_c(pcc) = Alloc E_L E_{size}\)
4. \(E_{size}, M_d, ddc, stc, pcc \downarrow v\)
5. \(E_L, M_d, ddc, stc, pcc \downarrow c\)
6. \(v \in \mathbb{Z}^+\)
7. \(\vdash c\)
8. \(\mathcal{M}'_d = M_d[c \mapsto (\delta, \text{nalloc} - v, \text{nalloc}, 0), i \mapsto 0 \forall i \in [\text{nalloc} - v, \text{nalloc}]\]
9. \(\text{nalloc}' = \text{nalloc} - v\)

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By unfolding assumption (CURR-COM-COMPILED) using Definition 59, we conclude:

\[ \text{commands}(Fd(pc.fid))(pc.n) = \text{Alloc } e_1 e_{size} \]

with \( E_L = \{ e_1 \}_{pc.fid.\text{moduleID}(pc.fid), \beta} \), and
\( E_{size} = \{ e_{size} \}_{pc.fid.\text{moduleID}(pc.fid), \beta} \)

To contradict (ASSM-NO-SRC-STEP), we have the following subgoals using rule Allocate:

- \((fid, n) = pc\), and
- \text{commands}(Fd(fid))(n) = \text{Alloc } e_1 e_{size}
  Proved above.
- \(e_1, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow (\delta, s, e, off)\), and
- \(e_{size}, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v\)
  Follow from Lemma 90, and we obtain \(v = v\) and \((\delta, s, e, off) = c\).
- \(s \leq s + off < e\)
  Immediate (after substitution) by assumption (7) (unfolding Definition 2).
- \(v \in \mathbb{Z}^+\)
  Immediate by assumption (6) after substitution using \(v = v\).
- \(nalloc - v > \nabla\)
  Immediate by assumption \(nalloc' > \nabla\) after substitution.
- \(nalloc' = nalloc - v\), and
- \(Mem' = Mem| s + off \mapsto (\delta, nalloc', nalloc, 0)|a \mapsto 0 \mid a \in [nalloc', nalloc]\)
  Nothing to prove.

Case jump0:

In this case, by inversion, we have the following assumptions:

1. \(\vdash_{\kappa} \text{pcc}\)
2. \(\text{M}_c(\text{pcc}) = \text{JumpIfZero } E_{\text{cond}} E_{\text{off}}\)
3. \(E_{\text{cond}}, \text{M}_d, ddc, stc, \text{pcc} \downarrow v\)
4. \(v = 0\)
5. \(E_{\text{off}}, \text{M}_d, ddc, stc, \text{pcc} \downarrow off\)
6. \(off \in \mathbb{Z}\)
7. \(\text{pcc}' = \text{inc}(\text{pcc}, off)\)

By unfolding assumption (CURR-COM-COMPILED) using Definition 59, we conclude:

\[ \text{commands}(Fd(pc.fid))(pc.n) = \text{JumpIfZero } e_c n_{dest} \]

with \( E_{\text{cond}} = \{ e_c \}_{pc.fid.\text{moduleID}(pc.fid), \beta} \), and \( E_{\text{off}} = \{ e_{off} \}_{pc.fid.\text{moduleID}(pc.fid), \beta} \)

To contradict (ASSM-NO-SRC-STEP), we have the following subgoals using rule Jump-zero:

- \((fid, n) = pc\), and
- \text{commands}(Fd(fid))(n) = \text{JumpIfZero } e_c e_{off}
  Proved above.
- \(e_c, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v\), and
- \(v = 0\)
  Follow from Lemma 90 by assumptions (3.) and (4.).

Case jump1:

In this case, by inversion, we have the following assumptions:
1. ⊢ κ pcc
2. \( \mathcal{M}_c(pcc) = \text{JumpIfZero} \ E_{\text{cond}} E_{\text{off}} \)
3. \( E_{\text{cond}}, \mathcal{M}_d, \text{ddc}, \text{stc}, \ pcc \Downarrow v \)
4. \( v \neq 0 \)
5. \( pcc' = \text{inc}(pcc, 1) \)

By unfolding assumption (CURR-COM-COMPILED) using Definition 59, we conclude:

\[
\text{commands} \left( \text{Fd}(\text{pc}. \text{fid}) \right)(\text{pc}.n) = \text{JumpIfZero} \ e_c \ n_{\text{dest}}
\]

with \( E_{\text{cond}} = \{ e_c \}_{\text{pc}, \text{fid}, \text{moduleID}(\text{pc}. \text{fid}), \beta} \) and \( E_{\text{off}} = \{ e_{\text{off}} \}_{\text{pc}, \text{fid}, \text{moduleID}(\text{pc}. \text{fid}), \beta} \)

To contradict (ASSM-NO-SRC-STEP), we have the following subgoals using rule Jump-non-zero:

– \((\text{fid}, n) = \text{pc}, \) and
– \( \text{commands} \left( \text{Fd}(\text{fid}) \right)(n) = \text{JumpIfZero} \ e_c \ n_{\text{dest}} \)
  Proved above.
– \( e_c, \Sigma, \Delta, \beta, \ MVar, \ Fd, \ Mem, \Phi, \text{pc} \Downarrow v, \) and
– \( v \neq 0 \)
  Follow from Lemma 90 by assumptions (3.) and (4.).

Case cinvoke:

In this case, by inversion, we have the following assumptions:

1. ⊢ κ pcc
2. \( \mathcal{M}_c(pcc) = \text{Cinvoke} \ mid \ fid \ \tau \)
3. \( \text{stk}' = \text{push}(\text{stk}, (\text{ddc}, \text{pcc}, \text{mid}, \text{fid})) \)
4. \( \phi(\text{mid}, \text{fid}) = (\text{nArgs}, \text{nLocal}) \)
5. \( (\delta, s, e, \text{off}) = \text{mstc}(\text{mid}) \)
6. \( \text{off}' = \text{off} + \text{nArgs} + \text{nLocal} \)
7. \( \text{stc}' = (\delta, s, e, \text{off}') \)
8. \( \tau(i), \mathcal{M}_d, \text{ddc}, \text{stc}, \ pcc \Downarrow v_i \ \forall i \in [0, n\text{Args}] \)
9. \( \forall i \in [0, n\text{Args}]. \ \tau(i) = v_i \implies v_i \cap \text{stc} = \emptyset \)
10. \( \mathcal{M}_d' = \mathcal{M}_d[s + \text{off} + i \mapsto v_i, \ \forall i \in [0, n\text{Args}]][s + \text{off} + \text{nArgs} + i \mapsto 0 \ \forall i \in [0, n\text{Local}] \]
11. \( \text{mstc}' = \text{mstc}(\text{mid} \mapsto \text{stc}') \)
12. \( (c, d, \text{offs}) = \text{imp}(\text{mid}) \)
13. \( \text{ddc}' = d \)
14. \( \text{pcc}' = \text{inc}(c, \text{offs}(\text{fid})) \)
15. \( \exists_c \text{stc}' \)

By unfolding assumption (CURR-COM-COMPILED) using Definition 59, we conclude:

\[
\text{commands} \left( \text{Fd}(\text{pc}. \text{fid}) \right)(\text{pc}.n) = \text{Call} \ fid_{\text{call}} \ \tau
\]

with \( \text{mid} = \text{moduleID}(\text{Fd}(\text{fid}_{\text{call}})), \)

\( \text{fid} = \text{fid}_{\text{call}}, \) and

\( \tau = \{ \tau \}_{\text{pc}, \text{fid}, \text{moduleID}(\text{pc}. \text{fid}), \beta} \)

To contradict (ASSM-NO-SRC-STEP), we have the following subgoals using rule Call:

– \((\text{fid}, n) = \text{pc}, \) and
– \( \text{commands} \left( \text{Fd}(\text{fid}) \right)(n) = \text{Call} \ fid_{\text{call}} \ \tau \)
  Proved above.
– \( \text{modID} = \text{moduleID}(\text{Fd}(\text{fid}_{\text{call}})), \)
– \( \text{argNames} = \text{args}(\text{Fd}(\text{fid}_{\text{call}})), \)

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– localIDs = localIDs(Fd(fid_call)),
– nArgs = length(aryNames) = length(τ),
– nLocal = length(localIDs),
– frameSize = frameSize(Fd(fid_call)),
– curFrameSize = frameSize(Fd(fid)), and
– curModID = moduleID(Fd(fid))

Nothing to prove.

– Σ(modID).1 + Φ(modID) + frameSize < Σ(modID).2

By unfolding assumption
K_{mod}; K_{fun}; Σ; Δ; β; MVar; Fd; (Mem, stk, pc, Φ, nalloc) ≃_{modIDs} (M_c, M_d, stk, imp, φ, ddc, stk, pc, mstc, nalloc) using Definition 64 then Definition 63, we obtain (*):
∀fid ∈ dom(Fd), mid. moduleID(Fd(fid)) = mid \implies (frameSize(Fd(fid)) + Σ(mid).1 + Φ(mid) < Σ(mid).2 \iff φ(mid, fid).1 + φ(mid, fid).2 + mstc(mid).σ + mstc(mid).off < mstc(mid).e)

We apply (*) to our goal, then it suffices to show (after substitution using fid = fid_call and mid = moduleID(Fd(fid_call))) that:
φ(mid, fid).1 + φ(mid, fid).2 + mstc(mid).σ + mstc(mid).off < mstc(mid).e

This is immediate by assumptions (4.), (5.), (6.), (7.), and (15.).

– Φ = Φ(modID) \implies Φ(modID) + frameSize], and
– φ' = Σ(modID).1 + Φ'(modID)

Nothing to prove.

– τ(i), Σ, Δ, β, MVar, Fd, Mem, Φ, pc ⊢ v_i \forall i ∈ [0, nArgs]

Follows from Lemma 90 after noticing that:
φ(modID, fid_call).1 = length(arys(Fd(fid_call)))

(from unfolding assumption
K_{mod}; K_{fun}; Σ; Δ; β; MVar; Fd; (Mem, stk, pc, Φ, nalloc) ≃_{modIDs} (M_c, M_d, stk, imp, φ, ddc, stk, pc, mstc, nalloc) using Definition 64 then Definition 63)

- ∃i ∈ [0, nArgs], s, e'. v_i = (s', e', _ ) \implies [s', e'] \cap Σ(curModID) = ∅

Follows from Lemma 90 and assumptions (9.) and “Σ(curModID) = [stc.σ, stc.e]” which is obtained by unfolding the assumptions using Definition 64.

- stk' = push(stk, pc),
- pc' = (fid_call, 0), and
- Mem' = Mem|φ' + s_i \mapsto v_i | β(aryNames(i)) = [s_i, _ ) \land i ∈ [0, nArgs] | [φ' + s_i \mapsto 0 | β(localIDs(i)) = [s_i, _ ) \land i ∈ [0, nLocal]

Nothing to prove.

Case **crettrue:**

In this case, by inversion, we have the following assumptions:

1. ⊢ pcc
2. M_c(pcc) = Creturn
3. stk'(ddc', pcc', mid, fid) = pop(stk)
4. φ(mid, fid) = (nArgs, nLocal)
5. (δ, s, e, off') = mstc(mid)
6. off' = off - nArgs - nLocal
7. mstc' = mstc[mid \mapsto (δ, s, e, off')]
8. ∃mid'. imp(mid').pc = pcc' ∧ stc' = mstc(mid')

By unfolding assumption (CURR-COM-COMPILED) using Definition 59, we conclude:

commands(Fd(pc.fid))(pc.n) = Return

To contradict (ASSM-NO-SRC-STEP), we have the following subgoals using rule **Return:**
By having considered all the possible cases for

Now we are required to prove: $K_{\text{mod}}: K_{\text{fun}}; \Sigma; \Delta; \beta; MVar; \text{Fd} \vdash (\text{Mem}, stk', pc', \Phi', nalloc') \equiv_{\text{modIDs}} (\mathcal{M}_c, \mathcal{M}_d', stk', \text{imp}, \phi, ddc', \text{stc'}, \text{pcc'}, \text{nalloc'})$

For this, we apply Lemma 97 obtaining the following subgoals:

1. $\models_{\text{mod}} \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}} = (\mathcal{M}_{c1}, \mathcal{M}_{d1}, \text{imp}_1, \text{mstc}_1, \phi_1)$
   Immediate by the corresponding assumption of our lemma.

2. $K_{\text{mod}}: K_{\text{fun}}; \models_{\text{mod2}} \times \models_{\text{mod1}}; \Sigma; \Delta; \beta; MVar; \text{Fd} \vdash_{\text{exec}} (\text{Mem}, stk, pc, \Phi, nalloc)$
   Immediate by the corresponding assumption of our lemma.

3. $t = (\mathcal{M}_{c2}, \mathcal{M}_{d2}, \text{imp}_2, \text{mstc}_2, \phi_2) \times (\mathcal{M}_{c1}, \mathcal{M}_{d1}, \text{imp}_1, \text{mstc}_1, \phi_1)$
   Immediate by the corresponding assumption of our lemma.

4. $t \vdash_{\text{exec}} (\mathcal{M}_c, \mathcal{M}_d, stk, \text{imp}, \phi, ddc, \text{stc}, \text{pcc}, \text{mstc}, \text{nalloc})$
   Immediate by the corresponding assumption of our lemma.

5. $\text{modIDs} = \{ \text{modID} \mid \{ \text{modID}, \_ , \_ \} \in \overline{\text{mods}} \}$
   Immediate by the corresponding assumption of our lemma.
6. \[\text{moduleID}(Fd(pc.fid)) \in modIDs\]
   Immediate by the corresponding assumption of our lemma.

7. \[K_{mod}; K_{fun}; \Sigma; \Delta; \beta; MVar; Fd; \langle Mem, stk, pc, \Phi, nalloc \rangle \cong_{modIDs} \langle M_c, M_d, stk, imp, \phi, ddc, stc, pcc, mstc, nalloc \rangle\]
   Immediate by the corresponding assumption of our lemma.

8. \[\Sigma; \Delta; \beta; MVar; Fd \vdash \langle Mem, stk, pc, \Phi, nalloc \rangle \rightarrow \langle Mem', stk', pc', \Phi', nalloc' \rangle\]
   Immediate by the previously proven subgoal (SUBGOAL-SRC-STEP-PROVED).

9. \[\text{moduleID}(Fd(pc'.fid)) \in modIDs\]
   Here, we prove it by case analysis on (SUBGOAL-SRC-STEP-PROVED):
   
   **Case Assign-to-var-or-arr:**
   
   **Case Allocate:**
   
   **Case Jump-non-zero:**
   
   **Case Jump-zero:**
   
   **Case Exit:**
   
   In these five cases, we observe that \(pc'.fid = pc.fid\).
   
   Thus, our goal (by substitution) becomes:
   
   \[\text{moduleID}(Fd(pc.fid)) \in modIDs\]
   
   But this is immediate by assumption.

   **Case Call:**
   
   Here, we obtain the following preconditions:
   
   \[\text{commands}(Fd(fid))(n) = \text{Call} \ fid_{call} \ \tau, \text{and} \]
   
   \[pc' = (fid_{call}, 0)\]
   
   By (CURR-COM-COMPILED), and the first precondition obtained above, we know:
   
   \[\mathcal{M}_c.(pcc) = \text{Cinvoke} \ \text{moduleID}(Fd(fid_{call})) \ fid_{call} \ \_\]
   
   From assumption (TRG-STEPS), and by inversion using rules cinvoke then cinvoke-aux, we know:
   
   \[(PCC'-BOUNDS):\]
   
   \[pc' \vdash \text{imp(moduleID}(Fd(fid_{call}))) \_\]
   
   Our goal (by substitution from the second precondition) becomes:
   
   \[\text{moduleID}(Fd(fid_{call})) \in modIDs\]
   
   which is immediate by assumptions.

   **Case Return:**
   
   Here, we deduce the following from the preconditions:
   
   \[pc' = stk(length(stk) - 1)\]
   
   Thus, our goal (by substitution) becomes:
   
   \[\text{moduleID}(Fd(stk(length(stk) - 1)).fid) \in modIDs\]
   
   By unfolding our lemma assumption using Definition 64 then Definition 62, we know that it suffices for our goal to prove:
   
   \[\exists mid \in modIDs. K_{mod}(mid) = (stk(length(stk) - 1).pcc.\sigma, stk(length(stk) - 1).pcc.e)\]
   
   - By inversion of our lemma assumption using rule creturn, we know
     
     \[(PCC'-IS-STK-TOP-ASSM):\]
     
     \[stk(length(stk) - 1).pcc = pcc', \text{and} \]
     
     \[(PCC'-IS-SOME-MODULE-CODE):\]
     
     \[\exists mid'. \text{imp(mid')}.pcc = pcc'\]
     
     - We obtain \(mid'\) from \((PCC'-IS-SOME-MODULE-CODE)\).
     
     - But then by Lemmas 91 and 92, and valid-linking, we know:
       
       \[mid' \in modIDs \land \text{imp(mid')}.pcc = (\kappa, K_{mod}(mid').1, K_{mod}(mid').2, 0)\]
       
     - By simple rewriting, we know:
       
       \[mid' \in modIDs \land K_{mod}(mid') = \text{imp(mid')}.pcc.\sigma, \text{imp(mid')}.pcc.e\]
\[
∀K_{\text{mod}}, K_{\text{fun}}, Σ; Δ; β; \text{MVar}; Fd, \langle \text{Mem}, stk, pc, Φ, \text{nalloc}, \text{mods} \rangle, t, (M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc).
\]
\[
\| \text{mods} \|_{Δ; Σ; \Delta; β; K_{\text{mod}}, K_{\text{fun}}} = t \wedge
\]
\[
K_{\text{mod}}; K_{\text{fun}}; \text{mods} \times \text{mods} = Σ; Δ; β; \text{MVar}; Fd \vdash_{\text{exec}} \langle \text{Mem}, stk, pc, Φ, \text{nalloc} \rangle \wedge t \vdash_{\text{exec}} (M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc) \wedge
\]
\[
\text{modIDs} = \{ \text{modID} \mid (\text{modID}, _{-}, _{-}) \in \text{mods} \} \wedge
\]
\[
K_{\text{mod}}; K_{\text{fun}}; Σ; Δ; β; MVar; Fd; \langle \text{Mem}, stk, pc, Φ, \text{nalloc} \rangle \overset{\text{modIDs}}{\approx} (M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc) \wedge
\]
\[
Σ; Δ; β; \text{MVar}; Fd; \langle \text{Mem}', stk', pc', Φ', \text{nalloc}' \rangle \overset{\text{modIDs}}{\approx} (M_c, M_d', stk', imp, φ, ddc', stk', pcc', nalloc') \wedge
\]
\[
\implies (M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc) \overset{*}{\rightarrow} (M_c', M_d', stk', imp, φ, ddc', stk', pcc', nalloc') \wedge
\]
\[
K_{\text{mod}}; K_{\text{fun}}; Σ; Δ; β; \text{MVar}; Fd; \langle \text{Mem}', stk', pc', Φ', \text{nalloc}' \rangle \overset{\text{modIDs}}{\approx} (M_c, M_d', stk', imp, φ, ddc', stk', pcc', nalloc')
\]

**Proof.**

We assume the antecedents, and we prove it by induction on the relation \( \rightarrow^* \).

- **Base case (reflexivity):**
  Here, our goal is immediate by the lemma assumptions.

- **Inductive case (transitivity):**
  Here, we obtain \( s''_t \) such that (ASSM1):
  \[
  Σ; Δ; β; \text{MVar}; Fd; s''_t \rightarrow (M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc) \rightarrow^* s''_t, \text{ and}
  Σ; Δ; β; \text{MVar}; Fd; s''_t \rightarrow (M_c', M_d', stk', imp', φ', ddc', stk', pcc', malloc').
  \]
  And by the inductive hypothesis, we have \( s''_t \) such that (ASSM2):
  \[
  (M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc) \rightarrow^* s''_t, \text{ and}
  Σ; Δ; β; \text{MVar}; Fd; s''_t \approx_{\text{modIDs}} s''_t
  \]

  By induction on the relation \( \rightarrow^* \) in (ASSM2) and by using Lemma 52, we know (**):
  \[
  t \vdash_{\text{exec}} s''_t
  \]

  By induction on the relation \( \rightarrow^* \) in (ASSM1) and by using Lemma 56, we know (**):
  \[
  K_{\text{mod}}; K_{\text{fun}}; \text{mods} \times \text{mods} = Σ; Δ; β; \text{MVar}; Fd; s''_t \approx_{\text{modIDs}} s''_t
  \]

  Our goal is:
  \[
  \exists M'_d, stk', ddc', stk', pcc', nalloc'. \ s''_t \rightarrow (M_c, M'_d, stk', imp, φ, ddc', stk', pcc', nalloc') \wedge
  K_{\text{mod}}; K_{\text{fun}}; Σ; Δ; β; \text{MVar}; Fd; \langle \text{Mem}', stk', pc', Φ', \text{nalloc}' \rangle \overset{\text{modIDs}}{\approx} (M_c, M'_d, stk', imp, φ, ddc', stk', pcc', nalloc')
  \]

  We apply Lemma 97 obtaining the following subgoals:
Theorem 1

This concludes the proof of Lemma 99.

Proof. Similar to the proof of Lemma 99. Follows from Lemma 98, Lemma 52, and Lemma 56.

Lemma 100 (Source and compiled initial states are cross-language related).

∀ω ∈ N, m, m, s, i, ∆, Σ, β, K_mod, K_fun, MVar, Fd, modIDs, t, t′ s_i.
modIDs = {modID | (modID, _, _) ∈ m
K_mod; K_fun; m; ∆ + ω; β; MVar; Fd ⊢ s_i ∧
t′ = m
K_mod; K_fun; ∆ + ω; β; MVar; Fd ⊢ s_i −> K_mod; K_fun; ∆ + ω; β; MVar; Fd; s_i ≡ modIDs s_i

Proof.

By inverting assumption t ⊢ s_i using rule initial-state, and by instantiating Lemma 5 then inversion using rule exec-state together with assumption s_i.pcc ∈ dom(t′.M_c), we know (ASSM1):

∃mainMod. (t′.imp + ω)(mainMod) = (p, d, offs) ∧ main ∈ dom(offs) ∧ s_i.pcc = (k, p, e, offs(main)) ∧ s_i.ddc = d ∧ s_i.stc = s_i.mstc(mainMod) ∧ t′.φ(mainMod, main) = (n_args, n_local)

s_i.stc = (δ, _, _, n_args + n_local)
Also, by inverting assumption $K_{mod}; \llbracket \text{fun} \rrbracket; \llbracket \Sigma \rrbracket; \Delta + \omega; \beta; MVar; \llbracket \text{Fd} \rrbracket \vdash s_i$ using rule Initial-state-src, we know (ASSM2):

\[
s_i, \text{pc} = (\text{main}, 0) \land \\
\Phi = \{\text{moduleID}(\text{Fd}(\text{main})) \mapsto \text{frameSize}(\text{Fd}(\text{main})) \cup \bigcup_{\text{mid} \in \text{dom}(\Delta) \setminus \{\text{moduleID}(\text{Fd}(\text{main}))\}} \{\text{mid} \mapsto 0\}
\]

Furthermore, by Lemma 91, and by inversion of the assumption $K_{mod}; \llbracket \text{fun} \rrbracket; \llbracket \Sigma \rrbracket; \Delta + \omega; \beta; MVar; \llbracket \text{Fd} \rrbracket \vdash s_i$ using rules Initial-state-src then Well-formed program and parameters then Well-formed program, we know mainMod of (*) is unique.

Our goal (by unfolding Definition 64) consists of the following subgoals:

- $s_i, \text{nalloc} = s_i, \text{nalloc}$
  From the assumptions and by inverting rules initial-state and Initial-state-src, we know $s_i, \text{nalloc} = s_i, \text{nalloc} = -1$.

- $A_s = \text{reachable_addresses}(\Sigma, \Delta + \omega, \text{modIDs}, s_i., \text{Mem})$ and $A_t = \text{reachable_addresses}(\bigcup_{\text{mid} \in \text{modIDs}} \{(t'.\text{imp} + \omega)(\text{mid}).\text{ddc}, t'.\text{mstc}(\text{mid})\}, t'.\text{M}_d + \omega)$

  From the assumptions, and by inverting rules initial-state, and Initial-state-src, we get the following substitutions:

  $t'.\text{M}_d + \omega = \{a \mapsto 0 \mid a \in \text{dom}(t'.\text{M}_d + \omega)\}$, and $s_i., \text{Mem} = \{a \mapsto 0 \mid a \in \bigcup_{\text{mid} \in \text{modIDs}} \Delta(\text{mid})\}$

  Thus, by Lemma 10 and Lemma 61, we observe that (

  $A_s = \text{static_addresses}(\Sigma, \Delta + \omega, \text{modIDs}),$ and $A_t = \{a \mid a \in [c.\sigma, c.e] \land c \in \bigcup_{\text{mid} \in \text{modIDs}} \{(t'.\text{imp} + \omega)(\text{mid}).\text{ddc}, t'.\text{mstc}(\text{mid})\}\}$

  By Definition 46, we thus know (**):

  $A_s = \{a \mid a \in (\Delta + \omega)(\text{mid}) \land \text{mid} \in \text{modIDs}\} \uplus \{a \mid a \in \Sigma(\text{mid}) \land \text{mid} \in \text{modIDs}\}$

  The first conjunct of our goal is $A_s = A_t$.

  Substituting using (*) and (**), it suffices to show that:

  $\forall \text{mid} \in \text{modIDs}, (\Delta + \omega)(\text{mid}) = [(t'.\text{imp} + \omega)(\text{mid}).\text{ddc.}\sigma, (t'.\text{imp} + \omega)(\text{mid}).\text{ddc.}\epsilon) \land \Sigma(\text{mid}) = [t'.\text{mstc}(\text{mid}).\sigma, t'.\text{mstc}(\text{mid}).\epsilon]$,

  By applying Definitions 15 and 44, and using simple arithmetic, it suffices to show that:

  $\forall \text{mid} \in \text{modIDs}, \Delta(\text{mid}) = [t'.\text{imp}(\text{mid}).\text{ddc.}\sigma, t'.\text{imp}(\text{mid}).\text{ddc.}\epsilon) \land \Sigma(\text{mid}) = [t'.\text{mstc}(\text{mid}).\sigma, t'.\text{mstc}(\text{mid}).\epsilon]$

  This follows immediately by Lemma 91.

- $(\Delta + \omega)(\text{moduleID}(\text{Fd}(s_i, \text{pc}.\text{fid}))) = (s_i, \text{ddc.}\sigma, s_i, \text{ddc.}\epsilon)$

  By (ASSM1) and (ASSM2), it suffices to show that:

  $(\Delta + \omega)(\text{main}) = [(t'.\text{imp} + \omega)(\text{mainMod}).\text{ddc.}\sigma, (t'.\text{imp} + \omega)(\text{mainMod}).\text{ddc.}\epsilon)$

  Again, by applying Definitions 15 and 44, and using simple arithmetic, it suffices to show that:

  $\Delta(\text{main}) = [t'.\text{imp}(\text{mainMod}).\text{ddc.}\sigma, t'.\text{imp}(\text{mainMod}).\text{ddc.}\epsilon]$

  By the uniqueness of mainMod argued above, this goal is immediate by Lemma 91.

- $\Sigma(\text{moduleID}(\text{Fd}(s_i, \text{pc}.\text{fid}))) = (s_i, \text{stc.}\sigma, s_i, \text{stc.}\epsilon)$

  By (ASSM1) and (ASSM2), and by rule initial-state giving $t'.\text{mstc} = s_i, \text{mstc}$, it suffices to show that:
\[\Sigma(\text{main}) = [t'.\text{mstc}(\text{mainMod}).\sigma, t'.\text{mstc}(\text{mainMod}).e]\]

By the uniqueness of \text{mainMod} argued above, this goal is immediate by Lemma 91.

- \(\Phi(\text{moduleID}(Fd(s_i,pc,fid))) = s_i.\text{stk}.\text{off}\)

  By (ASSM1) and (ASSM2), it suffices to show that:
  \[\text{frameSize}(Fd(\text{main})) = t'.\phi'(\text{mainMod},\text{main}).n\text{Args} + t'.\phi'(\text{mainMod},\text{main}).n\text{Local}\]

  By the definition of \text{frameSize}, it is equivalent to show that:
  \[\text{length}(\text{args}(Fd(\text{main}))) + \text{length}(\text{localIDs}(Fd(\text{main}))) = t'.\phi'(\text{mainMod},\text{main}).n\text{Args} + t'.\phi'(\text{mainMod},\text{main}).n\text{Local}\]

  By the uniqueness of \text{mainMod} argued above, this goal is immediate by Lemma 91.

- \(K_{\text{mod}}(\text{moduleID}(Fd(s_i,pc,fid))).1 + K_{\text{func}}(s_i,pc,fid).1 + s_i.\text{pcc}.\sigma + s_i.\text{pcc}.\text{off} \land K_{\text{mod}}(\text{moduleID}(Fd(s_i,pc,fid))) = [s_i.\text{pcc}.\sigma, s_i.\text{pcc}.e]\)

  By (ASSM1) and (ASSM2), it suffices to show that:
  \[K_{\text{mod}}(\text{moduleID}(Fd(\text{main}))).1 + K_{\text{func}}(\text{main}).1 + 0 = (t'.\text{imp} + \omega)(\text{mainMod}).\text{pcc}.\sigma + (t'.\text{imp} + \omega)(\text{mainMod}).\text{offs}(\text{main}) \land K_{\text{mod}}(\text{moduleID}(Fd(\text{main}))) = [t'.\text{imp} + \omega](\text{mainMod}).\text{pcc}.\sigma, (t'.\text{imp} + \omega)(\text{mainMod}).\text{pcc}.e]\]

  By Definition 15, it is equivalent to show that:
  \[K_{\text{mod}}(\text{moduleID}(Fd(\text{main}))).1 + \text{compilation} - \text{bounds} - \text{preserved} K_{\text{func}}(\text{main}).1 = t'.\text{imp}(\text{mainMod}).\text{pcc}.\sigma + t'.\text{imp}(\text{mainMod}).\text{offs}(\text{main}) \land K_{\text{mod}}(\text{moduleID}(Fd(\text{main}))) = [t'.\text{imp}(\text{mainMod}).\text{pcc}.\sigma, t'.\text{imp}(\text{mainMod}).\text{pcc}.e]\]

  By the uniqueness of \text{mainMod} argued above, this goal is immediate by Lemma 91.

- \(K_{\text{mod}}; K_{\text{func}}; \Sigma; \Delta + \omega; \beta; M\text{Var}; Fd; s_i.\text{stk} \triangleq_{\text{modIDs}} s_i.\text{stk}\)

  Here, by unfolding Definition 62, and choosing \(f = \emptyset\), we satisfy all the conjuncts of our goal because \(s_i.\text{stk} = \text{nil}\) and \(s_i.\text{stk} = \text{nil}\).

- \(K_{\text{mod}}; K_{\text{func}}; \Sigma; \Delta + \omega; \beta; M\text{Var}; Fd; s_i.\Phi \triangleq_{\text{modIDs}} s_i.\text{mstc}, s_i.\phi\)

  By unfolding Definition 63, it suffices to show:

  \[\forall \text{mid} \in \text{modIDs}. s_i.\Phi(\text{mid}) = s_i.\text{mstc}(\text{mid}).\text{off}\]

  Using the definition of \(s_i.\Phi\) given by (ASSM2), we distinguish two cases:

  * **Case mid = main:**
    In this case, our goal follows by (ASSM1), and the uniqueness of \text{mainMod} argued above together with Lemma 91.

  * **Case mid \neq main:**
    In this case, our goal is immediate by (ASSM1) and the precondition
    \[\forall \text{sc}. \text{sc} \in \text{range}(\text{mstc}) \setminus \{\text{stc}\} \implies \text{sc} = (\delta, \_ , \_ , 0)\]
    of rule \text{initial-state} which we get by inversion of our assumption \(t \vdash_1 s_i\).

  \[\forall \text{fid} \in \text{dom}(Fd), \text{mid}.\text{moduleID}(Fd(\text{fid})) = \text{mid} \implies (\text{frameSize}(Fd(\text{fid}))) + \Sigma(\text{mid}).1 + s_i.\Phi(\text{mid}) < \Sigma(\text{mid}).2 \iff s_i.\phi(\text{mid}.\text{fid}).1 + s_i.\phi(\text{mid}.\text{fid}).2 + s_i.\text{mstc}(\text{mid}).\sigma + s_i.\text{mstc}(\text{mid}).\text{off} < s_i.\text{mstc}(\text{mid}).e\]

  \[- \forall \text{fid} \in \text{dom}(Fd), \text{mid}.\text{moduleID}(Fd(\text{fid})) = \text{mid} \implies \text{length}(\text{args}(Fd(\text{fid}))) = s_i.\phi(\text{mid}.\text{fid}).1\]

  \[- \forall (\text{mid}.\text{fid}) \in \text{dom}(s_i.\phi), \text{fid} \in \text{dom}(Fd) \land \text{mid} = \text{moduleID}(Fd(\text{fid}))\]

  All of these three subgoals are immediate after substitution using Lemma 91.

This concludes the proof of Lemma 100. \(\square\)
**Definition 65** (Target empty context).

\[ \emptyset \overset{\text{def}}{=} \{ \}, \{ \}, \{ \}, \{ \}, \{ \} \]

**Lemma 101** (Target empty context is universally linkable).

\[ \forall t : \text{TragetSetup}. \emptyset \times t = [t] \]

**Proof.**
Immediate by Definition 65 and rule valid-linking. \(\square\)

**Definition 66** (Target whole-program convergence compatible with partial convergence).

\[ \omega, \nabla \vdash t \Downarrow \overset{\text{def}}{=} \omega, \nabla \vdash \emptyset \Downarrow [t] \]

**Definition 67** (Source empty context).

\[ \emptyset \overset{\text{def}}{=} \text{nil} \]

**Lemma 102** (Source empty context is universally linkable and universally order-preserving).

\[ \forall p : \text{Prog}. \text{wfp}(p) \implies \emptyset \times p = [p] \]

\[ \forall p, K_{mod}. \emptyset \triangleright K_{mod} p \]

\[ \forall p, \Delta. p \triangleright \emptyset \]

**Proof.**
Immediate by Definition 67 and (rule Valid-linking-src + definition 41). \(\square\)

**Definition 68** (Source whole-program convergence compatible with partial convergence).

\[ K_{mod}, K_{fun}, \Sigma, \Delta + \omega, \beta, \nabla \vdash m \Downarrow \overset{\text{def}}{=} K_{mod}, K_{fun}, \Sigma, \Delta + \omega, \beta, \nabla \vdash \emptyset \Downarrow [m] \Downarrow \]

**Lemma 103** (Cross-language relatedness implies equi-terminality).

\[ \forall K_{mod}, K_{fun}, \Sigma; \Delta; \beta; MVar; Fd, s_s, \overline{\text{mods}_1}, \overline{\text{mods}_2}, t_1, t_2, t, s_t. \]

\[ \begin{align*}
\overline{\text{mods}_1} &\|_{\Delta; \Sigma, \beta, K_{mod}, K_{fun}} = t_1 \land \\
K_{mod}; K_{fun}; \overline{\text{mods}_1} \times \overline{\text{mods}_2}; \Sigma; \Delta; \beta; MVar; Fd \vdash \text{exec} s_s \land \\
t &\overset{\text{exec}}{=} t_1 \times t_2 \land \\
t \vdash \text{exec} s_t \land \\
\text{modIDs} = \{ \text{modID} \mid (\text{modID}, \_ , \_) \in \overline{\text{mods}_1} \} \land \\
\text{moduleID}(Fd(s_s, pc.fid)) \in \text{modIDs} \land \\
K_{mod}; K_{fun}; \Sigma; \Delta; \beta; MVar; Fd; s_s \overset{\text{modIDs}}{=} s_t \\
\implies \\
\vdash t s_s \iff \vdash t s_t
\end{align*} \]

**Proof.**
We assume the antecedents.
• “$$\implies$$” direction:

We assume $$\vdash t \, s_t$$, and our goal by unfolding Definition 13 is to show that $$\mathcal{M}_c(s_t.pcc) = \text{Exit}$$.

Here, it suffices by assumption $$t = t_1 \times t_2$$ and rule valid-linking to show that:

$$t_1.\mathcal{M}_c(s_t.pcc) = \text{Exit}$$

assuming that:

$$s_t.pcc \in \text{dom}(t_1.\mathcal{M}_c)$$

The latter follows from the assumptions:

moduleID($$\text{Fd}(\text{pc.fid})$$) ∈ modIDs,

$$K_{\text{mod}}; K_{\text{fun}}; \text{mods}_1 \times \text{mods}_2; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd} \vdash_{\text{exec}} s_t$$, and

$$K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd}; s_t \cong_{\text{modIDs}} s_t$$ after unfolding Definitions 61 and 64.

For the former goal ($$t_1.\mathcal{M}_c(s_t.pcc) = \text{Exit}$$), we apply Lemma 93, to instead get the following three subgoals:

1. $$\exists \text{mods}, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}. \| \text{mods} \|_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}} = \langle t_1.\mathcal{M}_c, \vdash, \vdash, \vdash \rangle$$

   We choose $$\text{mods} = \text{mods}_1$$, and $$\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}$$ from our assumptions.

2. $$\exists \text{mid}, \text{fid}, n. s_t.pcc.\sigma + s_t.pcc.\text{off} = K_{\text{mod}}(\text{mid}).1 + K_{\text{fun}}(\text{fid}).1 + n$$

   which follows immediately by choosing $$\text{fid} = s_t.\text{pc.fid}, n = s_t.\text{pc.n}, \text{mid} = \text{moduleID}(s_t.\text{pc.fid})$$ from assumption $$K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd}; s_t \cong_{\text{modIDs}} s_t$$ after unfolding Definitions 61 and 64.

3. $$\langle \text{commands}(\text{Fd}(s_t.\text{pc.fid}))(s_t.\text{pc.n}) \rangle = \text{Exit}$$

   which is immediate by Definition 59 and by inverting assumption $$\vdash t \, s_t$$ using Terminal-state-src-exit.

This concludes the “$$\implies$$” direction.

• “$$\iff$$” direction:

Here, we assume $$\vdash t \, s_t$$, and our goal is to show $$\vdash t \, s_t$$.

(Similarly to the “$$\implies$$” direction, here we know $$s_t.pcc \in \text{dom}(t_1.\mathcal{M}_c)$$, and we know we have all the assumptions of Lemma 93.)

By inversion using rule Terminal-state-src-exit, our goal is to show that:

$$\text{commands}(\text{Fd}(s_t.\text{pc.fid}))(s_t.\text{pc.n}) = \text{Exit}$$

We assume for the sake of contradiction that ($$^*$$):

$$\text{commands}(\text{Fd}(s_t.\text{pc.fid}))(s_t.\text{pc.n}) \neq \text{Exit}$$

By Lemma 93 though, we know:

$$\langle \text{commands}(\text{Fd}(s_t.\text{pc.fid}))(s_t.\text{pc.n}) \rangle_{\text{Fd}, K_{\text{fun}}, s_t.\text{pc.fid} \text{moduleID}(\text{Fd}(s_t.\text{pc.fid})), \beta} = t_1.\mathcal{M}_c(\text{moduleID}(\text{Fd}(s_t.\text{pc.fid}))).1 + K_{\text{fun}}(s_t.\text{pc.fid}).1 + n$$

Equivalently, by assumptions

moduleID($$\text{Fd}(\text{pc.fid})$$) ∈ modIDs,

$$K_{\text{mod}}; K_{\text{fun}}; \text{mods}_1 \times \text{mods}_2; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd} \vdash_{\text{exec}} s_t$$, and

$$K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta; \beta; \text{MVar}; \text{Fd}; s_t \cong_{\text{modIDs}} s_t$$ after unfolding Definitions 61 and 64, we thus know:

$$\langle \text{commands}(\text{Fd}(s_t.\text{pc.fid}))(s_t.\text{pc.n}) \rangle_{\text{Fd}, K_{\text{fun}}, s_t.\text{pc.fid} \text{moduleID}(\text{Fd}(s_t.\text{pc.fid})), \beta} = t_1.\mathcal{M}_c(\text{pcc})$$

Equivalently, by assumption $$\vdash t \, s_t$$ after unfolding Definition 13, we thus know:

$$\langle \text{commands}(\text{Fd}(s_t.\text{pc.fid}))(s_t.\text{pc.n}) \rangle_{\text{Fd}, K_{\text{fun}}, s_t.\text{pc.fid} \text{moduleID}(\text{Fd}(s_t.\text{pc.fid})), \beta} = \text{Exit}$$

Thus, by inversion using Definition 59, we know:

$$\text{commands}(\text{Fd}(s_t.\text{pc.fid}))(s_t.\text{pc.n}) = \text{Exit}$$

This contradicts assumption ($$^*$$), so our goal is proved.

This concludes the proof of Lemma 103. □
3.2 Compositionality: linking-and-convergence-preserving homomorphism

Lemma 104 (Existence of an initial state is preserved and reflected by \( J \cdot K \))

\[
\forall \omega \in \mathbb{N}, \nabla < -1, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}, m, t, t'. \\
wfp_{\text{params}}(m, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}) \land t' = [m]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}} \land t = (t', M_c, t', M_d + \omega, t'.imp + \omega, t'.mstc, t'.\phi) \\
\implies (\exists s_i, MVar, Fd. K_{\text{mod}}; K_{\text{fun}}; m; \Sigma + \omega; \beta; MVar; Fd \vdash_{i} s_i) \\
\iff (\exists s_i, t \vdash_{i} s_i)
\]

Proof.

We assume the antecedents.

- “\( \implies \)” direction:

Here we have \( s_i, MVar, Fd \) with \( K_{\text{mod}}; K_{\text{fun}}; m; \Sigma + \omega; \beta; MVar; Fd \vdash_{i} s_i \).

By inversion using rules Initial-state-src and Exec-state-src, we obtain the following assumptions:

1. \( s_i.pc = (\text{main}, 0) \)
2. \( s_i.pc = (\text{funID}, _) \land \text{funID} \in \text{dom}(Fd) \)
3. \( wfp_{\text{params}}(m, \Delta + \omega, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}) \)

And our goal is to show \( \exists s_i, t \vdash_{i} s_i \).

We claim \( \exists \text{mainMod}. t.\text{imp}(\text{mainMod}) = (p, d, \text{offs}) \land \text{main} \in \text{dom}(\text{offs}) \).

This claim holds by assumptions 1 and 2 together with Lemma 91.

We pick:

\[
s_i = (t.M_c, t.M_d, \text{nil}, t.\text{imp}, t.\phi, t.\text{imp}(\text{mainMod}).\text{ddc}, t.\text{mstc}(\text{mainMod}), t.\text{imp}(\text{mainMod}).\text{pcc}, t.\text{mstc}, -1)
\]

Our goal using rules initial-state and exec-state consists of the following subgoals, all of which we prove below:

- \( s_i.pcc = (\kappa, _, _, _) \land s_i.dcc = (\delta, _, _, _) \land s_i.stc = (\delta, _, _, _) \land s_i.nalloc < 0 \).

This is immediate by Lemmas 91 and 92 which describe the range of \( t.\text{imp} \) (after unfolding Definition 15) and the range of \( t.\text{mstc} \).

- \( modIDs = \text{dom}(s_i.\text{imp}) = \text{dom}(s_i.\text{mstc}) = \text{dom}(t.\text{mstc}) \).

This is immediate by substitution then by Lemmas 91 and 92 which describe the domain of \( t.\text{imp} \) (after unfolding Definition 15).

- \( \forall mid \in modIDs. s_i.\text{mstc}(mid) = t.\text{mstc}(mid) \).

Immediate by substitution and the reflexivity of \( \equiv \).
∀sc ∈ range(s_i.mstc), c ∈ range(s_i.imp), sc = (δ, _, _, _) \land sc \cap c.2 = \emptyset:

The first conjunct is easy by Lemmas 91 and 92.

For the second conjunct, it is equivalent (after unfolding Definition 3, and unfolding the definition of s_i that we gave above) to show the following:

\[ \bigcup_{sc \in \text{range}(t'.mstc)} [sc.σ, sc.e) \cap c.2.σ, c.2.e) = \emptyset \]

By Definition 15, it is equivalent to show that:

\[ \bigcup_{sc \in \text{range}(t'.mstc)} [sc.σ, sc.e) \cap c.2.σ + c.2.e + \omega) = \emptyset \]

And by easy axions, it is equivalent to show that:

\[ \bigcup_{mid \in \text{dom}(t'.mstc)} [t'.mstc(mid).σ, t'.mstc(mid).e) \cap \bigcup_{mid \in \text{dom}(t'.imp)} [t'.imp(mid).2.σ + t'.imp(mid).2.e + \omega) = \emptyset \]

By Lemmas 91 and 92 (together with our assumption about t'), and by folding Definition 44, it is equivalent to show that:

\[ \bigcup_{mid \in \text{dom}(t'.mstc)} \Sigma(mid) \cap \bigcup_{mid \in \text{dom}(t'.imp)} (\Delta + \omega)(mid) = \emptyset \]

But by inverting assumption 3 using rule Well-formed program and parameters, we get the precondition (*)

\[ \bigcup(\Delta + \omega)(mid) \cap \bigcup \Sigma(mid) = \emptyset \]

(*) immediately satisfies our goal by Lemma 92 which describes dom(t'.mstc) and dom(t'.imp).

∀a, st, st ∈ range(s_i.mstc) \land a ∈ reachable_addresses\{st\}, s_i.M_d \implies a \geq s_i.nalloc:

Here, assuming the antecedents, by Lemma 10, we know a ∈ \{st.σ, st.e\}. And by Lemmas 91 and 92, we know a ∈ \bigcup \Sigma(mid).

And by condition \( (\bigcup \Delta(mid) \cup \bigcup \Sigma(mid)) \cap (-\infty, 0) = \emptyset \) which we get by inverting Module-list-translation then rule Well-formed program and parameters, we know \( a \geq 0 \).

Thus from \( 0 > s_i.nalloc \) which we proved above, we have our goal: \( a \geq s_i.nalloc \) by transitivity of ≥.

s_i.pcc \subseteq dom(s_i.M_c):

This holds by assumptions 1 and 2 together with Lemma 93.

∀a. s_i.M_d(a) = (k, σ, e, _) \implies [σ, e) \subseteq dom(s_i.M_c)

Vacuously true by noticing the definition of s_i.M_d.

∃mid \in modIDs. s_i.imp(mid) = (cc, dc, _, _) \land s_i.pcc \subseteq cc \land s_i.ddc = dc \land s_i.mstc(mid) = s_i.stc:

Pick mid = mainMod from above. Then this is immediate by the definition of s_i and reflexivity of =·

∀(cc, dc, _) ∈ range(s_i.imp). \( (cc = (k, σ, e, _) \land [σ, e) \subseteq dom(s_i.M_c)) \land (dc = (δ, σ, e, _) \land [σ, e) \subseteq dom(s_i.M_d)) \land \forall a \in reachable_addresses\{dc\}, M_d \implies a \geq s_i.nalloc

Fix arbitrary mid and (cc, dc, _) where s_i.imp(mid) = (cc, dc, _).

The first two conjuncts are immediate by Lemmas 91 to 93.

For the third conjunct, we fix arbitrary a ∈ reachable_addresses\{dc\}, M_d

Then by Lemma 10, we know a ∈ \{dc.σ, dc.e\}.

And by Lemmas 91 and 92, we know a ∈ \bigcup (\Delta + \omega)(mid).

And by condition \( (\bigcup (\Delta + \omega)(mid) \cup \bigcup \Sigma(mid)) \cap (-\infty, 0) = \emptyset \) which we get by inverting assumption 3 using rule Well-formed program and parameters, we know \( a \geq 0 \).

Thus from \( 0 > s_i.nalloc \) which we proved above, we have our goal: \( a \geq s_i.nalloc \) by transitivity of ≥.

∀_ ∈ elesa(s_i.stk), _:

Vacuously true because s_i.stk = nil.
This concludes the proof of the “\(\Rightarrow\)” direction.

• “\(\Leftarrow\)” direction:

Here we have \(s_i\) with \(t \vdash_i s_i\).

By inversion using rules initial-state and exec-state, we obtain the following assumptions:

1. \(\exists \text{mainMod}. \text{imp} (\text{mainMod}) = (p, d, offs) \land \text{main} \in \text{dom} (offs) \land \text{pcc} = (\kappa, p, \sigma, p, e, \text{offs} (\text{main})) \land \text{ddc} = d \land \text{stc} = \text{mstc} (\text{mainMod})\)

2. \(\forall sc \in \text{range} (s_i. \text{mstc}), c \in \text{range} (s_i. \text{imp}). \text{sc} = (\delta, _, _, _) \land \text{sc} \cap \{c\} = 0\)

And our goal is to show \(\exists s_i, \text{MVar}, Fd. K_{\text{mod}}; K_{\text{fun}}; \overline{m}; \Delta + \omega; \beta; \text{MVar}; Fd \vdash_i s_i\).

We pick \(s_i = \{a \mapsto 0 \mid a \in \bigcup_{\text{mid} \in \text{dom} (\Delta)} (\Delta + \omega)(\text{mid}) \cup \Sigma (\text{mid})\}, \text{nil}, (\text{main}, 0), \Phi, -1\)

where \(\Phi = \{\text{moduleID} (Fd (\text{main})) \mapsto \text{frameSize} (Fd (\text{main})) \cup \bigcup_{\text{mid} \in \text{dom} (\Delta) \setminus \{\text{moduleID} (Fd (\text{main}))\}} \{\text{mid} \mapsto 0\}\)

Our goal by inversion of rules Initial-state-src and Exec-state-src consists of the following subgoals, which we prove next (The preconditions of Initial-state-src are immediate by the definition of \(s_i\). The preconditions of Exec-state-src remain.):

- \(\text{wfp\_params}(\overline{m}, \Delta + \omega, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}})\)

Using rule Well-formed program and parameters, we need to prove the following subgoals:

* \(\forall \text{mid}, \text{mid}' \in \text{modIDs}. \text{mid} \neq \text{mid}' \implies (\Delta + \omega)(\text{mid}) \cap (\Delta + \omega)(\text{mid}') = 0\)

By unfolding the definition of \(\cap\) on intervals obtaining the characterizing inequalities, and by unfolding Definition 44, it is easy to show that it is equivalent to show that:

\(\forall \text{mid}, \text{mid}' \in \text{modIDs}. \text{mid} \neq \text{mid}' \implies (\Delta)(\text{mid}) \cap (\Delta)(\text{mid}') = 0\)

But the latter follows immediately from the assumption \(t' = [\overline{m}]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}}\) after inversion using rule Module-list-translation then rule Well-formed program and parameters.

* \(\bigcup_{\text{mid} \in \text{dom}(\Delta)} (\Delta + \omega)(\text{mid}) \cap \bigcup_{\text{mid} \in \text{dom}(\Delta)} \Sigma (\text{mid}) = 0\)

By Lemma 92 which describes \(\text{dom}(t'. \text{mstc})\) and \(\text{dom}(t'. \text{imp})\), together with the preconditions defining domains of \(\Sigma\) and \(\Delta\) which we get from the assumptions by inversion using rule Module-list-translation then rule Well-formed program and parameters, it is equivalent to show that:

\(\bigcup_{\text{mid} \in \text{dom}(t'. \text{mstc})} \Sigma (\text{mid}) \cap \bigcup_{\text{mid} \in \text{dom}(t'. \text{imp})} (\Delta + \omega)(\text{mid}) = 0\)

By Lemmas 91 and 92, and by unfolding Definition 44, it is equivalent to show that:

\(\bigcup_{\text{mid} \in \text{dom}(t'. \text{mstc})} [t'. \text{mstc}(\text{mid}).\sigma, t'. \text{mstc}(\text{mid}).\epsilon] \cap \bigcup_{\text{mid} \in \text{dom}(t'. \text{imp})} [t'. \text{imp}(\text{mid}).2.\sigma + \omega, t'. \text{imp}(\text{mid}).2.\epsilon + \omega] = 0\)

And by easy axioms about the domain and range of a function, it is equivalent to show that:

\(\bigcup_{\text{sc} \in \text{range}(t'. \text{mstc})} [\text{sc} \cdot \sigma, \text{sc} \cdot \epsilon] \cap \bigcup_{\text{c} \in \text{range}(t'. \text{imp})} [\text{c} \cdot 2.\sigma + \omega, \text{c} \cdot 2.\epsilon + \omega] = 0\)

By folding Definition 15, it is equivalent to show that:

\(\bigcup_{\text{sc} \in \text{range}(t'. \text{mstc})} [\text{sc} \cdot \sigma, \text{sc} \cdot \epsilon] \cap \bigcup_{\text{c} \in \text{range}(t'. \text{imp} + \omega)} [\text{c} \cdot 2.\sigma, \text{c} \cdot 2.\epsilon] = 0\)

But this is immediate by assumption 2.

* \((\bigcup_{\Delta + \omega}(\text{mid}) \cup \bigcup_{\Sigma (\text{mid})}) \cap (-\infty, 0) = 0\)

By assumption, \(\omega \geq 0\). Thus, this subgoal follows from the corresponding statement about \(\Delta\) which can be obtained from the assumption \(t' = [\overline{m}]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}}\) after inversion using rule Module-list-translation then rule Well-formed program and parameters.
\* ∀mid ∈ modIDs. \bigcup_{mVar(\mid mid)} β(vid, ‖, mid) = [0, (\Delta + \omega)(\mid mid).2 − (\Delta + \omega)(\mid mid).1)

By unfolding Definition 44, it is equivalent to show that:

∀mid ∈ modIDs. \bigcup_{mVar(\mid mid)} β(vid, ‖, mid) = [0, 2\Delta(\mid mid).2 + \omega − (\Delta(\mid mid).1 + \omega))

By simple arithmetic, it is equivalent to show:

∀mid ∈ modIDs. \bigcup_{mVar(\mid mid)} β(vid, ‖, mid) = [0, 2\Delta(\mid mid).2 - \Delta(\mid mid).1)

The latter is immediate from the assumption \( t' = \bigcap_{\mid mid} \bigcup_{\mid mid} (\Delta + \omega, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}) \) after inversion using rule Module-list-translation then rule Well-formed program and parameters.

The remaining subgoals are immediate from the assumption \( t' = \bigcap_{\mid mid} \bigcup_{\mid mid} (\Delta + \omega, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}) \) after inversion using rule Module-list-translation then rule Well-formed program and parameters.

- \( \text{modIDs} = \{\text{modID} \mid (\text{modID}, ‖, _) ∈ \overline{\text{m}}\} \land \text{funDefs} = \{\text{funDef} \mid \text{funDef} ∈ \text{funDefs} \land (\_, ‖, \text{modFunDefs}) ∈ \overline{\text{m}}\} \land \text{Fd} = \{\text{funID} ↦ \text{funDef} \mid \text{funDef} ∈ \text{funDefs} \land \text{funDef} = (\_, \text{funID}, ‖, \_, \_, \_, \_))\}

Nothing to prove.

- \( \text{dom}(K_{\text{mod}}) = \text{dom}(MVar) = \text{dom}(\Sigma) = \text{dom}(\Delta + \omega) = \text{modIDs} \)

After unfolding Definition 44, this subgoal is immediate from \( \text{wfp_params}(\overline{\text{m}}, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}) \)

which we get from the assumption \( t' = \bigcap_{\mid mid} \bigcup_{\mid mid} (\Delta + \omega, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}) \) after inversion using rule Module-list-translation.

- \( \text{MVar} = \{\text{modID} ↦ \text{varIDs} \mid (\text{modID}, \text{varIDs}, ‖, _) ∈ \overline{\text{m}}\}\)

Nothing to prove.

- \( s_i, \text{pc} = (\text{funID}, ‖, _) \land \text{funID} ∈ \text{dom}(\text{Fd}) \)

The first conjunct is immediate. The second conjunct follows from assumption 1 together with Lemma 92.

- \( \forall (\text{fid}, ‖, _) ∈ \text{elems}(s_i, \text{stk}). \text{fid} ∈ \text{dom}(\text{Fd}) \)

Vacuously true because \( s_i, \text{stk} = \text{nil} \) by construction.

- \( \text{static_addresses}(\Sigma, \Delta + \omega, \text{modIDs}) ⊆ \text{dom}(s_i, \text{Mem}) \)

By unfolding Definition 46, and by the choice of \( s_i, \text{Mem} \), it is immediate that\( \text{static_addresses}(\Sigma, \Delta + \omega, \text{modIDs}) = \text{dom}(s_i, \text{Mem}) \).

- \( \nabla < -1 \implies (s_i, \text{nalloc} > \nabla) \land \forall a ∈ \text{dom}(s_i, \text{Mem}). a > \nabla \land\)

\( \forall a, s, e, v. v ∈ \text{range}(s_i, \text{Mem}) \land v = (\delta, s, e, ‖) \land a ∈ [s, e) \implies a > \nabla) \)

Conjunct \( s_i, \text{nalloc} > \nabla \) is immediate by assumption \( \nabla < -1 \) and the choice \( s_i, \text{nalloc} = -1 \).

Conjunct \( \forall a ∈ \text{dom}(s_i, \text{Mem}). a > \nabla \) is immediate by the previously proved subgoal

\( (\bigcup(\Delta + \omega)(\text{mod}) \cup \bigcup(\Sigma)(\text{mod})) \cap (-\infty, 0) = \emptyset \)

and the definition of \( \text{dom}(s_i, \text{Mem}) \).

The last conjunct is vacuously true by noticing that \( \text{range}(s_i, \text{Mem}) = \{0\} \).

- \( \forall \text{mid} ∈ \text{modIDs}. \Sigma(\text{mid}).1 + s_i, \Phi(\text{mid}) ≤ \Sigma(\text{mid}).2 \)

Immediate by the interval type after noticing the definition of \( s_i, \Phi \) which ensures \( s_i, \Phi(\text{mid}) = 0 \).

- \( \forall \text{mid} ∈ \text{modIDs}. s_i, \Phi(\text{mid}) = ∑_{\text{fid} ∈ \{\text{fid} \mid \text{moduleID}(\text{Fd}(\text{fid})) = \text{mid}\}} \text{frameSize}(\text{Fd}(\text{fid})) \times \text{countIn}(\text{fid}, ‖, s_i, \text{stk}) + (s_i, \text{pc} = (\text{fid}, ‖) ? 1 : 0) \)

Here, first notice that the sub-term \( \text{countIn}(\text{fid}, ‖, s_i, \text{stk}) \) is always equal to 0 because \( s_i, \text{stk} = \text{nil} \) and \( \text{countIn}(\_, \_, \text{nil}) = 0 \).

Next, we distinguish two cases for \( \text{mid} \):
* Case \( \text{mid} = \text{moduleId}(Fd(\text{main})) \):
In this case, \( s_i.\Phi(\text{mid}) = \text{frameSize}(Fd(\text{main})) \).
The right-hand side evaluates also to a non-zero value that corresponds to:
\( \text{frameSize}(Fd(\text{main})) \)
due to the choice on the value of \( s_i.\text{pc} \).

* Case \( \text{mid} \neq \text{moduleId}(Fd(\text{main})) \):
In this case, the sub-term \( (s_i.\text{pc} = (\text{fid},_) ? 1 : 0) \) is 0 for all the summation terms.
Also, the \( \text{countIn}(\cdots) \) sub-term is 0 as explained above.
Thus in this case, both sides of the equality evaluate to 0: one side because \( s_i.\Phi(\text{mid}) = 0 \),
and the other as explained above.

- \( \text{stk} = \text{nil} \implies \text{pc.fid} = \text{main} \)
  Immediate by the choice of \( s_i.\text{pc} \) made above.
- \( s_i.\text{stk} \neq \text{nil} \implies s_i.\text{stk}(\theta).\text{fid} = \text{main} \)
  Vacuously true because \( s_i.\text{stk} = \text{nil} \).
- \( \forall \text{mid},a,\sigma,\epsilon,s_i.\text{Mem}(a) = (\delta,\sigma,\epsilon,_) \land (\sigma,\epsilon) \cap \Sigma(\text{mid}) \neq \emptyset \implies a \in \Sigma(\text{mid}) \)
  Vacuously true by choice of \( s_i.\text{Mem} \).
- \( s_i.\text{nalloc} < 0 \)
  Immediate by the choice \( s_i.\text{nalloc} = -1 \) made above.

This concludes the proof of the “\( \Leftarrow \)” direction.

This concludes the proof of Lemma 104.

Lemma 105 (Convergence is preserved and reflected by \( [\cdot] \)).
\[
\forall \omega \in \mathbb{N}, \nabla \in \mathbb{Z}^-, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}, m, t'.
  t' = [m]_{\Delta,\Sigma,\beta,K_{\text{mod}},K_{\text{fun}}} \implies
  (K_{\text{mod}}, K_{\text{fun}}, \Sigma, \Delta + \omega, \beta, \nabla \vdash m) \\
  \iff
  (\omega, \nabla, t' \Downarrow)
\]

Proof.

We assume \( t' = [m]_{\Delta,\Sigma,\beta,K_{\text{mod}},K_{\text{fun}}} \).

- We prove the “\( \implies \)” direction.
  Assume \( K_{\text{mod}}, K_{\text{fun}}, \Sigma, \Delta + \omega, \beta, \nabla \vdash m \).
  Thus, we have--by unfolding Definitions 43 and 68 and eliminating the tautologies resulting from Lemma 102 that:
\[
\exists s_t, \text{initial\_state}(m, \text{main\_module}(m)) \rightarrow^* s_t \land
\exists \text{MVar}, Fd. K_{\text{mod}}; K_{\text{fun}}; m; \Sigma; \Delta + \omega; \beta; \text{MVar}; Fd \vdash s_t
\]

Our goal (by unfolding Definitions 17 and 66 and eliminating the tautologies resulting from Lemma 101) is:
\[
\exists t. t = (t'.\text{M}_c, t'.\text{M}_d + \omega, t'.\text{imp} + \omega, t'.\text{mstc}, t'.\phi) \land
\exists s_i, t \vdash s_i \land
\forall s_i, t \vdash s_i \implies \exists s_t, s_t \rightarrow_{\text{V}} s_t \land \vdash_t s_t
\]
\section*{Subgoal} \( \exists t. t = (t'. M_c \lor t'. M_d + \omega; t'. \text{imp} + \omega; t'. \text{mstc}, t'. \phi) \): 

By the totality of the operator + \( \omega \) (Definitions 14 and 15), this subgoal is immediate.

\section*{Subgoal} \( \exists s_i. t \vdash s_i \): 

This follows immediately from Lemma 104.

\section*{Subgoal} \( \forall s_i. t \vdash s_i \implies \exists s_t. s_i \rightarrow^* s_t \land \vdash_t s_t \): 

Fix an arbitrary \( s_i \) and assume \( t \vdash s_i \).

From Proposition (1), we obtain \( s_i, \text{MVar}, \text{Fd} \) with:

\[ K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta + \omega; \beta; \text{MVar}; \text{Fd} \vdash_t s_i. \]

Thus, we can now conclude from Lemma 100 that (INIT-REL):

\[ K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta + \omega; \beta; \text{MVar}; \text{Fd}; s_i \equiv_{\text{modIDs}} s_t \]

with \( \text{modIDs} = \{ \text{modID} | (\text{modID}, _, _) \in \overline{\pi} \} \)

Now, again from Proposition (1), we obtain \( s_t \) with (SOURCE-STEPS):

\[ s_i \rightarrow^* \nu s_t. \]

For the first conjunct of our goal \( s_i \rightarrow^* s_t \), we apply Lemma 99.

The generated subgoals are provable by:

* (INIT-REL),
* (SOURCE-STEPS),
* obtaining the necessary source \( \vdash_{\text{exec}} \) statement through inversion of conjunct \( \vdash_t \) of Proposition (1) using rule Initial-state-src,
* obtaining the necessary target \( \vdash_{\text{exec}} \) statement through inversion of already proved statement \( t \vdash s_i \) using rule initial-state,
* choosing \( \text{mods}_1 = \overline{\pi} \),
* choosing \( \text{mods}_2 = \emptyset \) (Definition 67), and
* inversion of \( \vdash_{\text{exec}} \) (once before and once after using Lemma 56 to obtain the subgoals \( \text{moduleID}(\text{Fd}(s_i, \text{pc.fid})) \in \text{modIDs} \) and \( \text{moduleID}(\text{Fd}(s_t, \text{pc.fid})) \in \text{modIDs} \) respectively).

For the second conjunct of our goal \( (\vdash_t s_i) \), we apply Lemma 103.

The generated subgoals are provable by:

* (for subgoal \( K_{\text{mod}}; K_{\text{fun}}; \Sigma; \Delta + \omega; \beta; \text{MVar}; \text{Fd}; s_t \equiv_{\text{modIDs}} s_t \)) applying Lemma 99 which is possible as described above,
* choosing \( \text{mods}_1 = \overline{\pi} \),
* choosing \( \text{mods}_2 = \emptyset \) (Definition 67),
* (for subgoal \( t \vdash s_i \)) using Proposition (1),
* (for subgoal \( t \vdash_{\text{exec}} s_i \)) applying Lemma 52, and
* (for subgoal \( \vdash_{\text{exec}} s_i \)) applying Lemma 56.

Using Lemma 103, we conclude from \( \vdash_t s_t \) of Proposition (1) that \( \vdash_t s_t \) which satisfies our subgoal.

This concludes the proof of conjunct \( \forall s_i. t \vdash s_i \implies \exists s_t. s_i \rightarrow^* s_t \land \vdash_t s_t \).

This concludes all subgoals of the “\( \implies \)” direction.

\section*{We prove the “\( \Leftarrow \)” direction.}

Assume \( \omega, \nu \vdash (\overline{\pi})_{\Delta, \Sigma, \beta; K_{\text{mod}}, K_{\text{fun}}} \).

Thus, we have—by unfolding Definitions 17 and 66 and eliminating the tautologies resulting from Lemma 101—that:

\begin{align*}
\exists t. t &= (t'. M_c \lor t'. M_d + \omega; t'. \text{imp} + \omega; t'. \text{mstc}, t'. \phi) \\
\exists s_i. t \vdash s_i \land \\
\forall s_i. t \vdash s_i \implies \exists s_t. s_i \rightarrow^* s_t \land \vdash_t s_t
\end{align*}

(2)
Our goal (by unfolding Definitions 43 and 68 and eliminating the tautologies resulting from Lemma 102) is:

\[ \exists s_t. \text{initial\_state}(m, \text{main\_module}(m)) \rightarrow^v s_t \land \exists \text{MVar, Fd. } K_{\text{mod}}; K_{\text{fun}}; m; \Sigma; \Delta + \omega; \beta; \text{MVar}; Fd \vdash i s_t \]

- **Subgoal** \( \exists s_t. \text{MVar, Fd. } K_{\text{mod}}; K_{\text{fun}}; m; \Sigma; \Delta + \omega; \beta; \text{MVar}; Fd \vdash i s_t \)

Here, we apply Lemma 104.

The generated subgoals are proved using:

* Proposition (2),
* assumption \( t' = [m; \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}] \), and
* (for subgoal \( \text{wfp\_params}(m, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}) \)) inversion of assumption \( t' = [m; \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}] \) using rule Module-list-translation.

- **Subgoal** \( \forall s_t. \text{MVar, Fd. } K_{\text{mod}}; K_{\text{fun}}; m; \Sigma; \Delta + \omega; \beta; \text{MVar}; Fd \vdash i s_t \)

We fix arbitrary \( s_t, \text{MVar, Fd} \) and assume \( K_{\text{mod}}; K_{\text{fun}}; m; \Sigma; \Delta + \omega; \beta; \text{MVar}; Fd \vdash i s_t \).

From Proposition (2), we obtain \( s_t \) with \( t \vdash i s_t \).
This enables us to use Lemma 100 to conclude that (INIT-RELATED):
\( K_{\text{mod}}; K_{\text{fun}}; m; \Sigma; \Delta + \omega; \beta; \text{MVar}; Fd; s_t \cong_{\text{mod}\_IDs} s_t \)

Thus, instantiate Theorem 1 to obtain:
\( \exists s_t. s_t \rightarrow^v s_t \land K_{\text{mod}}; K_{\text{fun}}; m; \Sigma; \Delta + \omega; \beta; \text{MVar}; Fd; s_t \cong_{\text{mod}\_IDs} s_t \)

Now, the remaining conjunct follows from Lemma 103 as in the proof of the "\( \Rightarrow \)" direction.

This concludes all the subgoals of the "\( \Leftarrow \)" direction.

\[ \square \]

One key property of many (compositional) compilers is that they are compatible with source and target linking. In particular, our compiler is a linking-preserving homomorphism (Lemma 106).

**Lemma 106** (Compilation preserves linkability and convergence, i.e., \([m]\) is a linking-preserving homomorphism and more).

\[ \omega, \nabla \vdash [C]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}} \downarrow [m] \downarrow [C[m]]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}} \Leftrightarrow \omega, \nabla \vdash [C[m]]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}} \downarrow [m] \downarrow [C]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}} \]

**Proof.**

We let

\[ C = [C]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}} \]

\[ t_1 = [m]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}} \]

\[ t'_e = [C[m]]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}} \]

- *We prove the "\( \Rightarrow \)" direction."

From the assumption and by unfolding Definition 17 of convergence, we have the following:

\[ \exists t'. C \times t_1 = [t'] \land \exists t. t = (t' \cdot M_c, t' \cdot M_d + \omega, t' \cdot \text{imp} + \omega, t' \cdot \text{mstc}, t' \cdot \phi) \land \exists s_t, t \vdash i s_t \land \forall s_t, t \vdash i s_t \iff \exists s_t, s_t \rightarrow^v s_t \land t \vdash i s_t \]

(3)
Our goal, by unfolding Definitions 17 and 66 and after substituting using Lemma 101 is thus:

\[ \exists c. \, t_c = (t'_c, M_c, t'_c, M_d + \omega, t'_c, \text{imp} + \omega, t'_c, \text{mstc}, t'_c, \phi) \land \]
\[ \exists s'_i. \, t_c \vdash s'_i \land \]
\[ \forall s'_i. \, t_c \vdash s'_i \implies \exists s'_i. \, s'_i \rightarrow \top \land s'_i \]

The first conjunct of our goal is always true (see Definition 14).

For the second conjunct, we pick \( s'_i = s_i \) from Proposition (3), and we also pick \( t_c = t \) from Proposition (3). This allows conjunct \( t_c \vdash i \, s_i \) and conjunct \( \forall s'_i. \, t_c \vdash i \, s'_i = \implies \exists s'_i. \, s'_i \rightarrow \top \land t \, s'_i \) of our goal to follow immediately from the corresponding ones of Proposition (3).

So, it remains to show that \( t = (t'_c, M_c, t'_c, M_d + \omega, t'_c, \text{imp}, t'_c, \text{mstc}, t'_c, \phi) \). Here are all the subgoals:

- **Subgoal** \( (C \bowtie t_1). M_c = t'_c, M_c \):
  
  From rule valid-linking, we know:
  
  \( (C \bowtie t_1). M_c = C, M_c \uplus t_1, M_c \)
  
  By Lemma 93, and Definition 32 of source linking, we conclude our subgoal.

- **Subgoal** \( (C \bowtie t_1). M_d + \omega = t'_c, M_d + \omega \):
  
  After unfolding Definition 14, it suffices to show that:
  
  \( (C \bowtie t_1). M_d = t'_c, M_d \)
  
  From rule valid-linking, we know:
  
  \( (C \bowtie t_1). M_d = C, M_d \uplus t_1, M_d \)
  
  Our subgoal then follows from Definition 32 of source linking and rules Module-list-translation and Module-translation.

- **Subgoal** \( (C \bowtie t_1). \text{imp} + \omega = t'_c, \text{imp} + \omega \):
  
  After unfolding Definition 15, it suffices to show that:
  
  \( (C \bowtie t_1). \text{imp} = t'_c, \text{imp} \)
  
  From rule valid-linking, we know:
  
  \( (C \bowtie t_1). \text{imp} = C, \text{imp} \uplus t_1, \text{imp} \)
  
  By Lemma 91, and Definition 32 of source linking, we conclude our subgoal.

- **Subgoal** \( (C \bowtie t_1). \text{mstc} = t'_c, \text{mstc} \):
  
  From rule valid-linking, we know:
  
  \( (C \bowtie t_1). \text{mstc} = C, \text{mstc} \uplus t_1, \text{mstc} \)
  
  By Lemma 91, and Definition 32 of source linking, we conclude our subgoal.

- **Subgoal** \( (C \bowtie t_1). \phi = t'_c, \phi \):
  
  From rule valid-linking, we know:
  
  \( (C \bowtie t_1). \phi = C, \phi \uplus t_1, \phi \)
  
  By Lemma 91, and Definition 32 of source linking, we conclude our subgoal.
This concludes the proof of the "\( \implies \)" direction.

- We prove the "\( \iff \)" direction.

From the assumption and by unfolding Definitions 17 and 66 of whole program convergence and partial convergence, we obtain:

\[
\exists c', t_c = (t_c', M_c, t_c', M_d + \omega, t_c', \text{imp} + \omega, t_c', \text{mstc}, t_c', \phi) \land \\
\exists s'_i, t_c \vdash_i 's'_i \land \\
\forall s'_i, t_c \vdash_i 's'_i \implies \exists s'_i, 's'_i \rightarrow \phi 's'_i \land \vdash_i s'_i
\]

(4)

Also, by unfolding Definition 42 of layout-ordered linking, we obtain:

\[
\exists \_ t_1 = |t'| \land \\
\exists t, t = (t', M_c, t', M_d + \omega, t', \text{imp} + \omega, t', \text{mstc}, t', \phi) \land \\
\exists s_i, t \vdash_i s_i \land \\
\forall s_i, t \vdash_i s_i \implies \exists s_i, s_i \rightarrow \phi s_i \land \vdash_i s_i
\]

(5)

Our goal, after unfolding Definition 17, is:

\[
\exists t', \_ t_1 = |t'| \land \\
\exists t, t = (t', M_c, t', M_d + \omega, t', \text{imp} + \omega, t', \text{mstc}, t', \phi) \land \\
\exists s_i, t \vdash_i s_i \land \\
\forall s_i, t \vdash_i s_i \implies \exists s_i, s_i \rightarrow \phi s_i \land \vdash_i s_i
\]

To prove the first conjunct, we pick \( t' = t_c' \), the latter we have from Proposition (4) and we hence verify that all the assumptions of rule valid-linking hold:

- **Subgoal disjointness**
  \( (t_c' = (C, M_c \cup t_1.M_c, C, M_d \cup t_1.M_d, C, \text{imp} \cup t_1.\text{imp}, C, \text{mstc} \cup t_1.\text{mstc}, C, \phi \cup t_1.\phi)) \):
  Here, we apply Lemmas 91 to 93 to both the left-and-right-hand-sides of our goal and thus, we are left with disjointness subgoals that are provable by inversion of rules Valid-linking-src and Well-formed program (both we get by first inverting rule Module-list-translation).

- **Subgoal order condition** \( \min(\text{dom}(C, M_d)) > \max(\text{dom}(t_1, M_d)) \):
  Follows from conjunct \( \_t_1 > \_s_1 \) (Definition 41) of Proposition (5) after applying Lemma 92.

- **Subgoal distinct function IDs**
  \( \text{funIDs} = \{ \text{fid} \mid \text{fid} \in \text{dom}(\text{offs}) \land (\_, \_, \text{offs}) \in \text{range}(C, \text{imp}) \cup \text{range}(t_1, \text{imp}) \} \land \text{all_distinct} \): 
  Follows from the corresponding condition after inverting rule Well-formed program which we get by first applying Lemmas 91 and 92 and then inverting rule Module-list-translation then inverting the precondition \( \text{wfp_params}(m, \_\_\_\_\_\_) \) using rule Well-formed program and parameters.

- **Subgoal disjointness of capabilities**
  \( \forall c_1 \in \text{range}(C, \text{imp}), c_2 \in \text{range}(t_1, \text{imp}), c_1 \cap c_2 = \emptyset \):
  Follows from the checks obtained by inverting rule Module-list-translation and inverting the precondition \( \text{wfp_params}(m, \_\_\_\_\_) \) using rule Well-formed program and parameters after first applying Lemmas 91 and 92.

- **Subgoal disjointness of capabilities**
  \( \forall c_1 \in \text{range}(C, \text{mstc}), c_2 \in \text{range}(t_1, \text{mstc}), c_1 \cap c_2 = \emptyset \):
  Follows from the checks obtained by inverting rule Module-list-translation and inverting the precondition \( \text{wfp_params}(m, \_\_\_\_\_) \) using rule Well-formed program and parameters after first applying Lemmas 91 and 92.

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The next three conjuncts of our goal thus follow immediately from the corresponding conjuncts of Proposition (4).

This concludes the proof of Lemma 106.

Lemma 107 (Compiler is a linking-preserving homomorphism).

\[
\begin{align*}
&\left[ [C]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}}, [m]_{\Delta, \Sigma, \beta, K_{\text{mod}, K_{\text{fun}, l}}} \right] = \\
&\left[ [C[m]_{\Delta^{u}, \Sigma^{u}, \Sigma^{u}, \beta^{u}, \beta_{1}, K_{\text{mod}}, K_{\text{fun}, l}, K_{\text{fun}, l}}] \right]
\end{align*}
\]

Proof.

Similar to Lemma 106.
4 A sound trace semantics for CHERIExpress

We give a sound and complete trace semantics for CHERIExpress. In this section, we prove soundness only (Lemma 110). Completeness, on the other hand, follows as an immediate corollary (Corollary 12) from results about the compiler of Section 3.

We first give the trace actions $\lambda \in \Lambda$:

\[
\begin{align*}
\lambda & ::= \checkmark \quad \text{termination marker} \\
& \quad | \tau \quad \text{silent internal action} \\
& \quad | \text{call}(\text{mid}, \text{fid})?M_d,n \quad \text{receive a call} \\
& \quad | \text{ret}?M_d,n \quad \text{receive a return} \\
& \quad | \text{call}(\text{mid}, \text{fid})!M_d,n \quad \text{issue a call} \\
& \quad | \text{ret}!M_d,n \quad \text{issue a return}
\end{align*}
\]

We next state useful definitions and lemmas about the trace semantics which we give in Figure 9 and about CHERIExpress and the compiler.

Trace prefixes $\alpha \in \Lambda^+$ are finite sequences of actions. They describe an abstraction of the behavior of the program as given by a finite sequence of its reduction steps. The emphasis that is made by the abstraction is on the so-called “boundary-crossing” actions. In the interesting case when the boundary is set to be “compiled part of the program” vs. “arbitrary CHERIExpress linked context”, the trace behavior of a program helps in reasoning about the boundary-crossing actions which turn out to be sufficient to capture the observable behavior of compiled programs.

The action $\checkmark$ indicates that execution has reached a terminal state. Silent actions $\tau$ are actions that do not change ownership of the program counter capability $pcc$. Ownership of $pcc$ is whether it points to an address in one partition of the code memory (out of two designated partitions). Actions that are marked with a ? indicate incoming function calls or returns (with respect to a designated partition of the program), and actions that are marked with a ! indicate on the other hand the outgoing-directed function calls or returns. In our proofs, the partition is such that the actions performed by the part of the program that is compiled with our compiler are distinguished from the actions that are performed by the CHERIExpress context that is linked with the compiled program.

An incoming call action $\text{call}(\text{mid}, \text{fid})?M_d,n$ records, as indicated by rule cinvoke-context-to-compiled in Figure 9 that a Cinvoke command has been executed, where the function $\text{fid}$ in module $\text{mid}$ is being called, and the projection $M_d$ of the data memory is the recording of the values in all the data memory locations that have in the past been shared between the two parts of the program. The number $n$ indicates the memory consumption of the program so far. The return action $\text{ret}?M_d,n$ also records the same about the data memory and the memory consumption. And outgoing call and return actions are analogous to incoming ones.

**Alternating traces**

Let $\? ::= \text{call}(\text{mid}, \text{fid})?M_d,n \mid \text{ret}?M_d,n$ and $\! ::= \text{call}(\text{mid}, \text{fid})!M_d,n \mid \text{ret}!M_d,n$. And let $\alpha|\tau \defeq \pi_{\Lambda\setminus\{\tau\}}(\alpha)$. And define the set $\Alt$ of finite alternating traces as follows:

**Definition 69** (Alternatingly-communicating finite traces). We define the set $\Alt$ of finite traces where communication is alternating as follows: $\Alt \defeq (\?|\epsilon) (\!|\epsilon) (\?|\epsilon)$

**Claim 5** (Extending an alternating prefix to keep it alternating).

1. $(\alpha\lambda \in \Alt \land \lambda \in ? \land \lambda' \in !) \implies \alpha\lambda\lambda' \in \Alt$

2. $(\alpha\lambda \in \Alt \land \lambda \in ! \land \lambda' \in ?) \implies \alpha\lambda\lambda' \in \Alt$
Figure 9: Trace semantics for **CHERIExpress** for an arbitrary compiled component \( \tau : \text{TargetSetup} \)

\[
\begin{align*}
\mathcal{M}_c(\text{pcc}) &= \text{Assign } \mathcal{E}_L, \mathcal{E}_R \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{Alloc } \mathcal{E}_\text{size} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{JumpIfZero } \mathcal{E}_{\text{cond}}, \mathcal{E}_{\text{off}} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{Invoke mid fid } \mathcal{\tau} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{Invoke mid fid } \mathcal{\tau} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{Invoke mid fid } \mathcal{\tau} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{Invoke mid fid } \mathcal{\tau} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{Invoke mid fid } \mathcal{\tau} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{Invoke mid fid } \mathcal{\tau} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{Invoke mid fid } \mathcal{\tau} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{Invoke mid fid } \mathcal{\tau} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{Invoke mid fid } \mathcal{\tau} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{Invoke mid fid } \mathcal{\tau} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \\
\mathcal{M}_c(\text{pcc}) &= \text{Invoke mid fid } \mathcal{\tau} \quad \langle \mathcal{M}_c, \mathcal{M}_d, \text{stk, imp, } \phi, \text{ddc, stc, pcc, mstc, nalloc} \rangle \rightarrow \mathcal{\langle v \rangle, \mathcal{s}'}, \mathcal{\zeta} \end{align*}
\]
Figure 9 (Cont.): Trace semantics for \textbf{CHERIExpress} for an arbitrary compiled component \( \tau : \text{TargetSetup} \)

\[
\begin{align*}
\text{(return-silent-context)} \quad & 
\mathcal{M}_c(pcc) = \text{Creturn} \quad s = (\mathcal{M}_c, \mathcal{M}_d, \text{stk}, \text{imp}, \phi, \text{ddc}, \text{stc}, \text{pcc}, \text{mstc}, \text{nalloc}) \\
& s' = (\mathcal{M}_c, \mathcal{M}_d', \text{stk}', \text{imp}, \phi, \text{ddc}', \text{stc}', \text{pcc}', \text{mstc}', \text{nalloc}) \quad \text{pcc} \nsubseteq \text{dom}(\mathcal{M}_c) \quad \text{pcc}' \nsubseteq \text{dom}(\mathcal{M}_c') \\
\xrightarrow{s \rightarrow \nabla} s' &= (\mathcal{M}_c[,], \mathcal{M}_d[,], \text{imp}[]) \quad \text{pcc} \nsubseteq \text{dom}(\mathcal{M}_c) \quad \text{pcc}' \nsubseteq \text{dom}(\mathcal{M}_c') \\
& s, \zeta \xrightarrow{\tau} [\text{[nalloc \ldots]}] \nabla s', \zeta' \\
\text{(return-to-compiled)} \quad & 
\mathcal{M}_c(pcc) = \text{Creturn} \quad s = (\mathcal{M}_c, \mathcal{M}_d, \text{stk}, \text{imp}, \phi, \text{ddc}, \text{stc}, \text{pcc}, \text{mstc}, \text{nalloc}) \\
& s' = (\mathcal{M}_c, \mathcal{M}_d', \text{stk}', \text{imp}, \phi, \text{ddc}', \text{stc}', \text{pcc}', \text{mstc}', \text{nalloc}) \quad \text{pcc} \nsubseteq \text{dom}(\mathcal{M}_c) \quad \text{pcc}' \nsubseteq \text{dom}(\mathcal{M}_c') \\
\xrightarrow{s \rightarrow \nabla} s' &= (\mathcal{M}_c[,], \mathcal{M}_d[,], \text{imp}[]) \quad \text{pcc} \nsubseteq \text{dom}(\mathcal{M}_c) \quad \text{pcc}' \nsubseteq \text{dom}(\mathcal{M}_c') \\
& s, \zeta \xrightarrow{\tau} [\text{[nalloc \ldots]}] \nabla s', \zeta' \\
\text{(terminate-checkmark)} \quad & 
\mathcal{M}_c(pcc) = \text{Creturn} \quad s = (\mathcal{M}_c, \mathcal{M}_d, \text{stk}, \text{imp}, \phi, \text{ddc}, \text{stc}, \text{pcc}, \text{mstc}, \text{nalloc}) \\
& s \rightarrow \nabla s' \quad s' = (\mathcal{M}_c, \mathcal{M}_d', \text{stk}', \text{imp}, \phi, \text{ddc}', \text{stc}', \text{pcc}', \text{mstc}', \text{nalloc}) \quad \text{pcc} \nsubseteq \text{dom}(\mathcal{M}_c) \quad \text{pcc}' \nsubseteq \text{dom}(\mathcal{M}_c') \\
\xrightarrow{s \rightarrow \nabla} s' &= \text{reachable_addresses_closure}(\zeta, \mathcal{M}_d') \\
& s, \zeta \xrightarrow{\tau} [\text{[nalloc \ldots]}] \nabla s', \zeta'
\end{align*}
\]

**Definition 70** (Reflexive transitive closure of trace actions).

We write \( s \xrightarrow{\alpha*} s' \) where \( \xrightarrow{\alpha*} \subseteq (\text{TargetState} \times 2^\lambda) \times \lambda \times (\text{TargetState} \times 2^\lambda) \) to denote the reflexive transitive closure of the trace actions reduction relation \( \xrightarrow{\alpha} \subseteq (\text{TargetState} \times 2^\lambda) \times \lambda \times (\text{TargetState} \times 2^\lambda) \) where \( \alpha \) collects the individual trace actions in succession.

\[
\begin{align*}
\text{(trace-closure-ref)} \quad & 
\quad s, \zeta \xrightarrow{\tau*} [\text{[c]}], \nabla s', \zeta' \\
\text{(trace-closure-trans)} \quad & 
\quad s, \zeta \xrightarrow{\alpha*} [\text{[c]}], \nabla s'', \zeta'' \quad s'', \zeta'' \xrightarrow{\lambda} [\text{[c]}], \nabla s', \zeta' \\
\end{align*}
\]
where \( \xrightarrow{\alpha*} \subseteq (\text{TargetState} \times 2^\lambda) \times \lambda \times (\text{TargetState} \times 2^\lambda) \) is as defined in Figure 9.

**Definition 71** (Non-silent trace steps).

We write \( s \xrightarrow{\alpha} s' \) where \( \xrightarrow{\alpha} \subseteq (\text{TargetState} \times 2^\lambda) \times \lambda \times (\text{TargetState} \times 2^\lambda) \) to denote that execution on state \( s \) generates a sequence \( \alpha \) of non-silent trace actions (i.e., excluding \( \tau \) actions) and reaches state \( s' \). We sometimes drop the parameter \( \nabla \) (which is the upper limit on memory allocation) for convenience.

\[
\begin{align*}
\text{(trace-steps-lambda)} \quad & 
\quad s, \zeta \xrightarrow{\tau*} [\text{[c]}], \nabla s'', \zeta'' \quad s'', \zeta'' \xrightarrow{\lambda} [\text{[c]}], \nabla s', \zeta' \\
\quad & \quad \lambda \neq \tau \\
\quad & \quad s, \zeta \xrightarrow{\lambda} [\text{[c]}], \nabla s', \zeta'
\end{align*}
\]
Claim 6 (A non-silent trace is not the empty string).
\[
\forall c, \alpha, s, \zeta, s', \zeta', \nabla.
\]
\[
s, \zeta \xrightarrow{\alpha} [c], \nabla s', \zeta' \\
\implies 
|\alpha| > 1
\]

Claim 7 (\(\rightarrow\) eliminates \(\tau\) actions).
\[
\forall c, \alpha, s, \zeta, s', \zeta', \nabla.
\]
\[
s, \zeta \xrightarrow{\alpha \lambda} [c], \nabla s', \zeta' \\
\implies 
\lambda \neq \tau
\]

Claim 8 (\(\rightarrow\) is supported by \(\leftarrow\)).
\[
\forall c, \alpha, \lambda, s, \zeta, s', \zeta', \nabla.
\]
\[
s, \zeta \xrightarrow{\alpha \lambda} [c], \nabla s', \zeta' \\
\implies 
\exists s'', \zeta''. \\
\]
\[
s'', \zeta'' \xrightarrow{\lambda} [c], \nabla s', \zeta' \land \\
s, \zeta \xrightarrow{\alpha \lambda} [c], \nabla s'', \zeta''
\]

Claim 9 (\(\rightarrow\) decomposes).
\[
\forall c, \alpha_1, \alpha_2, s, \zeta, s', \zeta', \nabla.
\]
\[
s, \zeta \xrightarrow{\alpha_1 \alpha_2} [c], \nabla s', \zeta' \\
\implies 
\exists s_1, \zeta_1. \\
\]
\[
s, \zeta \xrightarrow{\alpha_1} [c], \nabla s_1, \zeta_1 \land \\
s_1, \zeta_1 \xrightarrow{\alpha_2} [c], \nabla s', \zeta' \land
\]

Claim 10 (Non-silent part of \(\rightarrow^*\) is supported by \(\rightarrow\)).
\[
\forall c, \alpha, s, \zeta, s', \zeta', \nabla.
\]
\[
|\alpha| \geq 1 \land \\

s, \zeta \xrightarrow{\alpha^*} [c], \nabla s', \zeta' \\
\implies 
\exists s'', \zeta''. s, \zeta \xrightarrow{\alpha|\tau} [c], \nabla s'', \zeta''
For a target program $\tau$: $\text{TargetSetup}$, we define the set $TR(\tau) \subseteq \Lambda^+$ of finite non-empty prefixes of $\tau$’s possible execution traces as follows:

**Definition 72** (A prefix of an execution trace is possible for a component).
A finite prefix $\alpha$ belonging to a component $\tau$’s set $TR(\omega,\tau(\tau))$ of possible execution trace prefixes is defined as:

$$\alpha \in TR(\omega,\tau(\tau)) \iff \exists C, t': \text{TargetSetup}, s': \text{TargetState}, \zeta': 2^\mathbb{Z}.
C \bowtie \tau = \{t'\} \land
\text{initial}_\tau(t' + \omega, \text{main}_\tau(t')); 0 \xrightarrow{\cdot} s', \zeta'$$

where $\leadsto_{[\cdot],\N} \subseteq (\text{TargetState} \times 2^\mathbb{Z}) \times \mathbb{N} \times (\text{TargetState} \times 2^\mathbb{Z})$ is as defined in Definition 71.

**Definition 73** (Trace equivalence).

$$\tau_1 \overset{T=\omega,\N}{=\tau_2} \overset{\text{def}}{=} TR(\omega,\tau(\tau_1)) = TR(\omega,\tau(\tau_2))$$

**Claim 11** (Termination markers appear only at the end of an execution trace).

$$\forall\tau, \alpha \in TR(\tau) \implies \alpha \in (\Lambda \setminus \{\checkmark\})^* \setminus \alpha \in (\Lambda \setminus \{\checkmark\})^* \checkmark$$

**Claim 12** (Prefix-closure of trace set membership).

$$\forall\tau, \alpha \in TR(\tau) \implies (\forall \alpha'. \alpha = \alpha' \alpha'' \implies \alpha' \in TR(\tau))$$

**Proof.**
Follows from Claim 9. Instantiate “$\implies$” direction of Definition 72 using the assumption, and apply its “$\iff$” direction to the goal. □

**Claim 13** (A state that is reachable by $\rightarrow$ reduction or by $\triangleright_{\equiv}$ is also reachable by $\rightarrow$).

$$\forall\tau, t, s, s', \zeta, \nabla.
(s \rightarrow_\nabla s' \lor s \triangleright_{\equiv} s')
\implies
\exists \lambda, \zeta'. s, \zeta \xrightarrow{\cdot} [\tau], \N s', \zeta'$$

**Claim 14** (A non-$\bot$ state that is reachable by $\rightarrow$ is also reachable by $\rightarrow$ reduction).

$$\forall t, \tau, s, s', \zeta, \zeta'. s', \mathcal{M}_c(s', \text{pcc}) \neq \bot \land
s, \zeta \xrightarrow{\cdot} [\tau], \N s', \zeta'
\implies
s \rightarrow_\nabla s'$$

**Claim 15** (Silent trace steps correspond to $\rightarrow$ steps).

$$\forall \tau, s, s', \zeta, \zeta', \nabla.
\tau \xrightarrow{\cdot} [\tau], \N s', \zeta'
\implies
s \rightarrow_\nabla s'$$
Claim 16 (Non-stuck trace steps correspond to → execution steps).
\[
\forall c, s, s', s'', \zeta, \zeta', \zeta'', \nabla.
\]
\[
s, \zeta \xrightarrow{\alpha} \star [c] \nabla s', \zeta' \land
\]
\[
s', \zeta' \xrightarrow{\lambda} [\zeta] \nabla s'', \zeta''
\]
\[
\implies
\]
\[
s \rightarrow^* \nabla s'
\]

Claim 17 (The set of shared addresses \(\zeta\) does not change by silent trace steps).
\[
\forall s, s', \zeta, \zeta', \nabla.
\]
\[
s, \zeta \xrightarrow{\tau} \star [c] \nabla s', \zeta'
\]
\[
\implies
\]
\[
\zeta = \zeta'
\]

Corollary 5 (Reachability by →* implies reachability by →*).
\[
\forall t_1, t_2, \omega, \nabla, s.
\]
\[
\text{initial\_state}(t_1 \times t_2 + \omega, \text{main\_module}(t_1 \times t_2)) \rightarrow^* s
\]
\[
\implies
\]
\[
\exists \kappa, \alpha. \text{initial\_state}(t_1 \times t_2 + \omega, \text{main\_module}(t_1 \times t_2)), \emptyset \xrightarrow{\alpha} [t_2] \nabla s, \zeta
\]

Corollary 6 (Reachability by →* implies reachability by →* when the state is non-⊥).
\[
\forall t_1, t_2, \omega, \nabla, s, \zeta, \alpha.
\]
\[
\text{initial\_state}(t_1 \times t_2 + \omega, \text{main\_module}(t_1 \times t_2)), \emptyset \xrightarrow{\alpha} [t_2] \nabla s, \zeta \land
\]
\[
s.\mathcal{M}_c(s.pcc) \neq \bot
\]
\[
\implies
\]
\[
\text{initial\_state}(t_1 \times t_2 + \omega, \text{main\_module}(t_1 \times t_2)) \rightarrow^* s
\]

Lemma 108 (Non-communication actions do not change context/compiled component’s ownership of pcc).
\[
\forall \tau, t : \text{TargetSetup}, s, s'.
\]
\[
t \times \tau \vdash_{\text{exec}} s \land
\]
\[
s \xrightarrow{\tau} [\tau] s'
\]
\[
\implies
\]
\[
(s.pcc \subseteq \text{dom}(\tau.\mathcal{M}_c) \iff s'.pcc \subseteq \text{dom}(\tau.\mathcal{M}_c))
\]

Proof. Fix arbitrary, \(\tau, t, s,\) and \(s',\) and assume the antecedents.

• **Subgoal** \(s.pcc \subseteq \text{dom}(\tau.\mathcal{M}_c) \implies s'.pcc \subseteq \text{dom}(\tau.\mathcal{M}_c):\)

  Assume \(s.pcc \subseteq \text{dom}(\tau.\mathcal{M}_c):\)

  Our goal is:
  \(s'.pcc \subseteq \text{dom}(\tau.\mathcal{M}_c)\)

  Distinguish the following cases for assumption \(s \xrightarrow{\tau} [\tau] s'.\)
– Case **assign-silent**:  
Here, by inversion of the preconditions using rule **assign**, obtain:
\[ s.pcc \xrightarrow{=} s'.pcc \]  
Thus, our goal follows by substitution using assumption \( s.pcc \subseteq \text{dom}(\tau.M_c) \).

– Case **alloc-silent**:  
Here, by inversion of the preconditions using rule **allocate**, obtain:
\[ s.pcc \xrightarrow{=} s'.pcc \]  
Thus, our goal follows by substitution using assumption \( s.pcc \subseteq \text{dom}(\tau.M_c) \).

– Case **jump-silent**:  
Here, distinguish two cases for inversion of \( s \rightarrow s' \):

* Case **jump0**:  
Here, obtain \( s'.pcc = \text{inc}(s.pcc, \_\_). \)  
Thus, have:
\[ s.pcc \xrightarrow{=} s'.pcc \]  
Thus, our goal follows by substitution using assumption \( s.pcc \subseteq \text{dom}(\tau.M_c) \).

* Case **jump1**:  
Here, obtain \( s'.pcc = \text{inc}(s.pcc, 1) \).  
Thus, have:
\[ s.pcc \xrightarrow{=} s'.pcc \]  
Thus, our goal follows by substitution using assumption \( s.pcc \subseteq \text{dom}(\tau.M_c) \).

– Case **cinvoke-silent-compiled**:  
Here, obtain:
\[ s.M_c(s.pcc) = \text{Cinvoke} \text{ mid fid } \tau, \mid \text{mid } \in \text{dom}(\tau.\text{imp}) \mid, \]  
and \( s \rightarrow s' \)  
Thus, by inversion using **cinvoke** then **cinvoke-aux**, have (*):
\[ s'.pcc \xrightarrow{=} s.\text{imp(mid)}.pcc \]  
By inversion of lemma antecedents using **valid-linking** and **valid-program**, we know:
\[ \text{mid } \in \text{dom}(\tau.\text{imp}) \implies s.\text{imp(mid)}.pcc \subseteq \text{dom}(\tau.M_c) \] (applied Lemma 2)
Instantiating the latter using our assumptions, and substituting using (*), we have our goal.

– Case **cinvoke-silent-context**:  
Precondition \( s.pcc \notin \text{dom}(\tau.M_c) \) contradicts our assumption. So, any goal is provable.

– Case **creturn-silent-compiled**:  
Goal is immediate by the precondition of rule **creturn-silent-compiled**.

– Case **creturn-silent-context**:  
Precondition \( s.pcc \notin \text{dom}(\tau.M_c) \) of rule **creturn-silent-context** contradicts our assumption. So, any goal is provable.

• **Subgoal** \( s.pcc \subseteq \text{dom}(\tau.M_c) \xrightarrow{\iff} s'.pcc \subseteq \text{dom}(\tau.M_c) \):  
Assume \( s'.pcc \subseteq \text{dom}(\tau.M_c) \)  
Our goal is:
\[ s.pcc \subseteq \text{dom}(\tau.M_c) \]  
Distinguish the following cases for assumption \( s \xrightarrow{\tau} s' \):

– Case **assign-silent**,  
– Case **alloc-silent**, and
– **Case jump-silent:**

Similar to the corresponding cases of the previous subgoal: Goal follows by substitution using the assumption after obtaining \(s.pcc = s'.pcc\).

– **Case cinvoke-silent-compiled:**

Goal is immediate by preconditions of cinvoke-silent-compiled.

– **Case cinvoke-silent-context:**

Obtain a contradiction to assumption \(s'.pcc \subseteq \text{dom}(\tau.M_c)\) by proving:

\[s'.pcc \not\subseteq \text{dom}(\tau.M_c)\]

First, obtain:

\[s.M_c(s.pcc) = \text{Cinvoke mid fid } \tau, mid \not\subseteq \text{dom}(\tau.\text{imp}), \text{ and } s \rightarrow s'\]

Thus, by inversion using cinvoke then cinvoke-aux, have (*):

\[s'.pcc \doteq s.\text{imp}(mid).pcc\]

By inversion of lemma antecedents using valid-linking and valid-program, we know:

\[mid \not\subseteq \text{dom}(\tau.\text{imp}) \implies s.\text{imp}(mid).pcc \not\subseteq \text{dom}(\tau.M_c)\] (applied Lemma 2)

Instantiating the latter using our assumptions, and substituting using (*), we have our goal.

– **Case creturn-silent-compiled:**

Goal is immediate by the preconditions of creturn-silent-compiled.

– **Case creturn-silent-context:**

Precondition \(s'.pcc \not\subseteq \text{dom}(\tau.M_c)\), so any goal is provable.

This concludes the proof of Lemma 108.

**Corollary 7** (Non-communication actions do not change ownership of \(pcc\) (star-closure)).

\[
\forall \tau, t : \text{TargetSetup}, s, s'. \quad t \vdash \tau \vdash_{\text{exec}} s \land
s, \varsigma \xrightarrow{\tau^*} s', \varsigma
\quad \implies
\quad (s.pcc \subseteq \text{dom}(\tau.M_c) \iff s'.pcc \subseteq \text{dom}(\tau.M_c))
\]

**Proof.** Follows by Lemma 108, Claim 15 and corollary 2.

Then, Lemma 109 states a restriction on the form of traces with respect to input actions ? and output actions !.

**Lemma 109** (Traces consist of alternating input/output actions).

\[
\forall \tau, \alpha. \alpha \in TR(\tau) \implies \alpha \in \text{Alt}^*\]

**Proof.**

* Fix arbitrary \(\tau\) and \(\alpha\), and assume the antecedents.
• By unfolding the assumptions using Definition 72, we obtain (*):

\[ \exists C, t', t : \text{TargetSetup}, s, s' : \text{TargetState}, \varsigma : 2^\mathbb{Z}. \]

\[ C \land \tau = [t'] \land \]

\[ t = (t'.M_c, t'.M_d + \omega, t'.imp, t'.mstc, t'.\phi) \land \]

\[ t \vdash s \land \]

\[ s, \emptyset \xrightarrow[\alpha'] C \nu s', \varsigma' \]

• Our goal is:

\[ \alpha \in \text{Alt}^* \]

• By inversion of the last conjunct of (*), we distinguish the following cases:

  – Case **trace-steps-lambda**:
    
    Here, we know \( \lambda \neq \tau \).
    
    And our goal becomes:
    
    \[ \lambda \in \text{Alt}^* \]
    
    This follows by regular language identities after unfolding Definition 69.

  – Case **trace-steps-alternating**:
    
    Here, we know (**):
    
    \[ s, \varsigma \xrightarrow[\lambda] [c] \nu s'', \varsigma'', \]
    
    \[ s'', \varsigma'' \xrightarrow[\tau] [\varsigma] \nu s'', \varsigma'', \]
    
    \[ s'', \varsigma'' \xrightarrow[\lambda'] [\varsigma] \nu s', \varsigma', \]
    
    And by the induction hypothesis, we know (IH):
    
    \[ \alpha' \in \text{Alt}^* \]
    
    By instantiating Claim 6 using (**), we obtain (LAST-ACTION-OF-ALPHA'):
    
    \[ \alpha' = \alpha'' \lambda' \]
    
    We prove our goal (\( \alpha'' \lambda' \in \text{Alt}^* \)) by distinguishing the following cases for \( \lambda' \):

    * **Case \( \lambda' = \tau \)**:
      
      By contradiction with (**), any goal is provable.

    * **Case \( \lambda' = \checkmark \)**:
      
      Here, our goal is immediate by regular language identities.

    * **Case \( \lambda' \in ? \)**:
      
      By regular language identities applied to our goal, it suffices to prove:
      
      \[ \alpha'' \lambda' \in \text{Alt}^* \]
      
      By applying Claim 5, we obtain the following subgoals:
      
      • \( \alpha'' \lambda' \in \text{Alt}^* \)
        
        Immediate by (IH) after substitution using (LAST-ACTION-OF-ALPHA').
      
      • \( \lambda' \in ? \)
        
        Unfolding the case condition (\( \alpha \in ? \)), distinguish the following cases:

        1. **Case \( \lambda = \text{call}(\_,\_ \_?\ _,\_\_) \)**:
           
           Here, by inversion of (**), we know:
           
           \[ s'' \pcc \not\subseteq \text{dom}(\pi.M_c) \]
           
           By instantiating Corollary 7 using (**), we obtain (S''-PCC-OWNERSHIP):
           
           \[ s'' \pcc \not\subseteq \text{dom}(\pi.M_c) \]
           
           And by instantiating Claim 8 using (**), after substitution using (LAST-ACTION-OF-ALPHA'), we obtain:
By inversion of the latter statement, we get the following cases:

(a) Case \( \lambda' = \tau \): (short for the cases that produce \( \tau \))

By instantiation of Claim 7 using (**), we know: \( \lambda' \neq \tau \)

This contradicts the assumption \( \lambda' = \tau \). So, our goal is provable.

(b) Case terminate-checkmark:

Here, we know \( \lambda' = \checkmark \).

Thus, after instantiating Claim 11 using \( \alpha \), we conclude using regular language identities that \( \lambda = \checkmark \).

This contradicts our case assumption \( \lambda \in \_ \). So, any goal is provable.

(c) Case cinvoke-compiled-to-context, and

(d) Case creturn-to-context:

Here, our goal \( (\lambda' \in !) \) is immediate by the obtained preconditions.

(e) Case cinvoke-context-to-compiled:

Here, we know:

\[ \text{mid} \in \text{dom}(\tau.\text{imp}) \]

and by inversion of the preconditions using cinvoke-aux, we know:

\[ s''\text{.pcc} = \text{inc}(s''\text{.imp(mid)}.\text{pcc}_,_) \]

Thus, by inversion of (*) using valid-linking and valid-program, we know:

\[ s''\text{.pcc} \subseteq \text{dom}(\tau.\text{M}_c) \]

This contradicts \( (S'\text{-PCC-OWNERSHIP}) \). So, any goal is provable.

(f) Case creturn-to-compiled:

Here, we have:

\[ s''\text{.pcc} \subseteq \text{dom}(\tau.\text{M}_c) \]

This contradicts \( (S'\text{-PCC-OWNERSHIP}) \). So, any goal is provable.

2. Case \( \lambda = \text{ret}?! \):

Here, by inversion of (***) using creturn-to-compiled, we know:

\[ s''\text{.pcc} \not\subseteq \text{dom}(\tau.\text{M}_c) \]

The proof proceeds as in the previous case. We omit it for brevity.

* Case \( \lambda \in \_ \):

This is dual to case \( \lambda \in \_ \). We omit the proof for brevity.

- This concludes the proof of Lemma 109.

\[ \square \]

4.1 Soundness

Lemma 110 (Soundness of trace equivalence with respect to contextual equivalence).

\[ t_1 \equiv_{\omega,\nabla} t_2 \implies t_1 \simeq_{\omega,\nabla} t_2 \]

Proof.

Equivalently, we prove the contra-positive, i.e., assuming (*):

\[ t_1 \not\equiv_{\omega,\nabla} t_2 \]

Our goal is now:

\[ t_1 \not\equiv_{\omega,\nabla} t_2 \]
Using (*) and by unfolding it using Definition 18, we know (without loss of generality) that:

$$\exists C, \omega, \nabla \vdash C[t_1] \downarrow \land \omega, \nabla \not\vdash C[t_2] \downarrow$$

By further unfolding using Definition 17, we know (**):

$$\exists C, t_1'. C \times t_1 = [t_1'] \land$$

$$\exists s_1, \text{initial\_state}(t_1' + \omega, \text{main\_module}(t_1')) \to^* s_1 \land \vdash s_1$$

$$\land$$

$$\forall t_2', s, C \times t_2 = [t_2'] \implies$$

$$\text{initial\_state}(t_2' + \omega, \text{main\_module}(t_2')) \to^* s \implies \nabla s$$

By unfolding our goal using Definition 73, our goal becomes:

$$TR_{\omega, \nabla}(t_1) \neq TR_{\omega, \nabla}(t_2)$$

For this, it suffices to prove (without loss of generality) that:

$$\exists \alpha. \alpha \in TR_{\omega, \nabla}(t_1) \land \alpha \notin TR_{\omega, \nabla}(t_2)$$

By unfolding using Definition 72, our goal becomes:

$$\exists \alpha, C_1, t_1', s_1', \varsigma_1'.$$

$$C_1 \times t_1 = [t_1'] \land$$

$$\text{initial\_state}(t_1' + \omega, \text{main\_module}(t_1')), \emptyset \overset{\alpha}{\nabla}_{[t_1]}, \nabla s_1', \varsigma_1' \land$$

$$\forall C_2, t_2', C_2 \times t_2 = [t_2'] \implies$$

$$\nexists s_2', \varsigma_2'. \text{initial\_state}(t_2' + \omega, \text{main\_module}(t_2')), \emptyset \overset{\alpha}{\nabla}_{[t_2]}, \nabla s_2', \varsigma_2'$$

From (**), we obtain C, t_1', and s_1. And by instantiating the \implies direction of Corollary 5, we know (#1):

$$\exists \varsigma, \alpha. \text{initial\_state}(t_1' + \omega, \text{main\_module}(t_1')), \emptyset \overset{\alpha}{\nabla}_{[t_1]}, \nabla s_1, \varsigma$$

By obtaining \varsigma from (#1), and by using conjunct \vdash s_1 of (** to instantiate rule terminate-checkmark, we know (#2):

$$s_1, \varsigma \overset{\varsigma}{\nabla}_{[t_1]}, \nabla s_1, \varsigma$$

Using (#1) and (#2), we instantiate rule trace-closure-trans to obtain (t_1✓):

$$\text{initial\_state}(t_1' + \omega, \text{main\_module}(t_1')), \emptyset \overset{\alpha✓}{\nabla}_{[t_1]}, \nabla s_1, \varsigma$$

To prove our existential goal, we pick \alpha✓ | | for \alpha. We have to prove each of the following subgoals (conjuncts):
\begin{itemize}
  \item **Subgoal** \( \exists s', \zeta'. \ C \times t_1 = [t_1'] \land initial\_state(t_1' + \omega, main\_module(t_1')), \emptyset \xrightarrow{\alpha'[\tau]}_{[t_1]} \forall s', \zeta' \):

  Here, we apply Claim 6 obtaining the following subgoals:

  \begin{itemize}
    \item \( | \alpha'[\tau] | \geq 1 \):
      Immediate because \( \emptyset \neq \tau \).
    \item \( \exists s, \varsigma. \ initial\_state(t_1' + \omega, main\_module(t_1)), \emptyset \xrightarrow{\alpha'[\varsigma]}_{[t_1]} \forall s, \varsigma \)
      Immediate by \( (t_1\emptyset) \).
  \end{itemize}

  \item **Subgoal** \( \forall C_2, t_2. \ C_2 \times t_2 = [t_2'] \implies \#s, \zeta. \ initial\_state(t_2' + \omega, main\_module(t_2')), \emptyset \xrightarrow{\alpha'[\varsigma]}_{[t_2]} \forall s, \zeta \):

  Pick arbitrary \( C_2, t_2 \) with \( C_2 \times t_2 = [t_2'] \).

  Our goal is to show:

  \( \#s, \zeta. \ initial\_state(t_2' + \omega, main\_module(t_2')), \emptyset \xrightarrow{\alpha'[\varsigma]}_{[t_2]} \forall s, \zeta \)

  For the sake of contradiction, assume the contrary, i.e.:

  \begin{itemize}
    \item Assume \( \exists s', \zeta'. \ initial\_state(t_2' + \omega, main\_module(t_2')), \emptyset \xrightarrow{\alpha'[\varsigma']}_{[t_2]} \forall s', \zeta' \)
    \item By simplification of the restriction operator, we know:
      \( \exists s', \zeta'. \ initial\_state(t_2' + \omega, main\_module(t_2')), \emptyset \xrightarrow{\alpha'[\varsigma']}_{[t_2]} \forall s', \zeta' \)
    \item Thus, by instantiating Claim 8, we know (CONTR):
      \( \exists s', \zeta', s'', \zeta''. \ initial\_state(t_2' + \omega, main\_module(t_2')), \emptyset \xrightarrow{\alpha'[\varsigma']}_{[t_2]} \forall s', \zeta', s'', \zeta'' \)
    \item By inversion of the first conjunct of (CONTR) using terminate-checkmark, we know (TERMINAL):
      \( \vdash s'' \).
    \item Using the second conjunct of (CONTR), and (TERMINAL) unfolding Definition 13, we instantiate Corollary 6 obtaining:
      \( initial\_state(t_2' + \omega, main\_module(t_2')) \rightarrow^\ast \forall s'' \)
    \item The latter together with (TERMINAL) are immediately a contradiction to the second conjunct of (**).
  \end{itemize}

  This concludes the proof of Lemma 110.
\end{itemize}
5 A complete trace semantics for ImpMod

We give a sound and complete trace semantics for ImpMod. In this section, we prove completeness only (Lemma 113). Soundness, on the other hand, follows as an immediate corollary (Corollary 13) from results about the compiler of Section 3.

The syntax of the traces is exactly the same as in Section 4. Figure 10 describes the trace semantics of ImpMod.

Definition 74 (Reflexive transitive closure of trace actions).
We write $s \xrightarrow{\alpha, s} s'$ where $\rightarrow_{[p], \nabla} \subseteq (\text{SourceState} \times 2^Z) \times \Lambda \times (\text{SourceState} \times 2^Z)$ to denote the reflexive transitive closure of the trace actions reduction relation $\rightarrow_{[p], \nabla} \subseteq (\text{SourceState} \times 2^Z) \times \Lambda \times (\text{SourceState} \times 2^Z)$ where $\alpha$ collects the individual trace actions in succession.

Definition 75 (Non-silent trace steps).
We write $s \xrightarrow{\alpha, s} s'$ where $\rightarrow_{[p], \nabla} \subseteq (\text{SourceState} \times 2^Z) \times \Lambda \times (\text{SourceState} \times 2^Z)$ to denote that execution on state $s$ generates a sequence $\alpha$ of non-silent trace actions (i.e., excluding $\tau$ actions) and reaches state $s'$. We sometimes drop the parameter $\nabla$ (which is the upper limit on memory allocation) for convenience.
Figure 10: Trace semantics for ImpMod for an arbitrary program p

\[
\text{commands}(Fd(pc.fid))(pc.n) = \text{Assign} e_i e_r, \Sigma; \Delta; \beta; \text{MVar}; Fd \vdash (\text{Mem, stk, pc, } \Phi, \text{ nalloc}) \to \varsigma s' \\
\text{commands}(Fd(pc.fid))(pc.n) = \text{Alloc} e_i e_{\text{size}}, \Sigma; \Delta; \beta; \text{MVar}; Fd \vdash (\text{Mem, stk, pc, } \Phi, \text{ nalloc}) \to \varsigma s', \varsigma \\
\text{commands}(Fd(pc.fid))(pc.n) = \text{JumpIfZero} e_r e_{\text{off}}, \Sigma; \Delta; \beta; \text{MVar}; Fd \vdash (\text{Mem, stk, pc, } \Phi, \text{ nalloc}) \to \varsigma s', \varsigma \\
\text{commands}(Fd(pc.fid))(pc.n) = \text{Call} \text{fid}_{\text{call}} \overrightarrow{\Sigma} \Sigma; \Delta; \beta; \text{MVar}; Fd \vdash (\text{Mem, stk, pc, } \Phi, \text{ nalloc}) \to \varsigma s' \\
\text{commands}(Fd(pc.fid))(pc.n) = \text{Call} \text{fid}_{\text{call}} \overrightarrow{\Sigma} \Sigma; \Delta; \beta; \text{MVar}; Fd \vdash (\text{Mem, stk, pc, } \Phi, \text{ nalloc}) \to \varsigma s' \\
\text{commands}(Fd(pc.fid))(pc.n) = \text{Return} s = (\text{Mem, stk, pc, } \Phi, \text{ nalloc}) \\
\text{commands}(Fd(pc.fid))(pc.n) = \text{Return} s = (\text{Mem, stk, pc, } \Phi, \text{ nalloc}) \\
\text{commands}(Fd(pc.fid))(pc.n) = \text{Return} s = (\text{Mem, stk, pc, } \Phi, \text{ nalloc})
\]
Figure 10 (Cont.): Trace semantics for ImpMod for an arbitrary program p

\[
\text{commands}(Fd(pc.fid))(pc.n) = \text{Return} \quad s = (\text{Mem}, stk, pc, \Phi, \text{alloc}) \\
\Sigma; \Delta; MVar; Fd \vdash s \rightarrow s' \quad \text{moduleID}(Fd(pc.fid)) \notin \text{moduleIDs}(p) \\
\text{moduleID}(Fd(pc'.fid)) \notin \text{moduleIDs}(p)
\]

\[
\Sigma; \Delta; \beta; MVar; Fd \vdash s, \varsigma \xrightarrow{\tau} \Pi \varphi_{[p]} \varphi_{[p]} s', \varsigma
\]

\[
\text{commands}(Fd(pc.fid))(pc.n) = \text{Return} \quad s = (\text{Mem}, stk, pc, \Phi, \text{alloc}) \\
\Sigma; \Delta; MVar; Fd \vdash s \rightarrow s' \quad \text{moduleID}(Fd(pc.fid)) \notin \text{moduleIDs}(p) \\
\text{moduleID}(Fd(pc'.fid)) \in \text{moduleIDs}(p) \\
\varsigma' = \text{reachable_addresses_closure}(\varsigma, \text{Mem'})
\]

\[
\Sigma; \Delta; \beta; MVar; Fd \vdash s, \varsigma \xrightarrow{\text{return}_1\text{Mem'}|_{\varsigma',\text{alloc}}} \Pi \varphi_{[p]} \varphi_{[p]} s', \varsigma'
\]

\[
\text{commands}(Fd(pc.fid))(pc.n) = \text{Return} \quad s = (\text{Mem}, stk, pc, \Phi, \text{alloc}) \\
\Sigma; \Delta; \beta; MVar; Fd \vdash s \rightarrow s' \quad \text{moduleID}(Fd(pc.fid)) \notin \text{moduleIDs}(p) \\
\text{moduleID}(Fd(pc'.fid)) \notin \text{moduleIDs}(p) \\
\varsigma = \text{reachable_addresses_closure}(\varsigma, \text{Mem'})
\]

\[
\Sigma; \Delta; \beta; MVar; Fd \vdash s, \varsigma \xrightarrow{\text{return}_1\text{Mem'}|_{\varsigma',\text{alloc}}} \Pi \varphi_{[p]} \varphi_{[p]} s', \varsigma'
\]

\[
\Sigma; \Delta; \beta; MVar; Fd \vdash s \xrightarrow{\text{terminate-checkmark-src}} \Pi \varphi_{[p]} \varphi_{[p]} s, \varsigma
\]

Claim 20 (\(\rightarrow\) is supported by \(\rightarrow\)).

\[
\forall p, \alpha, \lambda, s, \varsigma, s', \varsigma', \nabla.
\]

\[
s, \varsigma \xrightarrow{\lambda} \Pi \varphi_{[p]} \varphi_{[p]} s', \varsigma' \\
\implies \\
\exists s'', \varsigma''.
\]

\[
s'', \varsigma'' \xrightarrow{\lambda} \Pi \varphi_{[p]} \varphi_{[p]} s', \varsigma' \land \\
\exists s, \varsigma.
\]

Claim 21 (\(\rightarrow\) decomposes).

\[
\forall p, \alpha_1, \alpha_2, s, \varsigma, s', \varsigma', \nabla.
\]

\[
s, \varsigma \xrightarrow{\alpha_1, \alpha_2} \Pi \varphi_{[p]} \varphi_{[p]} s', \varsigma' \\
\implies \\
\exists s_1, \varsigma_1.
\]

\[
s, \varsigma \xrightarrow{\alpha_1} \Pi \varphi_{[p]} \varphi_{[p]} s_1, \varsigma_1 \land \\
s_1, \varsigma_1 \xrightarrow{\alpha_2} \Pi \varphi_{[p]} \varphi_{[p]} s', \varsigma' \land \\
\]
Claim 22 (Non-silent part of $\rightarrow^*$ is supported by $\rightarrow$).

$$\forall p, \alpha, s, \varsigma, s', \varsigma', \nabla. \\
| \alpha|_{\nabla} | \geq 1 \land \\
s, \varsigma \xrightarrow{\rightarrow}|_p \nabla s', \varsigma' \\
\implies \\
\exists s'', \varsigma''. s, \varsigma \xrightarrow{\rightarrow}|_p \nabla s'', \varsigma''$$

For a program $p$, we define the set $TR(p) \subseteq \Lambda^+$ of finite non-empty prefixes of $p$'s possible execution traces as follows:

**Definition 76** (A prefix of an execution trace is possible for a component).

A finite prefix $\alpha$ belonging to a component $p$'s set $TR_{\nabla, \Delta, \Sigma, \beta}(p)$ of possible execution trace prefixes is defined as:

$$\alpha \in TR_{\nabla, \Delta, \Sigma, \beta}(p) \\
\iff \\
\exists C, m, s', \varsigma', \Delta C, \Sigma C, \beta C. \\
\Delta' = \Delta \cup \Delta C \land \Sigma' = \Sigma \cup \Sigma C \land \beta' = \beta \cup \beta C \land \\
C[p]_{\Delta', \Sigma'} = m \land \\
\Sigma'; \Delta' + \omega; \beta'; \text{mvar}(m); \text{fd\_map}(m) \vdash \text{initial\_state}(m, \Delta' + \omega, \Sigma', \text{main\_module}(m)), 0 \xrightarrow{\rightarrow}_m \nabla s', \varsigma'$$

where $\rightarrow|_p \nabla \subseteq (\text{SourceState} \times 2^Z) \times A \times (\text{SourceState} \times 2^Z)$ is as defined in Definition 75.

**Definition 77** (Trace equivalence).

$$\beta_1, p_1 \xrightarrow{T} \omega, \nabla, \Delta, \Sigma, \beta_2, p_2 \overset{\text{def}}{=} TR_{\omega, \nabla, \Delta, \Sigma, \beta_1}(p_1) = TR_{\omega, \nabla, \Delta, \Sigma, \beta_2}(p_2)$$

**Claim 23** (Termination markers appear only at the end of an execution trace).

$$\forall p. \alpha \in TR(p) \implies \alpha \in (\Lambda \setminus \{\checkmark\})^* \lor \alpha \in (\Lambda \setminus \{\checkmark\})^* \checkmark$$

**Claim 24** (Prefix-closure of trace set membership).

$$\forall p, \alpha. \alpha \in TR(p) \implies (\forall \alpha'. \alpha = \alpha'\alpha'' \implies \alpha' \in TR(p))$$

**Proof.**

Follows from Claim 21. Instantiate “$\implies$” direction of Definition 76 using the assumption, and apply its “$\iff$” direction to the goal. 

**Claim 25** (A state that is reachable by $\rightarrow$ reduction or by $\triangleright_{\rightarrow}$ is also reachable by $\rightarrow$).

$$\forall p, s, s', \varsigma, \nabla. \\
(s \rightarrow \nabla s' \lor s \triangleright_{\rightarrow} s') \\
\implies \\
\exists \lambda, \varsigma'. s, \varsigma \xrightarrow{\rightarrow}_m |_p \nabla s', \varsigma'$$
Claim 26 (A non-$\perp$ state that is reachable by $\rightarrow$ is also reachable by $\rightarrow\text{reduction}$).

\[ \forall t, p, s, s', \varsigma, \varsigma'. \]
\[ s'.pc \neq \perp \land \]
\[ s, \varsigma \xrightarrow{[p], \nabla} s', \varsigma' \]
\[ \implies \]
\[ s \rightarrow\nabla s' \]

Claim 27 (Silent trace steps correspond to $\rightarrow$ steps).

\[ \forall p, s, s', \varsigma, \varsigma', \nabla. \]
\[ s, \varsigma \xrightarrow{\ast} [p], \nabla s', \varsigma' \land \]
\[ s', \varsigma' \xrightarrow{[p], \nabla} s'', \varsigma'' \]
\[ \implies \]
\[ s \rightarrow^\ast s' \]

Claim 28 (Non-stuck trace steps correspond to $\rightarrow$ execution steps).

\[ \forall p, s, s', s'', \varsigma, \varsigma', \varsigma'', \nabla. \]
\[ s, \varsigma \xrightarrow{\ast} [p], \nabla s', \varsigma' \land \]
\[ s', \varsigma' \xrightarrow{[p], \nabla} s'', \varsigma'' \land \]
\[ s'' \xrightarrow{\ast} [p], \nabla s, \varsigma \]
\[ \implies \]
\[ s \rightarrow^\ast s' \]

Claim 29 (The set of shared addresses $\varsigma$ does not change by silent trace steps).

\[ \forall s, s', \varsigma, \varsigma', \nabla. \]
\[ s, \varsigma \xrightarrow{\ast} [p], \nabla s', \varsigma' \land \]
\[ \implies \]
\[ \varsigma = \varsigma' \]

Corollary 8 (Reachability by $\rightarrow^\ast$ implies reachability by $\rightarrow^\ast$).

\[ \text{initial\_state}(C \cup p, \Delta, \Sigma, \text{main\_module}(C \cup p)) \rightarrow\nabla s \]
\[ \implies \]
\[ \exists \varsigma, \alpha. \text{initial\_state}(C \cup p, \Delta, \Sigma, \text{main\_module}(C \cup p)), [p], \emptyset \xrightarrow{\ast} \nabla s, \varsigma \]

Corollary 9 (Reachability by $\rightarrow^\ast$ implies reachability by $\rightarrow^\ast$ when the state is non-$\perp$).

\[ \text{initial\_state}(C \cup p, \Delta, \Sigma, \text{main\_module}(C \cup p), [p], \emptyset \xrightarrow{\ast} \nabla s, \varsigma \land \]
\[ s.pc \neq \perp \land \]
\[ \implies \]
\[ \text{initial\_state}(C \cup p, \Delta, \Sigma, \text{main\_module}(C \cup p)) \rightarrow^\ast s \]

Lemma 111 (Non-communication actions do not change context/compiled component’s ownership of $pc$).

\[ K_{\text{mod}}; K_{\text{fun}}; C \cup p; \Sigma; \Delta; \beta; MVar; Fd \vdash_{\text{exec}} s \land \]
\[ s \xrightarrow{[p]} s' \]
\[ \implies \]
\[ \text{(moduleID}(Fd(s.pc.fid)) \in \text{moduleIDs}(p) \iff \text{moduleID}(Fd(s'.pc.fid)) \in \text{moduleIDs}(p)) \]
Proof. Similar to the proof of Lemma 108.

Corollary 10 (Non-communication actions do not change ownership of $pc$ (star-closure)).

$$K_{mod}; K_{fun}; C \uplus p; \Sigma; \Delta; \beta; MVar; Fd \vdash_{exec} s \land$$

$$s, \varsigma \lessdot_{[p]} s', \varsigma \implies$$

$$(\text{moduleID}(Fd(s.pc.fid)) \in \text{moduleIDs}(p) \iff \text{moduleID}(Fd(s'.pc.fid)) \in \text{moduleIDs}(p))$$

Proof. Follows by Lemma 111, Claim 27 and corollary 4.

Then, Lemma 112 states a restriction on the form of traces with respect to input actions $?$ and output actions $!$.

Lemma 112 (Traces consist of alternating input/output actions).

$$\forall p, \alpha. \alpha \in TR(p) \implies \alpha \in \text{Alt}^\ast$$

Proof.

Similar to the proof of Lemma 109.

5.1 Completeness using back-translation

Lemma 113 (Completeness of trace equivalence with respect to contextual equivalence).

$$\forall m_1, m_2, \Delta, \beta_1, \beta_2, \Sigma, \nabla.$$ 

$$\text{dom}(\Sigma) = \{\text{moduleID}(m) \mid m \in m_1\} = \{\text{moduleID}(m) \mid m \in m_2\} \land$$

$$\text{dom}(\Delta) = \{\text{moduleID}(m) \mid m \in m_1\} = \{\text{moduleID}(m) \mid m \in m_2\} \land$$

$$\Delta, \beta_1, m_1 \simeq \Sigma, \omega, \nabla, \Delta, \beta_2, m_2$$

$$\implies$$

$$\exists \Delta, \Sigma, \beta_1, m_1 \vdash_{\omega, \nabla, \Delta, \Sigma} \beta_2, m_2$$

(Proof Sketch):

The proof of this lemma is similar to the correctness of the back-translation given by Lemma 163, and additionally relies on Lemma 115.

We omit the details to avoid repetition. The crucial difference is that back-translation is defined for the common prefix of two traces as follows: Back-translation is a function (denoted by $\langle\langle\cdot, \cdot\rangle\rangle$) that takes as input two traces $\alpha_1, \alpha_2$ of respectively two programs, $c_1$ and $c_2$, and produces a source (partial) program $c$ which is a distinguishing context. A distinguishing context satisfies either:

- when $c$ is linked with $c_1$, it constitutes a converging program, and when it is linked with $c_2$, it constitutes a diverging program, or
- when $c$ is linked with $c_1$, it constitutes a diverging program, and when it is linked with $c_2$, it constitutes a converging program.
Definition 78 (Distinguishing snippet for equi-flow trace actions).

\[
distinguishArgs : \mathcal{E} \rightarrow \mathcal{V} \rightarrow \mathcal{V} \rightarrow \text{Cmd}
\]
\[
distinguishArgs(e, v_1, v_2) \triangleq
\]
\[
capType(v_1) \neq \capT(v_2) \implies \text{if} \not \text{zero-then-else} \left( e - \capT(v_1), \text{converge}, \text{diverge} \right)
\]
\[
capType(v_1) = \capT(v_2) = \text{INTEGER} \implies \text{if} \not \text{zero-then-else} \left( e - v_1, \text{converge}, \text{diverge} \right)
\]
\[
capStart(v_1) \neq \capStart(v_2) \implies \text{if} \not \text{zero-then-else} \left( \text{start}(e) - \capStart(v_1), \text{converge}, \text{diverge} \right)
\]
\[
capEnd(v_1) \neq \capEnd(v_2) \implies \text{if} \not \text{zero-then-else} \left( \text{end}(e) - \capEnd(v_1), \text{converge}, \text{diverge} \right)
\]
\[
capOff(v_1) \neq \capOff(v_2) \implies \text{if} \not \text{zero-then-else} \left( \text{off}(e) - \capOff(v_1), \text{converge}, \text{diverge} \right)
\]

Lemma 114 (Value cross-relatedness on integers is compatible with ImpMod subtraction).

\[
\forall v_1, v_2, v_1, v_2, s.
\]
\[
v_1 \simeq v_1 \land v_2 \simeq v_2 \land v_1 - v_2, \ldots, \ldots, \ldots, \ldots, \downarrow v_s \implies v_s = 0
\]

Proof. Follows from Definition 60 and rule Evaluate-expr-binop.

Lemma 115 (If two target values are unequal, then distinguishArgs produces code that terminates on exactly one of them).

\[
\forall \Sigma; \Delta; \beta; MVar; Fd, s, e, v_1, v_2.
\]
\[
\text{upcoming \ commands}(s, \text{distinguishArgs}(e, v_1, v_2)) \land
\]
\[
v_1 \neq v_2 \land
\]
\[
\exists v, e, \Sigma; \Delta; \beta; MVar; Fd, s.Mem, s.\Phi, s.pc \downarrow v
\]
\[
\implies
\]
\[
(v \simeq v_2 \implies \exists t, \Sigma; \Delta; \beta; MVar; Fd \vdash s \rightarrow^t s_t \land \vdash t s_t) \land
\]
\[
(v \simeq v_1 \implies \exists t, \Sigma; \Delta; \beta; MVar; Fd \vdash s \rightarrow^t s_t \land \vdash t s_t)
\]

Proof. Follows by easy case distinction after unfolding Definition 78 from Lemmas 114, 154, 156 and 157.
6 Security guarantee about the compiler: full abstraction

To be convinced about the security of the compiler, we need:

1. a property for a compiler that captures security (for that, we use Definition 79 of full abstraction of a compiler),

2. and a proof that our compiler satisfies this property (Theorem 2).

To express compiler security, one de-facto standard exists: compiler full abstraction [5]. Informally, a compiler is fully abstract if the compilation from source programs to target programs preserves and reflects contextual/behavioral equivalence. In other words, a compiler is fully-abstract if for any two source programs \( m_1 \) and \( m_2 \) and in any possible execution environment, we have that they are behaviorally equivalent (\( m_1 \simeq m_2 \)) if and only if their compiled counterparts are behaviorally equivalent (\( \llbracket m_1 \rrbracket \simeq \llbracket m_2 \rrbracket \)). The notion of behavioral equivalence used here is the canonical notion of contextual equivalence: two terms are equivalent if they behave the same when plugged into any valid context.

Source and target contextual equivalence can be stated as in Definitions 18 and 45.

We say a compiler \( J \cdot K \) is fully abstract if in all execution environments, it preserves and reflects contextual equivalence. An execution environment determines (1) the stack region \( \hat{\Sigma}(\text{moduleID}(m)) \) that is allocated for a module \( m \) of the compiled program together with (2) the start address \( \omega \) of the data segment of the compiled program, and (3) the limit \( \nabla \) on dynamic memory allocation. So, effectively, full abstraction requires that for any fixed \( \hat{\Sigma} \): (1) the stack size allocated to any of the program’s modules (i.e., whether sufficient or not), (2) the offset in memory in which a program’s data segment lives, and (3) the heap space available for dynamic allocation (i.e., whether sufficient or not), the compiler should preserve and reflect the contextual equivalence of the source language programs. Thus, full abstraction of a compiler \( \llbracket \cdot \rrbracket \) denoted \( \text{FA}(\llbracket \cdot \rrbracket) \) is defined as follows.

**Definition 79 (Compiler full abstraction).**

\[
\text{FA}(\llbracket \cdot \rrbracket) \overset{\text{def}}{=} \forall m_1, m_2, \hat{\Delta}, \beta_1, \beta_2, K_{\text{mod}1}, K_{\text{fun}1}, K_{\text{mod}2}, K_{\text{fun}2}, \hat{\Sigma}, \nabla < -1, t_1, t_2.
\]

\[
\text{dom}(\hat{\Sigma}) = \{\text{moduleID}(m) \mid m \in m_1\} \land \{\text{moduleID}(m) \mid m \in m_2\} \\
\text{dom}(\hat{\Delta}) = \{\text{moduleID}(m) \mid m \in m_1\} \land \{\text{moduleID}(m) \mid m \in m_2\} \\
m_2 \sim m_2 \\
\llbracket m_1 \rrbracket \hat{\Delta}, \hat{\Sigma}, \beta_1, K_{\text{mod}1}, K_{\text{fun}1} = t_1 \\
\llbracket m_2 \rrbracket \hat{\Delta}, \hat{\Sigma}, \beta_2, K_{\text{mod}2}, K_{\text{fun}2} = t_2 \\
\implies \\
\hat{\Delta}, \beta_1, m_1 \simeq_{\hat{\Sigma}, \omega, \nabla} \hat{\Delta}, \beta_2, m_2 \iff t_1 \simeq_{\omega, \nabla} t_2
\]

Compiler full abstraction can be stated as follows:

**Theorem 2** (\( \llbracket \cdot \rrbracket \) is fully abstract). \( \llbracket \cdot \rrbracket \in \text{FA} \) where \( \llbracket \cdot \rrbracket \) is our compiler that is defined in rule Module-list-translation.

**Proof.**
Immediate by Lemmas 116 and 117.

Referring to Definition 79 of a translation being fully abstract, we call the \( \implies \) direction of the logical equivalence “preservation of contextual equivalence” (Lemma 117), and the other direction \( \iff \) “reflection of contextual equivalence” (Lemma 116).

The proof of Lemma 116 is easy given the correctness and compositionality results we proved in Section 3.
Lemma 116 ([[ ]] reflects contextual equivalence).

\[ \forall m_1, m_2, \Delta, \beta_1, \beta_2, K_{mod1}, K_{fun1}, K_{mod2}, K_{fun2}, \Sigma, \omega, \nabla. \]
\[ \text{dom}(\Sigma) = \{\text{moduleID}(m) \mid m \in m_1\} = \{\text{moduleID}(m) \mid m \in m_2\} \land \]
\[ \text{dom}(\Delta) = \{\text{moduleID}(m) \mid m \in m_1\} = \{\text{moduleID}(m) \mid m \in m_2\} \land \]
\[ \exists t_1, [m_1]_{\Delta, \Sigma, \beta_1, K_{mod}, K_{fun}} = t_1 \land \]
\[ \exists t_2, [m_2]_{\Delta, \Sigma, \beta_2, K_{mod}, K_{fun}} = t_2 \]
\[ \implies (\Delta, \beta_1, m_1 \simeq_{\Sigma, \omega, \nabla} \Delta, \beta_2, m_2 \iff t_1 \simeq_{\omega, \nabla} t_2) \]

Proof.

We fix the universally-quantified variables, and assume the antecedents.

Then, in order to prove the implication:

\[ \Delta, \beta_1, m_1 \simeq_{\Sigma, \omega, \nabla} \Delta, \beta_2, m_2 \iff t_1 \simeq_{\omega, \nabla} t_2 \]

we instead prove its contra-positive. Thus, we assume:

\[ \Delta, \beta_1, m_1 \not\simeq_{\Sigma, \omega, \nabla} \Delta, \beta_2, m_2 \]

(6)

And our goal becomes:

\[ t_1 \not\simeq_{\omega, \nabla} t_2 \]

From Proposition (6), and by unfolding Definition 45, we get (w.l.o.g.):

\[ \exists \Delta, \Sigma, \beta, K_{mod}, K_{fun}, C. \]
\[ \text{wfp}(C) \land \]
\[ K_{mod} \cup K_{mod1}, K_{fun} \cup K_{fun1}, \Sigma \cup \Sigma, \Delta \cup \Delta, \omega \cup \omega, \beta \cup \beta_1, \nabla \vdash C[m_1] \downarrow \land \]
\[ K_{mod} \cup K_{mod2}, K_{fun} \cup K_{fun2}, \Sigma \cup \Sigma, \Delta \cup \Delta, \omega \cup \omega, \beta \cup \beta_2, \nabla \not\vdash C[m_2] \downarrow \]

(7)

and our goal (by unfolding Definition 18) is to show that:

\[ \exists C, \omega, \nabla \vdash C[t_1] \downarrow \land \omega, \nabla \not\vdash C[t_2] \downarrow \]

In order to show this goal, we pick:

\[ C = \llbracket C \rrbracket_{\Delta, \Sigma, \beta, K_{mod}, K_{fun}} \]

(8)

which we know from rule Module-list-translation that it exists because of conjunct wfp(C) of Proposition (7). By substitution from the assumptions and from Proposition (8), our goal is thus to show that:

\[ \omega, \nabla \vdash \llbracket C \rrbracket_{\Delta, \Sigma, \beta, K_{mod}, K_{fun}} [\llbracket m_1 \rrbracket_{\Delta, \Sigma, \beta_1, K_{mod1}, K_{fun1}}] \downarrow \land \omega, \nabla \not\vdash \llbracket C \rrbracket_{\Delta, \Sigma, \beta, K_{mod}, K_{fun}} [\llbracket m_2 \rrbracket_{\Delta, \Sigma, \beta_2, K_{mod2}, K_{fun2}}] \downarrow \]

By applying Lemma 106, it suffices to instead prove:

\[ \omega, \nabla \vdash \llbracket C[m_1] \rrbracket_{\Delta \cup \Delta, \Sigma \cup \Sigma, \beta \cup \beta_1, K_{mod} \cup K_{mod1}, K_{fun} \cup K_{fun}} \downarrow \land \]
\[ \omega, \nabla \not\vdash \llbracket C[m_2] \rrbracket_{\Delta \cup \Delta, \Sigma \cup \Sigma, \beta \cup \beta_2, K_{mod} \cup K_{mod2}, K_{fun} \cup K_{fun}} \downarrow \]

By Lemma 105, we immediately have the two conjuncts of our goal following from respectively the two conjuncts of Proposition (7). This concludes the proof of Lemma 116. \[ \square \]
Now, we turn to Lemma 117, which states that the compiler preserves contextual equivalence of ImpMod programs.

To prove this lemma, we rely on trace equivalence of CHERIExpress (Definition 73), and trace equivalence of ImpMod as a go-between. Thus, preservation of contextual equivalence follows immediately by the following three lemmas:

- Soundness of target trace equivalence (Lemma 110)
- Compilation preserves trace equivalence (Lemma 118)
- Completeness of source trace equivalence (Lemma 113)

**Lemma 117** ([\(\llbracket\cdot\rrbracket\) preserves contextual equivalence].)

\[
\forall m_1, m_2, \tilde{\Delta}, \beta_1, \beta_2, K_{mod1}, K_{fun1}, K_{mod2}, K_{fun2}, \tilde{\Sigma}, \omega \in \mathbb{N}, \nabla \in \mathbb{Z}^-. \\
dom(\tilde{\Sigma}) = \{\text{moduleID}(m) | m \in m_1\} = \{\text{moduleID}(m) | m \in m_2\} \land \\
dom(\tilde{\Delta}) = \{\text{moduleID}(m) | m \in m_1\} = \{\text{moduleID}(m) | m \in m_2\} \land \\
\exists t_1. \llbracket m_1 \rrbracket_{\Delta, \Sigma, \beta_1, K_{mod1}, K_{fun1}} = t_1 \land \\
\exists t_2. \llbracket m_2 \rrbracket_{\Delta, \Sigma, \beta_2, K_{mod2}, K_{fun2}} = t_2 \\
\implies (\Delta, \beta_1, m_1 \equiv_{\tilde{\Sigma}, \omega, \nabla} \Delta, \beta_2, m_2 \implies t_1 \equiv_{\omega, \nabla} t_2)
\]

**Proof.**
Immediate by Lemmas 110, 113 and 118.

**Lemma 118** (Compilation preserves trace equivalence).

\[
\beta_1, p_1 =_{\omega, \nabla, \Delta, \Sigma, \beta_2, p_2 \implies \llbracket p_1 \rrbracket_{\Delta, \Sigma, \beta_1, K_{mod1}, K_{fun1}} =_{\omega, \nabla} \llbracket p_2 \rrbracket_{\Delta, \Sigma, \beta_2, K_{mod2}, K_{fun2}}
\]

**Proof.**
Unfolding using Definitions 73 and 77, we need to prove:

\[
Tr_{\omega, \nabla, \Delta, \Sigma, \beta_1}(p_1) = Tr_{\omega, \nabla, \Delta, \Sigma, \beta_2}(p_2) \implies Tr_{\omega, \nabla}(\llbracket p_1 \rrbracket_{\Delta, \Sigma, \beta_1, K_{mod1}, K_{fun1}}) = Tr_{\omega, \nabla}(\llbracket p_2 \rrbracket_{\Delta, \Sigma, \beta_2, K_{mod2}, K_{fun2}})
\]

This is immediate by Lemmas 127 and 168.

**Lemma 127** follows by lifting compiler forward simulation to the trace semantics.
6.1 Lifting compiler forward and backward simulation to trace semantics

Lemma 119 (Forward simulation of call attempt).

\[∀K_{mod}, K_{fun}, Σ; Δ; β; MVar; Fd, \langle Mem, stk, pc, Φ, nalloc\rangle, modIDs, m, λ, ω, ρ\]
\[t, (M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc)\]
\[\langle M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc, \rangle \]
\[\Rightarrow (M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc, \rangle \]

Proof.

Similar to case Call of Lemma 97.

Lemma 120 (Forward simulation of call attempt).

\[∀K_{mod}, K_{fun}, Σ; Δ; β; MVar; Fd, \langle Mem, stk, pc, Φ, nalloc\rangle, modIDs, m, λ, ω, ρ\]
\[t, (M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc)\]
\[\langle M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc, \rangle \]
\[\Rightarrow (M_c, M_d, stk, imp, φ, ddc, stk, pcc, mstc, nalloc, \rangle \]

Proof.

Similar to case cinvoke of Lemma 98.
Lemma 121 (Compiler forward simulation lifted to a trace step).

\[
\forall K_{mod}, K_{fun}, \Delta; \beta; MVar; Fd; \langle Mem, stk, pc, \Phi, nalloc \rangle, \overline{m}, m, \lambda, \varsigma, \varsigma' \\
t, \langle M_c, M_d, stk, imp, \phi, ddc, stc, pcc, mstc, nalloc \rangle.
\]

\[
m \subseteq \overline{m} \Lambda \\
\frak{K}_{mod}; \frak{K}_{fun}; \Delta; \beta; MVar; Fd \vdash_{exec} \langle Mem, stk, pc, \Phi, nalloc \rangle \Lambda \\
t \vdash_{exec} \langle M_c, M_d, stk, imp, \phi, ddc, stc, pcc, mstc, nalloc \rangle \Lambda \\
modIDs = \{modID | (modID, _, _) \in \overline{m} \} \Lambda \\
K_{mod}; K_{fun}; \Delta; \beta; MVar; Fd; \langle Mem, stk, pc, \Phi, nalloc \rangle \equiv_{modIDs} \langle M_c, M_d, stk, imp, \phi, ddc, stc, pcc, mstc, nalloc \rangle \Lambda \\
\Sigma; \Delta; \beta; MVar; Fd \vdash \langle Mem, stk, pc, \Phi, nalloc \rangle \varsigma \xrightarrow{\Delta m} \langle Mem', stk', pc', \Phi', nalloc' \rangle, \varsigma' \Lambda \\
K_{mod}; K_{fun}; \Delta; \beta; MVar; Fd; \langle Mem', stk', pc', \Phi', nalloc' \rangle \equiv_{modIDs} \langle M_c, M_d, stk', imp, \phi, ddc', stc', pcc', mstc, nalloc' \rangle \Lambda
\]

Proof.
We distinguish two cases for \( \lambda \):

- **Case** \( \lambda = \tau \):

Here, after instantiating Claim 27 using the given trace step

\[
\Sigma; \Delta; \beta; MVar; Fd \vdash \langle Mem, stk, pc, \Phi, nalloc \rangle \varsigma \xrightarrow{\Delta m} \langle Mem', stk', pc', \Phi', nalloc' \rangle, \varsigma' \Lambda
\]

we obtain our goal immediately by applying Lemma 97.

- **Case** \( \lambda \neq \tau \):

Here, distinguish two cases:

  - **Case** \( s'.pc = \bot \):

Here, the goal is immediate by applying Lemma 119.

  - **Case** \( s'.pc \neq \bot \):

Here, after instantiating Claim 26,
we obtain our goal immediately again by applying Lemma 97.

This concludes the proof of Lemma 121. \(\square\)
Lemma 122 (Compiler backward simulation lifted to a trace step).

\[
\forall t, (M_c, M_d, stk, imp, \phi, ddc, stc, pcc, mstc, nalloc).
\]

Case 1: \(\lambda = \tau\):

Here, after instantiating Claim 15 using the given trace step

\[
(M_c, M_d, stk, imp, \phi, ddc, stc, pcc, mstc, nalloc), \lambda \vdash (M_c', M_d', stk', imp', \phi', ddc', stc', pcc', mstc', nalloc')
\]

we obtain our goal immediately by applying Lemma 98.

Case 2: \(\lambda \neq \tau\):

Here, distinguish two cases:

1. Case \(\lambda' \cdot M_c(s', pcc) = \bot\):

   Here, the goal is immediate by applying Lemma 120.

2. Case \(\lambda' \cdot M_c(s', pcc) \neq \bot\):

   Here, after instantiating Claim 14,

   we obtain our goal immediately again by applying Lemma 98.

This concludes the proof of Lemma 122.
Lemma 123 (Compiler forward simulation lifted to many trace steps).

∀K′ mod, K′ fun, Σ; Δ; β; MVar; Fd; \( \langle Mem, stk, pc, Φ, nalloc \rangle \), \( \overline{\text{mods}}_1 \), α, ζ, ζ′

\( t, (M_c, M_d, stk, imp, φ, ddc, stc, pcc, mstc, nalloc) \).

\( \overline{\text{mods}}_1 \subseteq \text{mods} \) ∧

\( \left[ \text{mods} \right]_{Δ, Σ, δ, K′ \text{mod}, K′ \text{fun}} = t \) ∧

\( K_{mod}; K_{fun}; \overline{\text{mods}}_1; Σ; Δ; β; MVar; Fd \vdash \text{exec} \langle Mem, stk, pc, Φ, nalloc \rangle \) ∧

\( t \vdash \text{exec} \langle M_c, M_d, stk, imp, φ, ddc, stc, pcc, mstc, nalloc \rangle \) ∧

\( \text{modIDs} = \{ \text{modID} \mid (\text{modID}, _, _) \in \overline{\text{mods}}_1 \} \) ∧

\( K_{mod}; K_{fun}; Σ; Δ; β; MVar; Fd; \langle Mem, stk, pc, Φ, nalloc \rangle \equiv_{\text{modIDs}} \langle M_c, M_d, stk, imp, φ, ddc', stc', pcc', nalloc' \rangle \) ∧

\( Σ; Δ; β; MVar; Fd \vdash \langle Mem', stk', pc', Φ', nalloc' \rangle \equiv_{\text{modIDs}} \langle M_c, M_d'; stk', imp, φ, ddc', stc', pcc', nalloc' \rangle \) ∧

\( \left[ \text{mods} \right]_{Δ, Σ, δ, K′ \text{mod}, K′ \text{fun}} \equiv_{\text{modIDs}} \langle M_c, M_d', stk', imp, φ, ddc', stc', pcc', nalloc' \rangle \), ζ' \)

\[ \implies \langle M_c, M_d, stk, imp, φ, ddc, stc, pcc, mstc, nalloc \rangle, ζ \xrightarrow{α \ast \left[ \text{mods} \right]_{Δ, Σ, δ, K′ \text{mod}, K′ \text{fun}}} \langle M_c, M_d', stk', imp, φ, ddc', stc', pcc', nalloc' \rangle, ζ' \]

\[ \implies \langle M_c, M_d', stk', imp, φ, ddc', stc', pcc', nalloc' \rangle \]

Proof.

Follows from Lemma 121:

In the inductive step (case \text{trace-closure-trans}),

the necessary assumptions about the source, and target execution invariants \( t \vdash \text{exec} \) and \( t \vdash \text{exec} \) follow from Corollary 4 and Corollary 2 respectively,

after instantiating Claim 16, and Claim 28.

Lemma 124 (Compiler backward simulation lifted to many trace steps).

∀K′ mod, K′ fun, Σ; Δ; β; MVar; Fd; \( \langle Mem, stk, pc, Φ, nalloc \rangle \), \( \overline{\text{mods}}_1 \), α, ζ, ζ'

\( t, (M_c, M_d, stk, imp, φ, ddc, stc, pcc, mstc, nalloc) \).

\( \overline{\text{mods}}_1 \subseteq \text{mods} \) ∧

\( \left[ \text{mods} \right]_{Δ, Σ, δ, K′ \text{mod}, K′ \text{fun}} = t \) ∧

\( K_{mod}; K_{fun}; \overline{\text{mods}}_1; Σ; Δ; β; MVar; Fd \vdash \text{exec} \langle Mem, stk, pc, Φ, nalloc \rangle \) ∧

\( t \vdash \text{exec} \langle M_c, M_d, stk, imp, φ, ddc, stc, pcc, mstc, nalloc \rangle \) ∧

\( \text{modIDs} = \{ \text{modID} \mid (\text{modID}, _, _) \in \overline{\text{mods}}_1 \} \) ∧

\( K_{mod}; K_{fun}; Σ; Δ; β; MVar; Fd; \langle Mem, stk, pc, Φ, nalloc \rangle \equiv_{\text{modIDs}} \langle M_c, M_d, stk, imp, φ, ddc', stc', pcc', nalloc' \rangle \) ∧

\( Σ; Δ; β; MVar; Fd \vdash \langle Mem', stk', pc', Φ', nalloc' \rangle \equiv_{\text{modIDs}} \langle M_c, M_d', stk', imp, φ, ddc', stc', pcc', nalloc' \rangle \) ∧

\( \langle M_c, M_d, stk, imp, φ, ddc, stc, pcc, mstc, nalloc \rangle, ζ \xrightarrow{α \ast \left[ \text{mods} \right]_{Δ, Σ, δ, K′ \text{mod}, K′ \text{fun}}} \langle M_c, M_d', stk', imp, φ, ddc', stc', pcc', nalloc' \rangle, ζ' \)

\[ \implies \langle M_c, M_d', stk', imp, φ, ddc', stc', pcc', nalloc' \rangle \]

\[ \implies \langle M_c, M_d', stk', imp, φ, ddc', stc', pcc', nalloc' \rangle \]

Proof.

Follows from Lemma 122:

In the inductive step (case \text{trace-closure-trans-src}),

the necessary assumptions about the source, and target execution invariants \( t \vdash \text{exec} \) and \( t \vdash \text{exec} \) follow from Corollary 4 and Corollary 2 respectively,

after instantiating Claim 16, and Claim 28.
Lemma 125 (Compiler forward simulation lifted to compressed trace steps).

\[
\forall K_{mod}, K_{fun}, \Sigma; \Delta; \beta; MVar; Fd, \langle Mem, stk, pc, \Phi, nalloc \rangle, \overline{\text{mods}_1}, m, \alpha, \varsigma, \varsigma' \\
\therefore (M_c, M_d, stk, imp, \phi, ddc, stkc, pcc, mstc, nalloc).
\]

\[
\text{modIDs} = \{ \text{modID} \mid (\text{modID}, \_ , \_ ) \in \text{mods}_1 \} \land \\
K_{mod}; K_{fun}; \overline{\text{mods}_1}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{exec} \langle Mem, stk, pc, \Phi, nalloc \rangle \land \\
t \vdash_{exec} \langle M_c, M_d, stk, imp, \phi, ddc, stkc, pcc, mstc, nalloc \rangle \land \\
\text{modIDs} = \{ \text{modID} \mid (\text{modID}, \_ , \_ ) \in \text{mods}_1 \} \land \\
K_{mod}; K_{fun}; \overline{\text{mods}_1}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{exec} \langle Mem, stk, pc, \Phi, nalloc \rangle \land \\
t \vdash_{exec} \langle M_c, M_d, stk, imp, \phi, ddc, stkc, pcc, mstc, nalloc \rangle \land \\
\text{modIDs} = \{ \text{modID} \mid (\text{modID}, \_ , \_ ) \in \text{mods}_1 \} \land \\
K_{mod}; K_{fun}; \overline{\text{mods}_1}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{exec} \langle Mem', stk', pc', \Phi', nalloc' \rangle \land \\
\Sigma; \Delta; \beta; MVar; Fd \vdash \langle Mem', stk', pc', \Phi', nalloc' \rangle, \varsigma \overset{\alpha [\overline{\text{mods}_1}]}{\Rightarrow} \langle Mem', stk', pc', \Phi', nalloc' \rangle, \varsigma'
\]

Proof.
Follows from Lemmas 121 and 123.

Lemma 126 (Compiler backward simulation lifted to compressed trace steps).

\[
\forall K_{mod}, K_{fun}, \Sigma; \Delta; \beta; MVar; Fd, \langle Mem, stk, pc, \Phi, nalloc \rangle, \overline{\text{mods}_1}, m, \alpha, \varsigma, \varsigma' \\
\therefore (M_c, M_d, stk, imp, \phi, ddc, stkc, pcc, mstc, nalloc).
\]

\[
\text{modIDs} = \{ \text{modID} \mid (\text{modID}, \_ , \_ ) \in \text{mods}_1 \} \land \\
K_{mod}; K_{fun}; \overline{\text{mods}_1}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{exec} \langle Mem, stk, pc, \Phi, nalloc \rangle \land \\
t \vdash_{exec} \langle M_c, M_d, stk, imp, \phi, ddc, stkc, pcc, mstc, nalloc \rangle \land \\
\text{modIDs} = \{ \text{modID} \mid (\text{modID}, \_ , \_ ) \in \text{mods}_1 \} \land \\
K_{mod}; K_{fun}; \overline{\text{mods}_1}; \Sigma; \Delta; \beta; MVar; Fd \vdash_{exec} \langle Mem, stk, pc, \Phi, nalloc \rangle \land \\
\Sigma; \Delta; \beta; MVar; Fd \vdash \langle Mem', stk', pc', \Phi', nalloc' \rangle, \varsigma \overset{\alpha [\overline{\text{mods}_1}]}{\Rightarrow} \langle Mem', stk', pc', \Phi', nalloc' \rangle, \varsigma'
\]

Follows from Lemmas 122 and 124.

Lemma 127 (No trace is removed by compilation).

\[
\alpha \in Tr_c, \forall, \Delta, \Sigma, \beta (p) \implies \alpha \in Tr_c, \forall (\overline{p})_{\Delta, \Sigma, \beta, K_{mod}, K_{fun}}
\]

Proof.
Immediate by Lemma 125 after unfolding Definitions 72 and 76.

6.2 Strong and weak similarity

Definition 80 (Component-controlled memory region).

In a given trace-execution state \( s, \varsigma \) of a program \( t \times \tau \) (i.e., \( t \times \tau \downarrow_{exec} s \)), we define the function
\( \rho_{\overline{c}} : (\text{TargetState} \times 2^{Z}) \rightarrow 2^{Z} \) which computes the set of memory addresses on which the similarity relation applies. For strong similarity, this set is all the memory that is reachable by \( \overline{c} \). For weak similarity, this set is only the set of addresses that are private to \( \overline{c} \).

\[
\rho_{\overline{c}}(s, \varsigma) \overset{\text{def}}{=} \begin{cases} \text{reachable_addresses}(\{s.\text{mstc}(mid), \overline{c}.\text{imp}(mid).\text{ddc}\}, s.\mathcal{M}) \\ \text{reachable_addresses}(\{s.\text{mstc}(mid), \overline{c}.\text{imp}(mid).\text{ddc}\}, s.\mathcal{M}) \setminus \varsigma \end{cases}
\]

Claim 30 (Controlled-region equality implies reachability equality).

\[
\forall \overline{c}, s_1, s_2, \varsigma_1, \varsigma_2.
\dom(s_1.\mathcal{M}) = \dom(s_2.\mathcal{M}) \land \\
\varsigma_1 = \varsigma_2 \land \\
s_1.\text{pcc} = s_2.\text{pcc} \land \\
\rho_{\overline{c}}(s_1, \varsigma_1) = \rho_{\overline{c}}(s_2, \varsigma_2) \Rightarrow \\
\text{reachable_addresses}(\{s_1.\text{stc}, s_1.\text{ddc}\}, s_1.\mathcal{M}) = \\
\text{reachable_addresses}(\{s_2.\text{stc}, s_2.\text{ddc}\}, s_2.\mathcal{M})
\]

Definition 81 (Similarity of stack capabilities). Two stack capability maps \( \text{mstc}_1 \) and \( \text{mstc}_2 \) are similar up to with respect to a component \( \overline{c} \) iff all the \( \overline{c} \) modules have the same stack capability value given by \( \text{mstc}_1 \) as that given by \( \text{mstc}_2 \). Formally:

\[
\text{mstc}_1 \approx_{\overline{c}} \text{mstc}_2 \overset{\text{def}}{=} \forall \text{mid}. \text{mid} \in \dom(\overline{c}.\text{imp}) \implies \text{mstc}_1(\text{mid}) = \text{mstc}_2(\text{mid})
\]

Claim 31 (Similarity of \( \text{mstc} \) is an equivalence relation).

Proof. Immediate by Definition 81.

6.3 Stack similarity (successor-preserving isomorphism)

Two stacks \( \text{stk}_1 \) and \( \text{stk}_2 \) (of two executions of a program \( \overline{c} \)) are related whenever the number of alternations of program frames and context frames is the same in \( \text{stk}_1 \) as in \( \text{stk}_2 \), and each two corresponding program stack frames (i.e., a program stack-frame from \( \text{stk}_1 \) that corresponds to one from \( \text{stk}_2 \)) are equal. The correspondence and the guarantee on the number of alternations are given by a function \( f \) between indexes of \( \text{stk}_1 \) and indexes \( \text{stk}_2 \). The function \( f \) satisfies the following conditions:

1. Domain of \( f \) is exhaustive of \( \overline{c} \) call sites in \( \text{stk}_1 \), and contains top and bottom sentinel values.
2. Range of \( f \) is exhaustive of \( \overline{c} \) call sites in \( \text{stk}_2 \) and contains top and bottom sentinel values.
3. \( f \) is sentinel-value preserving.
4. \( f \) is strictly monotone.
5. \( f \) is compatible with stack-frame equality (i.e., corresponding frames are equal).
6. \( f \) is a successor-preserving homomorphism.

A more formal definition is given by Definitions 82 and 83 which differ only in the condition on sentinel values. Weak stack-similarity (Definition 83) drops the top-sentinel-value requirement.

Conditions for strengthening and weakening are given next.
Definition 82 (Strong stack-similarity).

\[ stk_1 \approx_{[c]} stk_2 \]

\[ \exists f : Z \rightarrow Z, \]

\[ \text{dom}(f) = \{ i \in \text{dom}(stk_1) \mid stk_1(i).pcc \subseteq \text{dom}(\tau.M_c) \} \uplus \{-1, \text{length}(stk_1)\} \land \]

\[ \text{range}(f) = \{ i \in \text{dom}(stk_2) \mid stk_2(i).pcc \subseteq \text{dom}(\tau.M_c) \} \uplus \{-1, \text{length}(stk_2)\} \land \]

\[ f(-1) = -1 \land \]

\[ f(\text{length}(stk_1)) = \text{length}(stk_2) \land \]

\[ \forall i,j. i > j \implies f(i) > f(j) \land \]

\[ \forall i \in \text{dom}(f) \setminus \{-1, \text{length}(stk_1)\}. f(i) = j \implies stk_1(i) = stk_2(j) \land \]

\[ \forall i,j \in \text{dom}(f). j = i + 1 \iff f(j) = f(i) + 1 \]

Definition 83 (Weak stack-similarity).

\[ stk_1 \sim_{[c]} stk_2 \]

\[ \exists f : Z \rightarrow Z, \]

\[ \text{dom}(f) = \{ i \in \text{dom}(stk_1) \mid stk_1(i).pcc \subseteq \text{dom}(\tau.M_c) \} \uplus \{-1\} \land \]

\[ \text{range}(f) = \{ i \in \text{dom}(stk_2) \mid stk_2(i).pcc \subseteq \text{dom}(\tau.M_c) \} \uplus \{-1\} \land \]

\[ f(-1) = -1 \land \]

\[ \forall i,j. i > j \implies f(i) > f(j) \land \]

\[ \forall i \in \text{dom}(f) \setminus \{-1, \text{length}(stk_1)\}. f(i) = j \implies stk_1(i) = stk_2(j) \land \]

\[ \forall i,j \in \text{dom}(f). j = i + 1 \iff f(j) = f(i) + 1 \]

Notice that the functions \( f \) used in Definitions 82 and 83 are injective because they are strictly monotone.

Lemma 128 (A strictly-monotone function is injective).

\[ \forall f. \]

\[ (\forall i,j. i > j \implies f(i) > f(j)) \implies (\forall i,j. i \neq j \implies f(i) \neq f(j)) \]

Proof. Immediate by the anti-reflexivity and asymmetry of the \( < \) relation.

Definition 84 (Trace-state similarity).

Given two trace states \( s_1, s_1 \) and \( s_2, s_2 \), we define between them two similarity relations: strong similarity \( s_1, s_1 \approx_{[\tau]} s_2, s_2 \), and weak similarity \( s_1, s_1 \sim_{[\tau]} s_2, s_2 \) where both relations are parametrized with a component \( \tau \) for which the trace is collected. The intuition is that strong similarity holds as long as \( \tau \) is executing, and weak similarity holds as long as the context is executing. Strong similarity satisfies lock-step simulation, and weak similarity satisfies option simulation.
Formally:

\[ s_1, \varsigma_1 \approx[\rho] s_2, \varsigma_2 \equiv \]
\[ \rho[\rho](s_1, \zeta_1) = \rho[\rho](s_2, \zeta_2) = r \land \]
\[ s_1, \stk \approx[\rho] s_2, \stk \land \]
\[ s_1, \mstc \approx[\rho] s_2, \mstc \land \]
\[ \varsigma_1 = \varsigma_2 \land \]
\[ s_1, \Md|_r = s_2, \Md|_r \land \]
\[ s_1, \ddc = s_2, \ddc \land \]
\[ s_1, \stc = s_2, \stc \land \]
\[ s_1, \pcc = s_2, \pcc \land \]
\[ s_1, \nalloc = s_2, \nalloc \]

and

\[ s_1, \varsigma_1 \sim[\rho, \pr] s_2, \varsigma_2 \equiv \]
\[ (s_1, \pcc \cap \dom(\tau, \Md) = \emptyset) \iff \]
\[ s_2, \pcc \cap \dom(\tau, \Md) = \emptyset \land \]
\[ s_1, \stk \sim[\rho] s_2, \stk \land \]
\[ s_1, \mstc \approx[\rho] s_2, \mstc \land \]
\[ \varsigma_1 = \varsigma_2 \land \]
\[ s_1, \Md|_{\pr} = s_2, \Md|_{\pr} \]

**Lemma 129** (Strong stack-similarity is an equivalence relation).

• **Reflexivity:** \( \forall \stk, \tau. \, \stk \approx[\rho] \stk \)

• **Symmetry:** \( \forall \stk_1, \stk_2, \tau. \, \stk_1 \approx[\rho] \stk_2 \implies \stk_2 \approx[\rho] \stk_1 \)

• **Transitivity:** \( \forall \stk_1, \stk_2, \stk_3, \tau. \, \stk_1 \approx[\rho] \stk_2 \land \stk_2 \approx[\rho] \stk_3 \implies \stk_1 \approx[\rho] \stk_3 \)

**Proof.**

• For reflexivity, pick the identity function \( f(x) = x \).

• For symmetry, obtain \( f \) by unfolding the assumption using Definition 82.

Then, pick \( f^{-1} \) such that \( \dom(f^{-1}) := \range(f) \) and \( f^{-1}(f(x)) := x \).

By injectivity of \( f \) (Lemma 128), notice that \( f^{-1}(f(x)) \) is well defined, and that \( \range(f^{-1}) = \dom(f) \).

The “frame-relatedness” condition for \( f^{-1} \) follows by symmetry of the frame relation from the frame-relatedness condition on \( f \).

The remaining conditions are easy.

• For transitivity, obtain \( f_1 \) and \( f_2 \) by unfolding the assumption using Definition 82.

Then, pick \( f_{1,3} := f_2 \circ f_1 \). Notice that \( f_{1,3} \) has the desired domain and range.

The “frame-relatedness” condition for \( f_{1,3} \) follows by transitivity of the frame relation from the frame-relatedness conditions on \( f_1 \) and \( f_2 \).

The remaining conditions are easy.
Claim 32 (Weak stack-similarity is an equivalence relation).

- **Reflexivity**: \( \forall stk, \bar{c}. \ stk \sim_{[\bar{c}]} stk \)

- **Symmetry**: \( \forall stk_1, stk_2, \bar{c}. \ stk_1 \sim_{[\bar{c}]} stk_2 \implies stk_2 \sim_{[\bar{c}]} stk_1 \)

- **Transitivity**: \( \forall stk_1, stk_2, stk_3, \bar{c}. \ stk_1 \sim_{[\bar{c}]} stk_2 \land stk_2 \sim_{[\bar{c}]} stk_3 \implies stk_1 \sim_{[\bar{c}]} stk_3 \)

Proof. Similar to the proof of Lemma 129.

Claim 33 (State similarity is an equivalence relation).

The relation \( \approx_{[\bar{c}]} \) is reflexive, symmetric, and transitive.

- \( \forall s, \zeta, \bar{c}. \ s, \zeta \approx_{[\bar{c}]} s, \zeta \)

- \( \forall s_1, s_2, \zeta_1, \bar{c}. \ s_1, \zeta_1 \approx_{[\bar{c}]} s_2, \zeta_2 \implies s_2, \zeta_2 \approx_{[\bar{c}]} s_1, \zeta_1 \)

- \( \forall s_1, s_2, \zeta_1, \zeta_2, \zeta_3, \bar{c}. \ s_1, \zeta_1 \approx_{[\bar{c}]} s_2, \zeta_2 \land s_2, \zeta_2 \approx_{[\bar{c}]} s_3, \zeta_3 \implies s_1, \zeta_1 \approx_{[\bar{c}]} s_3, \zeta_3 \)

Proof. Follows from Claim 31 and Lemma 129.

Lemma 130 (Similarity of stack capabilities compatible with uniform substitution).

\( \forall mstc_1, mstc_2, mid, stc. \ mstc_1 \approx_{[\bar{c}]} mstc_2 \implies mstc_1[mid \mapsto \text{stc}] \approx_{[\bar{c}]} mstc_2[mid \mapsto \text{stc}] \)

Proof. Immediate by unfolding Definition 81, and a case distinction on the map’s key entry.

Lemma 131 (Initial states of the program of interest are strongly related).

\[
\begin{align*}
s_1 &= \text{initial}_\text{state}(C_1 \times p, \text{main}_\text{module}(C_1 \times p)) \land \\
 s_2 &= \text{initial}_\text{state}(C_2 \times p, \text{main}_\text{module}(C_2 \times p)) \land \\
 s_1.pcc &\subseteq \text{dom}(p.M_c) \land \\
 s_1.pcc &\subseteq \text{dom}(p.M_c) \\
 &\implies s_1, \emptyset \approx_{[p]} s_2, \emptyset
\end{align*}
\]

Proof.
Follows by Definition 84.

Lemma 132 (Initial states of the context are weakly related).

\[
\begin{align*}
s_1 &= \text{initial}_\text{state}(C_1 \times p, \text{main}_\text{module}(C_1 \times p)) \land \\
 s_2 &= \text{initial}_\text{state}(C_2 \times p, \text{main}_\text{module}(C_2 \times p)) \land \\
 s_1.pcc &\nsubseteq \text{dom}(p.M_c) \land \\
 s_1.pcc &\nsubseteq \text{dom}(p.M_c) \\
 &\implies s_1, \emptyset \sim_{[p]} \rho(p)[s_1, \emptyset] \ s_2, \emptyset
\end{align*}
\]

Proof.
Follows by Definition 84.
Lemma 133 (Equality of expression evaluation between strongly-similar states).

\[ \forall t_1, t_2, s_1, s_2, \varsigma_1, \varsigma_2, \mathcal{E}, r. \]
\[ t_1 \vdash_{\text{exec}} s_1 \land t_2 \vdash_{\text{exec}} s_2 \land \]
\[ r = \text{reachable\_addresses}\left(\{s_1.\text{stc}, s_1.\text{ddc}\}, s_1.\mathcal{M}_d\right) \land \]
\[ s_1.\text{stc} = s_2.\text{stc} \land \]
\[ s_1.\text{ddc} = s_2.\text{ddc} \land \]
\[ s_1.\mathcal{M}_d|_r = s_2.\mathcal{M}_d|_r \land \]
\[ \mathcal{E}, s_1.\mathcal{M}_d, s_1.\text{ddc}, s_1.\text{stc}, s_1.\text{pcc} \downarrow v \]
\[ \implies \]
\[ \mathcal{E}, s_2.\mathcal{M}_d, s_2.\text{ddc}, s_2.\text{stc}, s_2.\text{pcc} \downarrow v \]

Proof.
We assume the antecedents, and prove our goal by induction on the evaluation \( \mathcal{E}, s_1.\mathcal{M}_d, s_1.\text{ddc}, s_1.\text{stc}, s_1.\text{pcc} \downarrow v \):

1. Case evalconst:
   Here, observe that \( n, \_ , \_ , \_ , \_ \downarrow n \), so our goal follows.

2. Case evalddc:

3. Case evalstc:
   Here, we obtain our goals by conjuncts \( s_1.\text{ddc} = s_2.\text{ddc} \), and \( s_1.\text{stc} = s_2.\text{stc} \) of the antecedent respectively.

4. Case evalCapType:

5. Case evalCapStart:

6. Case evalCapEnd:

7. Case evalCapOff:

8. Case evalBinOp:

9. Case evalIncCap:

10. Case evalLim:
    Here, our goals follow by inverting the corresponding rule, applying the induction hypothesis, and re-applying the rule for the \( s_2 \) components.

11. Case evalDeref:
    - Here, we have \( \mathcal{E} = \text{deref}(\mathcal{E}') \), and we obtain the preconditions \( \mathcal{E}', s_1.\mathcal{M}_d, s_1.\text{ddc}, s_1.\text{stc}, s_1.\text{ddc} \downarrow v, t_3 v \), and \( v' = s_1.\mathcal{M}_d(v.s + v.\text{off}) \).
    - The induction hypothesis gives us that \( \mathcal{E}', s_2.\mathcal{M}_d, s_2.\text{ddc}, s_2.\text{stc}, s_2.\text{ddc} \downarrow v \).
    - So, we need to show that \( s_2.\mathcal{M}_d(v.s + v.\text{off}) = v' = s_1.\mathcal{M}_d(v.s + v.\text{off}) \).
    - But we have by assumption that \( s_2.\mathcal{M}_d|_r = s_1.\mathcal{M}_d|_r \).
      So it suffices to show that \( v.s + v.\text{off} \in r \).
    - But by Lemma 25 about completeness of \( \text{reachable\_addresses} \), and the definition of \( r \) from the assumption we have that \( [v.s, v.e] \subseteq r \).
So our sufficient goal \( "v.s + v.off \in r" \) follows by the definition of \( \subseteq \) because from the above-obtained precondition \( \vdash_{\delta} v \), and by Definition 2, we know that \( v.s + v.off \in [v.s, v.e] \).

(Notice that Lemma 25 is applicable by the preconditions of rule \( \text{exec-state} \) of conjunct \( t_1 \times \tau \vdash_{\text{exec}} s_1 \) of the assumption, and the preconditions \( \mathcal{E}', s_1.M_d, s_1.ddc, s_1.stc, s_1.ddc \downarrow v \) and \( \vdash_{\delta} v.\))

\[\] 

Lemma 134 (The empty stack is in a singleton equivalence class of strong stack-similarity).

\[\forall stk, \tau.\]

\[\text{nil} \approx_{\tau} stk \]

\[\implies \]

\[stk = \text{nil}\]

Proof.

By unfolding the assumption using Definition 82, obtain \( f \) where the following hold:

\[f(-1) = -1, \quad \text{and} \quad f(0) = \text{length}(stk).\]

But by instantiating the successor-preservation assumption, know that \( f(0) = 0 \), hence \( \text{length}(stk) = 0 \), thus it must be that \( stk = \text{nil} \).

\[\] 

Lemma 135 (Adequacy of strong stack-similarity (syncing border-crossing return to non-\( \tau \) call-site)).

\[\forall stk_1, stk_2, \tau, pcc_1, pcc_2.\]

\[pcc_1 \not\subseteq \text{dom}(\tau.M_c) \land \]

\[stk_1++[pcc_1] \approx_{\tau} stk_2++[pcc_2] \]

\[\implies \]

\[pcc_2 \not\subseteq \text{dom}(\tau.M_c)\]

Proof.

• Suppose the negation were true: \( pcc_2 \subseteq \text{dom}(\tau.M_c) \).

• Then, by assumption (unfolding Definition 82), we obtain (*):

\[f(\text{length}(stk_1)) = \text{length}(stk_2).\]

• But we also know by the sentinel-preservation assumption that (**):

\[f(\text{length}(stk_1) + 1) = \text{length}(stk_2) + 1.\]

• But then using (*) and (**) to instantiate the \( \iff \) direction of the successor-preservation assumption, we know that

\[f(\text{length}(stk_1)) = \text{length}(stk_2).\]

• This last assertion together with the assumption that defines \( \text{dom}(f) \) gives us \( pcc_1 \subseteq \text{dom}(\tau.M_c) \).

• This last assertion in turn immediately contradicts our assumption.
Lemma 136 (Weak stack-similarity is preserved by a unilateral silent return).

\[
\forall stk_1, stk_2, \overline{c},
\]

\[
stk_1 \sim_{\overline{c}} stk_2 \land
\]

\[
top(stk_1).pcc \not\subseteq \text{dom}(\overline{c}.M_c)
\]

\[
\implies
\]

\[
pop(stk_1), stk \sim_{\overline{c}} stk_2
\]

Proof.

By unfolding Definition 83, we obtain \( f \) satisfying:

\[
\text{dom}(f) = \{ i \in \text{dom}(stk_1) \mid \text{stk}_1(i).pcc \subseteq \text{dom}(\overline{c}.M_c) \} \cup \{-1\}
\]

Moreover, we infer from our assumption about \( \text{top}(stk_1) \) that (*):

\[
\text{length}(stk_1) - 1 \notin \text{dom}(f).
\]

We also know by the spec. of \( \text{pop} \) that (**):

\[
\text{dom}(\text{pop}(stk_1).stk) = \text{dom}(stk_1) \cup \{\text{length}(stk_1) - 1\}
\]

By unfolding our goal using Definition 83, it suffices to pick the same \( f \) obtained above, if we prove all the following:

1. Domain of \( f \) is exhaustive of \( \overline{c} \) call sites in \( \text{pop}(stk_1).stk \).
   
   Immediate by assumption after noticing by (**) and (*) that \( \text{dom}(stk_1) = \text{dom}(\text{pop}(stk_1).stk) \).

2. Range of \( f \) is exhaustive of \( \overline{c} \) call sites in \( stk_2 \)
   
   Immediate by assumption.

3. \( f \) is sentinel-value preserving.
   
   Immediate by assumption.

4. \( f \) is strictly monotone.
   
   Immediate by assumption.

5. \( f \) is compatible with stack-frame equality.
   
   Immediate by assumption.

6. \( f \) is successor-preserving.
   
   Immediate by assumption.

This concludes our proof of Lemma 136.

Lemma 137 (Weak stack-similarity is preserved by a unilateral silent call).

\[
\forall stk_1, stk_2, \overline{c}, pcc.
\]

\[
stk_1 \sim_{\overline{c}} stk_2 \land
\]

\[
pcc \not\subseteq \text{dom}(\overline{c}.M_c)
\]

\[
\implies
\]

\[
\text{push}(stk_1, (\_, pcc, \_, \_)) \sim_{\overline{c}} stk_2
\]

Proof. Similar to the proof of Lemma 136. We avoid repetition.

Lemma 138 (Weakening of strong stack-similarity).

\[
\forall stk_1, stk_2, \overline{c}.
\]

\[
stk_1 \approx_{\overline{c}} stk_2
\]

\[
\implies
\]

\[
stk_1 \sim_{\overline{c}} stk_2
\]

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Proof.
By unfolding the assumption using Definition 82, we obtain \( f \).
Then, by unfolding the goal using Definition 83, we pick:
\[
f' := f \setminus \{ \text{length}(stk_1) \mapsto \text{length}(stk_2) \}
\]
Thus, it remains to prove all of the following:

1. Domain of \( f' \) is exhaustive of \( \tau \) call sites in \( stk_1 \)
   \[
   \text{dom}(f') = \{ i \in \text{dom}(stk_1) \mid stk_1(i).pcc \subseteq \text{dom}(\tau,M_c) \} \cup \{-1\}
   \]
   Immediate by the corresponding assumption about \( f \), and the choice of \( f' \).

2. Range of \( f' \) is exhaustive of \( \tau \) call sites in \( stk_2 \)
   \[
   \text{range}(f') = \{ i \in \text{dom}(stk_2) \mid stk_2(i).pcc \subseteq \text{dom}(\tau,M_c) \} \cup \{-1\}
   \]
   Immediate by the corresponding assumption about \( f \), and the choice of \( f' \).

3. \( f' \) is sentinel-value preserving
   \[
   f'(-1) = -1.
   \]
   Immediate by the corresponding assumption about \( f \) and the choice of \( f' \).

4. \( f' \) is strictly monotone
   \[
   \forall i, j. i > j \implies f'(i) > f'(j).
   \]
   Pick arbitrary \( i, j \in \text{dom}(f') \).
   Notice that \( i, j \in \text{dom}(f) \).
   Thus, our goal is immediate by the corresponding assumption about \( f \).

5. \( f' \) is compatible with stack-frame equality
   \[
   \forall i \in \text{dom}(f') \setminus \{-1, \text{length}(stk_1)\}. f'(i) = j \implies stk_1(i) = stk_2(j).
   \]
   Proof is the same as the previous subgoal.

6. \( f' \) is successor-preserving
   \[
   \forall i, j \in \text{dom}(f'). j = i + 1 \iff f'(j) = f'(i) + 1.
   \]
   Proof is the same as the previous subgoal.

This concludes the proof of Lemma 138.

Lemma 139 (Strong stack-similarity is preserved by a bilateral call (from same \( \tau \)-call-site)).

\[
\forall stk_1, stk_2, \tau, pcc.
\]  
\[
stk_1 \approx_{[\tau]} stk_2 \land
\]  
\[
pcc \subseteq \text{dom}(\tau,M_c)
\]
\[
\implies
\]
\[
push(stk_1, (_, pcc, _, _)) \approx_{[\tau]} push(stk_2, (_, pcc, _, _))
\]

Proof.
By unfolding the assumption using Definition 82, we obtain \( f \).

Then, by unfolding the goal using Definition 82, we pick:
\[
f' := f \cup \{ \text{length}(stk_1) + 1 \mapsto \text{length}(stk_2) + 1 \}.
\]
It thus remains to prove all of the following:

1. Domain of \( f' \) is exhaustive of \( \tau \) call sites in \( push(stk_1, (_, pcc, _, _)) \), and
2. Range of \( f' \) is exhaustive of \( \tau \) call sites in \( stk_2 \)
   Immediate by the corresponding assumptions and by the choice of \( f' \).
3. \( f \) is sentinel-value preserving.
   The bottom sentinel value is preserved: \( f'(-1) = -1 \) follows from \( f(-1) = -1 \).
   The top sentinel value is preserved by choice of \( f' \).

4. \( f \) is strictly monotone.
   Pick arbitrary \( i, j \in \text{dom}(f') \) where \( i < j \).
   Show \( f'(i) < f'(j) \).
   Distinguish three cases:
   
   - **Case** \( i, j \in \text{dom}(f) \)
     Immediate by strict monotonicity of \( f \).
   
   - **Case** \( i \notin \text{dom}(f) \):
     Know \( i = \text{length}(stk_1) + 1 \).
     Thus, \( j > \text{length}(stk_1) + 1 \).
     Thus, this case is impossible by the definition of \( \text{dom}(f') \).
   
   - **Case** \( j \notin \text{dom}(f) \):
     Know \( j = \text{length}(stk_1) + 1 \), and
     know \( i \in \text{dom}(f) \) (by choice of \( f' \)).
     Thus, the goal becomes
     \( f(i) < f'(\text{length}(stk_1)) + 1 \)
     By choice of \( f' \), the goal becomes \( f(i) < \text{length}(stk_2) + 1 \)
     This is immediate by the definition of \( \text{range}(f) \).

5. \( f \) is compatible with stack-frame equality.
   Immediate by the choice of \( f' \), and the corresponding assumption about \( f \).

6. \( f \) is successor-preserving.
   Pick arbitrary \( i, j \in \text{dom}(f) \) with \( i = j + 1 \).
   Show \( f'(i) = f'(j) + 1 \).
   Distinguish the following cases:
   
   - **Case** \( i, j \in \text{dom}(f) \):
     Immediate by the corresponding assumption about \( f \).
   
   - **Case** \( i \notin \text{dom}(f) \):
     Know \( i = \text{length}(stk_1) + 1 \)
     Goal becomes \( \text{length}(stk_2) = f'(\text{length}(stk_1)) \).
     Immediate by the choice of \( f' \).
   
   - **Case** \( j \notin \text{dom}(f) \):
     Know \( j = \text{length}(stk_1) + 1 \).
     Thus, \( i = \text{length}(stk_1) + 2 \) which is impossible by the definition of \( \text{dom}(f') \).

This concludes the proof of Lemma 139. \( \square \)

**Lemma 140** (Strong stack-similarity is weakened by a bilateral return to a non-\( \tau \)-call-site).

\[
\forall stk_1, stk_2, \tau, pcc_1, pcc_2 .
\]

\[
stk_1[++[pcc_1]] \approx_{[\tau]} stk_2[++[pcc_2]] \land
pcc_1 \notin \text{dom}(\tau, M_e)
\]

\[
\Rightarrow
stk_1 \sim_{[\tau]} stk_2
\]

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Proof.
Assume the antecedents.
By instantiating Lemma 135 using the assumptions, we know that
\[ pcc_2 \notin \text{dom}(\pi.M_c) \] (*).
Also, by instantiating Lemma 138 using the assumptions, we know
\[ stk_1 \text{++}[pcc_1] \sim_{[\pi]} stk_2 \text{++}[pcc_2] \] (**).
Thus, by instantiating Lemma 136 using (*) and (**), we know
\[ stk_1 \sim_{[\pi]} stk_2 \text{++}[pcc_2] \] (POPPED-LEFT).
By instantiating symmetry (Claim 32) with (POPPED-LEFT), we thus know
\[ stk_2 \text{++}[pcc_2] \sim_{[\pi]} stk_1. \]
Now again by instantiating Lemma 136, we know
\[ stk_2 \sim_{[\pi]} stk_1. \]
Finally, by instantiating symmetry (Claim 32), we know
\[ stk_1 \sim_{[\pi]} stk_2, \] which is our goal. \qed

Lemma 141 (Strong stack-similarity is preserved by a bilateral return to a \( \pi \)-call-site).

\[ \forall stk_1, stk_2, \pi, pcc_1, pcc_2.
\text{stk}_1 \text{++}[pcc_1] \equiv_{[\pi]} \text{stk}_2 \text{++}[pcc_2] \wedge
pcc_1 \subseteq \text{dom}(\pi.M_c)
\[ \implies
\text{stk}_1 \equiv_{[\pi]} \text{stk}_2 \]

Proof.
Assume the antecedents (unfold by Definition 82 to obtain \( f \)).
By the assumptions, know that \( pcc_2 \subseteq \text{dom}(\pi.M_c) \):

- Suppose the negation were true: \( pcc_2 \not\subseteq \text{dom}(\pi.M_c) \).

- By instantiating symmetry (Lemma 129) using our assumption, then instantiating Lemma 135,
  we know \( pcc_1 \not\subseteq \text{dom}(\pi.M_c) \) which contradicts the case condition.

In particular, by instantiating the definition of \( \text{dom}(f) \) using the assumption, we know that
\[ f(\text{length}(stk_1)) = \text{length}(stk_2) \] (*)
by instantiating the \( \implies \) direction of the successor-preservation assumption (about \( f \)) using the sentinel-value preservation assumption (about \( f \)).

For our goal (unfolding Definition 82), we pick
\[ f' := f \setminus \{ \text{length}(stk_1) + 1 \mapsto \text{length}(stk_2) + 1 \}. \]

1. Domain of \( f' \) is exhaustive of \( \pi \) call sites in \( stk_1 \).
   Follows from the corresponding assumption about \( f \) and from the choice of \( f' \).
   The sentinel value follows from \( pcc_1 \subseteq \text{dom}(\pi.M_c) \).
2. Range of \( f' \) is exhaustive of \( \pi \) call sites in \( stk_2 \).
   Follows from the corresponding assumption about \( f \) and from the choice of \( f' \).
   The sentinel value follows from \( pcc_2 \subseteq \text{dom}(\pi.M_c) \).
3. \( f' \) is sentinel-value preserving.
   Follows from the corresponding assumption about \( f \) and from the choice of \( f' \).
4. $f'$ is strictly monotone:
   
   \[(\forall i, j. i > j \implies f'(i) > f'(j))\].

   Notice that $f' \subseteq f$, so for arbitrary $i, j \in \text{dom}(f')$, the consequent holds by instantiating the strict-monotonicity assumption about $f$.

5. $f'$ is compatible with stack-frame equality.
   
   Pick an arbitrary $i$ where $i \in \text{dom}(f') \setminus \{-1, \text{length}(stk_1)\}$.
   
   Show that $\text{stk}_1(i) = \text{stk}_2(f'(i))$.
   
   This is immediate by instantiating the corresponding assumption (compatibility with stack-frame equality) for $f$.

6. $f'$ is successor-preserving.
   
   Pick arbitrary $i, j \in \text{dom}(f')$.
   
   Show that $j = i + 1 \iff f'(j) = f'(i) + 1$.
   
   Observe that $\text{dom}(f') \subseteq \text{dom}(f)$.
   
   Thus, the goal is immediate successor preservation about $f$.

This concludes the proof of Lemma 141.

\[\square\]

**Lemma 142** (Strengthening of weak stack-similarity by a bilateral call from non-$\tau$ call-sites).

\[
\forall \text{stk}_1, \text{stk}_2, \tau, \text{pcc}_1, \text{pcc}_2. \\
\text{stk}_1 \sim[\tau] \text{stk}_2 \land \\
\text{pcc}_1 \not\subseteq \text{dom}(\tau.M_c) \land \\
\text{pcc}_2 \not\subseteq \text{dom}(\tau.M_c) \\
\implies \\
\text{push}(\text{stk}_1, (_, \text{pcc}_1, _, _)) \approx[\tau] \text{push}(\text{stk}_2, (_, \text{pcc}_2, _, _))
\]

**Proof.**

By unfolding the assumption using Definition 83, we obtain $f$.

Then, by unfolding the goal using Definition 82, we pick:

\[f' := f \uplus \text{length}(\text{stk}_1) + 1 \mapsto \text{length}(\text{stk}_2) + 1\] .

It thus remains to prove all of the following:

1. Domain of $f'$ is exhaustive of $\tau$ call sites in $\text{push}(\text{stk}_1, (_, \text{pcc}_1, _, _))$:
   
   \[
   (\text{dom}(f') = \{i \in \text{dom}(\text{push}(\text{stk}_1, (_, \text{pcc}_1, _, _))) | \text{push}(\text{stk}_1, (_, \text{pcc}_1, _, _))(i).\text{pcc} \subseteq \text{dom}(\tau.M_c)\} \uplus \\
   \{-1, \text{length}(\text{push}(\text{stk}_1, (_, \text{pcc}_1, _, _))))\}).
   \]
   
   Immediate by choice of $f'$ after noticing the corresponding assumption about $f$, the assumption about $\text{pcc}_1$, and that $\text{length}(\text{push}(\text{stk}_1, (_, \text{pcc}_1, _, _))) = \text{length}(\text{stk}_1) + 1$.

2. Range of $f'$ is exhaustive of $\tau$ call sites in $\text{push}(\text{stk}_2, (_, \text{pcc}_2, _, _))$:
   
   \[
   (\text{range}(f') = \{i \in \text{dom}(\text{push}(\text{stk}_2, (_, \text{pcc}_2, _, _))) | \text{push}(\text{stk}_2, (_, \text{pcc}_2, _, _))(i).\text{pcc} \subseteq \text{dom}(\tau.M_c)\} \uplus \\
   \{-1, \text{length}(\text{push}(\text{stk}_2, (_, \text{pcc}_2, _, _))))\}).
   \]
   
   Proof is similar to the previous subgoal.

3. $f'$ is sentinel-value preserving:
   
   \[
   f'(-1) = -1 \land f'(\text{length}(\text{stk}_1) + 1) = \text{length}(\text{stk}_2 + 1)
   \]
   
   Immediate by the choice of $f'$ and by the corresponding assumption about $f$.
4. \( f' \) is strictly monotone: \( (\forall i, j. i > j \implies f'(i) > f'(j)) \).

Pick arbitrary \( i, j \in \text{dom}(f) \), and distinguish these cases:

- **Case** \( i, j \in \text{dom}(f) \):
  Here, our goal is immediate by the corresponding assumption about \( f \).

- **Case** \( i \notin \text{dom}(f) \):
  Infer \( i = \text{length}(\text{stk}_1) + 1 \).
  Thus, infer \( f'(i) = \text{length}(\text{stk}_2) + 1 \).
  Thus, the goal becomes:
  \[
  \forall j. j < \text{length}(\text{stk}_1) + 1 \implies \text{length}(\text{stk}_2) + 1 > f'(j)
  \]
  But assuming \( j < \text{length}(\text{stk}_1) + 1 \) gives us \( j \in \text{dom}(f) \).
  Thus, \( f'(j) = f(j) \).
  But then by the assumption about the range of \( f \), we have our goal.

- **Case** \( j \notin \text{dom}(f) \):
  Infer \( j = \text{length}(\text{stk}_1) + 1 \).
  Thus, goal follows vacuously because no index \( i \in \text{dom} (\text{push}(\text{stk}_1, _)) \) satisfies \( i > \text{length}(\text{stk}_1) + 1 \).

5. \( f' \) is compatible with stack-frame equality:
\[
(\forall i \in \text{dom}(f') \setminus \{-1, \text{length}(\text{stk}_1) + 1\}. f'(i) = j \implies \text{push}(\text{stk}_1, (_{, pcc_1, _, _}))(i) = \text{push}(\text{stk}_2, (_{, pcc_2, _, _}))(j))
\]
Fix \( i \in \text{dom}(f') \setminus \{-1, \text{length}(\text{stk}_1) + 1\} \), and distinguish two cases:

- **Case** \( i \in \text{dom}(f) \):
  Know by the assumption about \( \text{dom}(f) \) from unfolding Definition 83 that \( i \in \text{dom}(\text{stk}_1) \).
  Thus, our goal follows after instantiating the corresponding assumption about \( f \) (i.e., compatibility of \( f \) with stack-frame equality), and substitution using simple facts about \( \text{push} \).

- **Case** \( i \notin \text{dom}(f) \):
  By choice of \( f' \), and the condition on the fixed \( i \), this case is impossible.

6. \( f' \) is successor-preserving:
\[
(\forall i, j \in \text{dom}(f'). j = i + 1 \iff f'(j) = f'(i) + 1)
\]
Fix arbitrary \( i, j \in \text{dom}(f') \), and distinguish the following cases:

- **Case** \( i, j \in \text{dom}(f) \):
  Here, the goal is immediate by the corresponding assumption about \( f \) (after noticing the choice of \( f' \)).

- **Case** \( i \notin \text{dom}(f) \):
  Know by the choice of \( f' \) that \( i = \text{length}(\text{stk}_1) + 1 \).
  \[
  \implies : \quad \text{Here, know } j = \text{length}(\text{stk}_1) + 2.
  \text{Thus, our goal is immediate by deriving a contradiction to } j \in \text{dom}(f').
  \]

- **Case** \( j \notin \text{dom}(f) \):
  Know by the choice of \( f' \) that \( j = \text{length}(\text{stk}_1) + 1 \).
\[ \forall \tau, t_1, s_1, \varsigma_1, t_2, s_2, \varsigma_2, s'_1, \varsigma'_1. \]
\[ \tau \in \text{range}(\llbracket \square \rrbracket) \land \]
\[ t_1 \times \tau \vdash \text{exec} s_1 \land \]
\[ t_2 \times \tau \vdash \text{exec} s_2 \land \]
\[ s_1.pcc \subseteq \text{dom}(\tau, M_c) \land \]
\[ s_1, \varsigma_1 \approx \llbracket \tau \rrbracket s_2, \varsigma_2 \land \]
\[ s_1, \varsigma_1 \tau \Rightarrow s'_1, \varsigma'_1 \]
\[ \Rightarrow \exists s'_2, \varsigma'_2. \]
\[ s_2, \varsigma_2 \tau \Rightarrow s'_2, \varsigma'_2 \land \]
\[ s'_1, \varsigma'_1 \approx \llbracket \tau \rrbracket s'_2, \varsigma'_2 \]

**Proof.** We fix arbitrary \( \tau, t_1, s_1, \varsigma_1, t_2, s_2, \varsigma_2, s'_1, \varsigma'_1 \), and assume the antecedent:
\[ \tau \in \text{range}(\llbracket \square \rrbracket) \land \]
\[ t_1 \times \tau \vdash \text{exec} s_1 \land \]
\[ t_2 \times \tau \vdash \text{exec} s_2 \land \]
\[ s_1, \varsigma_1 \approx \llbracket \tau \rrbracket s_2, \varsigma_2 \land \]
\[ s_1, \varsigma_1 \tau \Rightarrow s'_1, \varsigma'_1 \]

From conjunct \( s_1, \varsigma_1 \approx \llbracket \tau \rrbracket s_2, \varsigma_2 \) of Proposition (9) and by Definition 84, we have (after substituting
\( s_1.\text{pcc} \subseteq \text{dom}(\tau.M_c) \) in Definition 80) the following assumptions:

\[
\begin{align*}
  s_1.\text{pcc} &\subseteq \text{dom}(\tau.M_c) \land \\
  s_2.\text{pcc} &\subseteq \text{dom}(\tau.M_c) \land \\
  \bigcup_{\text{mid} \in \text{dom}(\tau.\text{imp})} \text{reachable_addresses}\{s_1.\text{mstc}(\text{mid}), \tau.\text{imp}(\text{mid}).\text{ddc}\}, s_1.M_d = r \land \\
  \bigcup_{\text{mid} \in \text{dom}(\tau.\text{imp})} \text{reachable_addresses}\{s_2.\text{mstc}(\text{mid}), \tau.\text{imp}(\text{mid}).\text{ddc}\}, s_2.M_d = r \land \\
  s_1.\text{ddc} &= s_2.\text{ddc} \land \\
  s_1.\text{stc} &= s_2.\text{stc} \land \\
  s_1.\text{pcc} &= s_2.\text{pcc} \land \\
  s_1.\text{nalloc} &= s_2.\text{nalloc} \land \\
  s_1.\text{stk} &\approx_{[\tau]} s_2.\text{stk} \land \\
  s_1.\text{mstc} &\approx_{[\tau]} s_2.\text{mstc} \land \\
  \varsigma_1 &= \varsigma_2 \land \\
  s_1.\text{M}_d|_r &= s_2.\text{M}_d|_r
\end{align*}
\] (10)

From \( s_1.\text{pcc} \subseteq \text{dom}(\tau.M_c) \) and \( s_2.\text{pcc} \subseteq \text{dom}(\tau.M_c) \) of Proposition (10), and by substitution in Proposition (9) after inversion using \textit{exec-state} and \textit{valid-linking}, we know:

\[
s_1.\text{M}_e(s_1.\text{pcc}) = s_2.\text{M}_e(s_2.\text{pcc})
\] (11)

Our goal \( \exists s_2',\varsigma_2'. s_2,\varsigma_2 \xrightarrow{\tau}\varsigma_2' \) \( s_2',\varsigma_2' \) \( s_1',\varsigma_1' \) consists by unfolding it using Definition 84 then Definition 80 of the following subgoals:

\[
\begin{align*}
  \exists s_2',\varsigma_2'. s_2,\varsigma_2 \xrightarrow{\tau}\varsigma_2' \land \\
  s_1'.\text{pcc} &\subseteq \text{dom}(\tau.M_c) \land \\
  s_2'.\text{pcc} &\subseteq \text{dom}(\tau.M_c) \land \\
  \bigcup_{\text{mid} \in \text{dom}(\tau.\text{imp})} \text{reachable_addresses}\{s_1'.\text{mstc}(\text{mid}), \tau.\text{imp}(\text{mid}).\text{ddc}\}, s_1'.M_d = r \land \\
  \bigcup_{\text{mid} \in \text{dom}(\tau.\text{imp})} \text{reachable_addresses}\{s_2'.\text{mstc}(\text{mid}), \tau.\text{imp}(\text{mid}).\text{ddc}\}, s_2'.M_d = r \land \\
  s_1'.\text{ddc} &= s_2'.\text{ddc} \land \\
  s_1'.\text{stc} &= s_2'.\text{stc} \land \\
  s_1'.\text{pcc} &= s_2'.\text{pcc} \land \\
  s_1'.\text{nalloc} &= s_2'.\text{nalloc} \land \\
  s_1'.\text{stk} &\approx_{[\tau]} s_2'.\text{stk} \land \\
  s_1'.\text{mstc} &\approx_{[\tau]} s_2'.\text{mstc} \land \\
  \varsigma_1' &= \varsigma_2' \land \\
  s_1'.\text{M}_d|_r &= s_2'.\text{M}_d|_r
\end{align*}
\]

Notice that subgoals
\( s_1'.\text{pcc} \subseteq \text{dom}(\tau.M_c) \) and \( s_2'.\text{pcc} \subseteq \text{dom}(\tau.M_c) \)
follow by Lemma 108 from respectively
the assumption \( s_1,\varsigma_1 \xrightarrow{\tau}\varsigma_1' \) \( s_1',\varsigma_1' \)

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We prove the remaining subgoals by considering all the possible cases of the rule $s_1, \varsigma_1 \xrightarrow{\tau} [s] s'_1, \varsigma'_1$ of Proposition (9):

1. **Case assign-silent:**

   - We obtain the precondition $s_1, \mathcal{M}_r(s_1, \text{pcc}) = \text{Assign } \mathcal{E}_l \mathcal{E}_r$, so by Proposition (11), we have $s_2, \mathcal{M}_r(s_2, \text{pcc}) = \text{Assign } \mathcal{E}_l \mathcal{E}_r$. So, the only rule possibly-applicable to $s_2, \varsigma_2 \xrightarrow{\tau} [s] s'_2, \varsigma'_2$ is assign-silent. So, if $\lambda'$ exists, then $\lambda' = \tau$.

   - Now, we show that indeed $s'_2, \varsigma'_2$ exist by showing that $s_2 \rightarrow s'_2$ using rule assign.

     - By Lemma 133, and given $\mathcal{E}_l, s_1, \mathcal{M}_d, s_1, \text{ddc}, s_1, \text{stc}, s_1, \text{pcc} \downarrow c_1$ (which we do have by inversion), we have that $\mathcal{E}_l, s_2, \mathcal{M}_d, s_2, \text{ddc}, s_2, \text{stc}, s_2, \text{pcc} \downarrow c_1$. Also by Lemma 133, and given $\mathcal{E}_r, s_1, \mathcal{M}_d, s_1, \text{ddc}, s_1, \text{stc}, s_1, \text{pcc} \downarrow v_1$ (which we do have by inversion), we have that $\mathcal{E}_r, s_2, \mathcal{M}_d, s_2, \text{ddc}, s_2, \text{stc}, s_2, \text{pcc} \downarrow v_1$.

     - The preconditions on $s_2, \text{pcc}$ and on $s_2, \text{stc}$ then follow by substitution using respectively conjuncts $s_1, \text{pcc} = s_2, \text{pcc}$ and $s_1, \text{stc} = s_2, \text{stc}$ of Proposition (10).

     - Thus, we can now conclude that $s_2 \rightarrow s'_2$ since all the preconditions of rule assign hold.

     - Thus, by rule assign-silent, we have the first conjunct of our goal: $\exists s'_2, \varsigma'_2$. $s_2, \varsigma_2 \xrightarrow{\tau} [s] s'_2, \varsigma'_2$.

   - We show the remaining subgoals:

     - We observe from rule assign that $s'_2, \text{ddc} = s_2, \text{ddc}$, which by Proposition (10) gives $s'_2, \text{ddc} = s_1, \text{ddc}$, which by rule assign gives us $s'_2, \text{ddc} = s'_1, \text{ddc}$

     - A similar argument shows that $s'_2, \text{stk} = s'_1, \text{stk}$, $s'_2, \text{mstc} = s'_1, \text{mstc}$, $s'_2, \text{stc} = s'_1, \text{stc}$, and $s'_2, \text{nalloc} = s'_1, \text{nalloc}$.

     - Using the necessary preconditions $s'_1, \text{pcc} = \text{inc}(s_1, \text{pcc}, 1)$ and $s'_2, \text{pcc} = \text{inc}(s_2, \text{pcc}, 1)$ of rule assign, and by substitution using $s_1, \text{pcc} = s_2, \text{pcc}$ of Proposition (10), we get $s'_2, \text{pcc} = s'_1, \text{pcc}$.

     - Moreover, we have by rule assign-silent, that $\varsigma'_2 = \varsigma_2$, which by Proposition (10) gives us that $\varsigma'_2 = \varsigma_1$, which by rule assign-silent gives us $\varsigma'_2 = \varsigma'_1$.

     - From the above, we have obtained the following conjuncts:

       - $s'_1, \text{stk} \approx[s] s'_2, \text{stk} \land s'_1, \text{mstc} \approx[s] s'_2, \text{mstc}$ by reflexivity of both the $\approx[s]$ overloaded relations after substituting from $s'_1, \text{stk} = s'_2, \text{stk}$, and $s'_1, \text{mstc} = s'_2, \text{mstc}$ respectively.

       - $s'_1, \text{ddc} = s'_2, \text{ddc} \land s'_1, \text{stc} = s'_2, \text{stc} \land s'_1, \text{pcc} = s'_2, \text{pcc} \land s'_1, \text{nalloc} = s'_2, \text{nalloc} \land \varsigma'_1 = \varsigma'_2$

       - which we obtained successively by the arguments detailed above.

     - Thus, it remains to show that $\rho(s'_1, \varsigma'_1) = \rho(s'_2, \varsigma'_2)$ and $s'_1, \mathcal{M}_d|_\tau = s'_2, \mathcal{M}_d|_\tau$.

     - We show that $(\text{S1}'-\text{PCC-SUBSET-C})$:

       - $s'_1, \text{pcc} \subseteq \text{dom}(\tau, \mathcal{M}_d)$

       - To prove this, we apply Lemma 108 obtaining subgoals that are provable by the assumptions.

       - From $(\text{S1}'-\text{PCC-SUBSET-C})$, we obtain by substitution using the previously proven subgoals:

       - $(\text{S2}'-\text{PCC-SUBSET-C})$:

       - Now, by substituting $(\text{S1}'-\text{PCC-SUBSET-C})$, and $(\text{S2}'-\text{PCC-SUBSET-C})$ in our goal after unfolding it using Definition 80, our goal becomes:

         \[
         \bigcup_{mid \in \text{dom}(\tau, \text{imp})} \text{reachable_addresses}\{s'_1, \text{mstc}(\text{mid}), \tau, \text{imp}(\text{mid}), \text{ddc}\}, s'_1, \mathcal{M}_d) =
         \]

         \[
         \bigcup_{mid \in \text{dom}(\tau, \text{imp})} \text{reachable_addresses}\{s'_2, \text{mstc}(\text{mid}), \tau, \text{imp}(\text{mid}), \text{ddc}\}, s'_2, \mathcal{M}_d)
         \]

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By additivity of reachable_addresses (Lemma 18), it suffices to show that:
reachables ≡ \bigcup_{mid \in \text{dom}(\tau, \text{imp})} \{ s'_{1}.\text{mstc}(mid), \tau.\text{imp}(mid).\text{ddc} \}, s_{1}.Md =
reachables ≡ \bigcup_{mid \in \text{dom}(\tau, \text{imp})} \{ s'_{2}.\text{mstc}(mid), \tau.\text{imp}(mid).\text{ddc} \}, s_{2}.Md

By conjunct \{ s'_{1}.\text{mstc} \approx \{ \}, s'_{2}.\text{mstc} \} that we already proved above, it suffices to show that:
reachables \equiv \bigcup_{mid \in \text{dom}(\tau, \text{imp})} \{ s'_{1}.\text{mstc}(mid), \tau.\text{imp}(mid).\text{ddc} \}, s_{1}.Md =
reachables \equiv \bigcup_{mid \in \text{dom}(\tau, \text{imp})} \{ s'_{1}.\text{mstc}(mid), \tau.\text{imp}(mid).\text{ddc} \}, s_{2}.Md.

So, we would like to use Lemma 29 about preservation of reachability equivalence with the instantiation \( C := \bigcup_{mid \in \text{dom}(\tau, \text{imp})} \{ s'_{1}.\text{mstc}(mid), \tau.\text{imp}(mid).\text{ddc} \} \), but we have first to satisfy the premise: \( C, s_{1}.Md \models v \lor v \notin \{ \delta \} \times Z \times Z \).

We know \{ \text{sc}, s_{1}.ddc \}, s_{1}.Md \models v \lor v \notin \{ \delta \} \times Z \times Z \).

The latter follows immediately by Lemma 25 about completeness of reachable_addresses, and by simplifying Definition 23 of \{ \text{sc}, s_{1}.ddc \}, s_{1}.Md \models v.

(Note that the premises of Lemma 25 are satisfied by conjunct \( t_{1} \times c \vdash \text{exec} s_{1} \) of Proposition (9).)

By Lemma 27, we thus have the premise \( C, s_{1}.Md \models v \lor v \notin \{ \delta \} \times Z \times Z \) for Lemma 29.

So, now we can use Lemma 29 which gives us (**):
reachables \equiv \bigcup_{mid \in \text{dom}(\tau, \text{imp})} \{ s'_{1}.\text{mstc}(mid), \tau.\text{imp}(mid).\text{ddc} \}, s_{1}.Md =
reachables \equiv \bigcup_{mid \in \text{dom}(\tau, \text{imp})} \{ s'_{1}.\text{mstc}(mid), \tau.\text{imp}(mid).\text{ddc} \}, s_{2}.Md.

This was sufficient for proving the subgoal \( r' = \rho_{\text{exec}}(s_{1}', s_{1}') = \rho_{\text{exec}}(s_{2}', s_{2}') \).

Now, it remains to show the subgoal \( s_{1}.Md|_{r'} = s_{2}.Md|_{r'} \).

By the precondition \( \vdash_{\Delta} c_{1} \), we can apply Lemma 25 to conclude that \( c_{1}.s + c_{1}.off \in r \).

Thus, by Definition 23, we have the premises for Lemma 38.

By Lemma 38, in order to show that \( s_{1}.Md|_{r'} = s_{2}.Md|_{r'} \), it suffices to show that \( s_{1}.Md|_{r} = s_{2}.Md|_{r} \).

We show that \( \forall a \in r, s_{1}.Md(a) = s_{2}.Md(a) \) by distinguishing two cases:

* **Case** \( a = c_{1}.s + c_{1}.off \):

Here, address \( a \) is the one assigned in both reduction rules \( (s_{1} \rightarrow s_{1}') \) and \( (s_{2} \rightarrow s_{2}') \).

So, the preconditions \( s'_{1}.Md = s_{1}.Md|_{c_{1} \rightarrow v_{1}} \) and \( s'_{2}.Md = s_{2}.Md|_{c_{1} \rightarrow v_{1}} \) clearly show our goal in this case because they update this address with the same value \( v_{1} \).

* **Case** \( a \neq c_{1}.s + c_{1}.off \):

In this case, similarly to above, we obtain the preconditions \( s'_{1}.Md = s_{1}.Md|_{c_{1} \rightarrow v_{1}} \)
and \( s'_{2}.Md = s_{2}.Md|_{c_{1} \rightarrow v_{1}} \) which show that in this case, the memories \( s'_{1}.Md \)
and \( s'_{2}.Md \) at address \( a \) are not updated.

So, our goal follows from the assumption \( s_{1}.Md|_{r} = s_{2}.Md|_{r} \) of Proposition (10).

This concludes case assign-silent. Cases alloc-silent and jump-silent are not surprisingly different; a so-far-convinced reader may well skip them.

2. **Case** alloc-silent:

- We obtain the precondition \( s_{1}.M_{c}(s_{1}.pcc) = \text{Alloc } E_{1} E_{size} \), so by Proposition (11), we have \( s_{2}.M_{c}(s_{2}.pcc) = \text{Alloc } E_{1} E_{size} \). So, the only rule possibly-applicable to \( s_{2}, s_{2}' \) of alloc-silent. So, if \( \lambda' \) exists, then \( \lambda' = \tau \).
Now, it remains to show that it is indeed applicable (i.e., $\exists s_2, \zeta_2, s_2, \zeta_2 \vdash \framebox[s]{\zeta_2}, s_2$) and that $s_1, \zeta_1 \approx \framebox[s]{\zeta_2}, s_2$.

We show that $s_2 \rightarrow s_2'$ for some $s_2'$, and in particular that rule allocate is applicable.

By Lemma 133, and given $\Delta, s_1, M_d, s_1, ddc, s_1, stc, s_1, pcc \downarrow c_1$ (which we do have by inversion), we have that $\Delta, s_1, M_d, s_2, ddc, s_2, stc, s_2, pcc \downarrow c_1$. Also by Lemma 133, and given $E_{\text{size}}(s_1, M_d, s_1, ddc, s_1, stc, s_1, pcc \downarrow v_1$ (which we do have by inversion), we have that $E_{\text{size}}(s_2, M_d, s_2, ddc, s_2, stc, s_2, pcc \downarrow v_1$.

The preconditions on $s_2, pcc$ and on $s_2, nalloc$ then follow by substitution using respectively conjuncts $s_1, pcc = s_2, pcc$ and $s_1, nalloc = s_2, nalloc$ of Proposition (10).

Thus, we can now conclude that $s_2 \rightarrow s_2'$ since all the preconditions of rule allocate hold.

Moreover, by the precondition $\vdash c_1$, we can apply Lemma 25 to conclude that $c_1, s + c_1, \text{off} \in r$.

We observe from rule allocate that $s_2', ddc = s_2, ddc$, which by Proposition (10) gives $s_2', ddc = s_1, ddc$, which by rule allocate gives us $s_2', ddc = s_1$.d

A similar argument shows that $s_2', stk = s_1'.stk$, $s_2', mstc = s_1'.mstc$ (thus, implying the desired stack and stack-capability-map similarities (definitions 81 and 82) respectively), and $s_2', stc = s_1'.stc$.

Using the necessary preconditions $s_1'.pcc = \text{inc}(s_1, pcc, 1)$ and $s_2'.pcc = \text{inc}(s_2, pcc, 1)$ of rule allocate, and by substitution using $s_1, pcc = s_2, pcc$ of Proposition (10), we get $s_2, pcc = s_1'.pcc$.

Also, we have that $s_2', nalloc = s_1'.nalloc$ by substituting conjunct $s_1, nalloc = s_2, nalloc$ of Proposition (10) in the preconditions $s_2', nalloc = s_2, nalloc - v_1$ and $s_1', nalloc = s_1, nalloc - v_1$, where the same $v_1$ appears in both expressions due to the equal-evaluation that is shown above of the expression $E_{\text{size}}$.

Moreover, we have by rule alloc-silent, that $\zeta_2' = \zeta_2$, which by Proposition (10) gives us that $\zeta_2' = \zeta_1$, which by rule alloc-silent gives us $\zeta_2' = \zeta_1'$.

Next, we show that $r' = \rho[s_1', \zeta_1'] = \rho[s_1', \zeta_2']$ by the same argument as in case assign. We avoid repetition.

Now, it remains to show that $s_1', M_d|r' = s_2', M_d|r'$.

By Lemma 40, it suffices to show that $s_1', M_d|r = s_2', M_d|r$.

We show that $\forall a \in r s_1', M_d(a) = s_2', M_d(a)$ by distinguishing three cases that are exhaustive (we do not prove that they are mutually exclusive because that is not needed, although we believe them to be mutually exclusive):

- **Case** $a = c_1, s + c_1, \text{off}$:
  Here, address $a$ is updated in both reduction rules ($s_1 \rightarrow s_1'$ and $s_2 \rightarrow s_2'$). So, the preconditions $s_1'.M_d(c_1) = (\delta, s_1, nalloc - v_1, s_1, nalloc, 0)$ and $s_2'.M_d(c_1) = (\delta, s_2, nalloc - v_1, s_2, nalloc, 0)$ show our goal in this case because by substitution using conjunct $s_1, nalloc = s_2, nalloc$ of Proposition (10), they update address $a$ with the same value.

- **Case** $a \in (s_2, nalloc - v_1, s_2, nalloc)$:
  Here, similarly to the previous case, address $a$ is one that is assigned in both reduction rules ($s_1 \rightarrow s_1'$ and $s_2 \rightarrow s_2'$ because $s_2, nalloc = s_1, nalloc$ by Proposition (10)). So, the updated value 0 of both $s_1', M_d(a)$ and $s_2', M_d(a)$ is the same, so we have our goal.

- **Case** $a \neq c_1, s + c_1, \text{off} \land a \notin (s_2, nalloc - v_1, s_2, nalloc)$:
  In this case, similarly to above, we obtain the preconditions $s_1'.M_d = s_1, M_d[c_1 \rightarrow v_1]$ and $s_2'.M_d = s_2, M_d[c_1 \rightarrow v_1]$ which show that in this case, the memories $s_1', M_d$ and $s_2'.M_d$ at address $a$ are not updated.

So, our goal follows from the assumption $s_1, M_d|r = s_2, M_d|r$ of Proposition (10).

This concludes case alloc-silent.
3. Case \textbf{jump-silent}:

- We obtain the precondition $s_1.\mathcal{M}_c(s_1.\text{pcc}) = \text{Jump0Zero} \ pcc \mathcal{E}_{\text{cond}} \mathcal{E}_{\text{cap}}$, so by Proposition (11), we have $s_2.\mathcal{M}_c(s_2.\text{pcc}) = \text{Jump0Zero} \ pcc \mathcal{E}_{\text{cond}} \mathcal{E}_{\text{cap}}$. So, the only rule possibly-applicable to $s_2,\varsigma_2 \xrightarrow{\lambda'}[\mathcal{E}] s_2',\varsigma_2'$ is \textbf{jump-silent}. So, if $\lambda'$ exists, then $\lambda' = \lambda = \tau$.

- Now, it remains to show that it is indeed applicable (i.e., $\exists s_2',\varsigma_2'. s_2,\varsigma_2 \xrightarrow{\lambda'}[\mathcal{E}] s_2',\varsigma_2'$) and that $s_1',\varsigma_1' \approx[\mathcal{E}] s_2',\varsigma_2'$.

- We show that $s_2 \rightarrow s_2'$ for some $s_2'$, and in particular that either rule \textbf{jump1} or \textbf{jump0} is applicable.

- For that, we distinguish the two possible cases for $s_1 \rightarrow s_1'$:
  
  - \textbf{Case jump1}:
    * By Lemma 133, and given $\mathcal{E}_{\text{cond}}, s_1.\mathcal{M}_d, s_1.\text{ddc}, s_1.\text{stc}, s_1.\text{pcc} \downarrow v_1$ (which we do have by inversion), we have that $\mathcal{E}_{\text{cond}}, s_2.\mathcal{M}_d, s_2.\text{ddc}, s_2.\text{stc}, s_2.\text{pcc} \downarrow v_1$.
    * The precondition on $s_2.\text{pcc}$ then follows by substitution using conjunct $s_1.\text{pcc} = s_2.\text{pcc}$ of Proposition (10) and the precondition on $v_1$ still holds as well because $\mathcal{E}_{\text{cond}}$ evaluates to the same $v_1$ as in rule $s_1 \rightarrow s_1'$ as shown above.
    * Thus, we can now conclude that $s_2 \rightarrow s_2'$ since all the preconditions of rule \textbf{jump1} hold.
    * The similarities $s_1'.\text{stk} \approx[\mathcal{E}] s_2'.\text{stk} \wedge s_1'.\text{mstc} \approx[\mathcal{E}] s_2'.\text{mstc}$ hold by substitution using the corresponding equalities in Proposition (10).
    * Also, we have that all the required equalities (namely, $\varsigma'_1 = \varsigma'_2, s_1'.\mathcal{M}_d|\tau' = s_2'.\mathcal{M}_d|\tau'$, and $s_2'.\text{ddc} = s_2'.\text{ddc}$) follow from the corresponding ones in Proposition (10) by noticing that $s_2'.\mathcal{M}_d = s_2.\mathcal{M}_d$ and $s_1.\mathcal{M}_d = s_1'.\mathcal{M}_d$ and similarly for $\varsigma'_2, s_2'.\text{ddc}, s_2'.\text{stc}$, and $s_2'.\text{alloc}$.
    * So all conjuncts of our goal are proved.

  - \textbf{Case jump0}:
    This case is exactly the same as \textbf{jump1}, except that $s_2 \rightarrow s_2'$ holds by rule \textbf{jump0}.

This concludes case \textbf{jump-silent}.

4. Case \textbf{invoke-silent-compiled}:

- We obtain the precondition $s_1.\mathcal{M}_c(s_1.\text{pcc}) = \text{Invoke mid fid }\tau$, so by Proposition (11), we have $s_2.\mathcal{M}_c(s_2.\text{pcc}) = \text{Invoke mid fid }\tau$.

  Also, by $s_1.\text{pcc} = s_2.\text{pcc}$ of Proposition (10), we know that the precondition $s_2.\text{pcc} \in \text{dom}(\tau.\mathcal{M}_c)$ holds.

  Thus, this, together with the precondition $\text{mid} \in \text{dom}(\tau.\text{imp})$ give us that the only rule possibly-applicable to $s_2,\varsigma_2 \xrightarrow{\lambda'}[\mathcal{E}] s_2',\varsigma_2'$ is \textbf{invoke-silent-compiled}. So, if $\lambda'$ exists, then $\lambda' = \lambda = \tau$.

- Now, it remains to show that it is indeed applicable (i.e., $\exists s_2',\varsigma_2'. s_2,\varsigma_2 \xrightarrow{\lambda'}[\mathcal{E}] s_2',\varsigma_2'$) and that $s_1',\varsigma_1' \approx[\mathcal{E}] s_2',\varsigma_2'$.

- We show that $s_2 \rightarrow s_2'$ for some $s_2'$, and in particular that rule \textbf{invoke} is applicable.

- We obtain the preconditions $s_1.\phi(\text{mid}, \text{fid}) = (n\text{Args}, n\text{Local})$, and $(c, d, \text{offs}) = s_1.\text{imp}(\text{mid})$.

  So, by Lemma 2, and by our earlier statement $s_2.\mathcal{M}_c(s_2.\text{pcc}) = \text{Invoke mid fid }\tau$, we notice that we have $s_2.\phi(\text{mid}, \text{fid}) = (n\text{Args}, n\text{Local})$, and $(c, d, \text{offs}) = s_2.\text{imp}(\text{mid})$.

  This gives us the equalities $s_1'.\text{ddc} = s_2'.\text{ddc}$ and $s_1'.\text{stc} = s_2'.\text{stc}$, and $s_1'.\text{pcc} = s_2'.\text{pcc}$ of our goal.
• We also conclude that expression evaluation of the arguments in state \(s_2\) gives the same values as evaluation in state \(s_1\).

I.e., given \(\pi(i), s_1, M_d, s_1, ddc, s_1, stc, s_1, pcc \vdash \pi(i) \forall i \in [0, n\text{Args})\) (which we get by inverting \(s_1 \succ s'_1\) using \text{cinvoke-aux}), we have by Lemma 133 that
\[
\pi(i), s_2, M_d, s_2, ddc, s_2, stc, s_2, pcc \vdash \pi(i) \forall i \in [0, n\text{Args}).
\]
This, consequently, gives us that \(s'_{2,M_d\mid\pi} = s'_{1,M_d\mid\pi}\) by case distinction on the updated vs. non-updated locations and substitution in both cases.

Similarly to case \text{assign-silent}, this suffices to prove subgoal \(s'_{2,M_d\mid\pi} = s'_{1,M_d\mid\pi}\).

• We obtain subgoal \(s'_1.mstc \approx_{\pi} s'_2.mstc\) by Lemma 130.

• We would like to prove \(s'_1.stk \approx_{\pi} s'_2.stk\).

This is immediate by Lemma 139

• The equalities \(s'_1.nalloc = s'_2.nalloc\) and \(\varsigma' = \varsigma'_2\) follow immediately by substitution and the equalities of Proposition (10).

• All subgoals are proved.

5. Case \text{cinvoke-silent-context}:

We obtain the precondition \(s_1.pcc \not\in \text{dom}(\pi,M_c),\) which immediately contradicts conjunct \(s_1.pcc \subseteq \text{dom}(\pi,M_c)\) of Proposition (9).

So, any goal is provable.

6. Case \text{return-silent-compiled}:

• We obtain the precondition \(s_1.M_c(s_1.pcc) = \text{Creturn},\) so by Proposition (11), we have \(s_2.M_c(s_2.pcc) = \text{Creturn}\).

Also, by \(s_1.pcc = s_2.pcc\) of Proposition (10), we know that the precondition \(s_2.pcc \in \text{dom}(\pi,M_c)\) holds.

Now, we have the precondition \(s'_1.pcc \in \text{dom}(\pi,M_c),\) and we argue that \(s'_2.pcc \in \text{dom}(\pi,M_c)\) holds. But first, we show \(s'_2\) exists.

In particular, we argue that \(s_2 \rightarrow s'_2\) using rule \text{return}.

• For that, we need to ensure that the precondition
\[
s'_2.stk, (s'_2.ddc, s'_2.pcc, _, _) = \text{pop}(s_2.stk)
\]
holds, i.e., we need to show that the computation \(\text{pop}(s_2.stk)\) is not stuck.

• We know by \(s_1 \rightarrow s'_1\) that \(s_1.stk \neq \text{nll}\).

• For showing non-stuckness of \(\text{pop}(s_2.stk)\), we use conjunct \(s_1.stk \approx_{\pi} s_2.stk\) of Proposition (10), where by unfolding Definition 82, we have by \(s'_1.pcc \in \text{dom}(\pi,M_c)\) that
\[
\text{top}(s_1.stk) = \text{top}(s_2.stk) = (s'_1.ddc, s'_1.pcc, _, _).
\]

• The above suffices to prove that \(s_2 \rightarrow s'_2\) using rule \text{return}, and that \(s'_2.ddc = s'_1.ddc, s'_2.stc = s'_1.stc,\) and \(s'_2.pcc = s'_1.pcc\).

• It is also immediate by substitution and transitivity of equality that \(s'_2.nalloc = s'_1.nalloc\)

• Thus, this, together with the precondition \(s'_1.pcc \in \text{dom}(\pi,M_c)\) give us that \(s'_2.pcc \in \text{dom}(\pi,M_c)\).

So, the only rule possibly-applicable to \(s_2, \varsigma_2 \xrightarrow{\lambda'} s'_2, \varsigma'_2\) is \text{return-silent-compiled}. So \(\lambda' = \lambda = \tau\).

• And thus, we have \(\varsigma'_2 = \varsigma'_1\).

• Thus, it remains to show that \(s'_1.stk \approx_{\pi} s'_2.stk, s'_1.mstc \approx_{\pi} s'_2.mstc,\) and \(s'_1.M_d|\tau = s'_2.M_d|\tau'.\)

• The former follows by obtaining from \(s_1.stk \approx_{\pi} s_2.stk\) the isomorphism \(f\) by unfolding Definition 82.

This is immediate by instantiating Lemma 141.
• For $s_1'.mstc \approx_\pi s_2'.mstc$ we notice that the definition of $off' = off - nArgs - nLocal$ is the same in both $s_1 \rightarrow s_1'$ and $s_2 \rightarrow s_2'$ (by in-turn the similarity of the definitions of $off$, $nArgs$ and $nLocal$).

And thus, by Lemma 130, we have that $s_1'.mstc \approx_\pi s_2'.mstc$.

• Conject $s_1'.M_d|_r = s_2'.M_d|_r$ follows immediately by $s_1.M_d|_r = s_2.M_d|_r$ of Proposition (10) and substitution.

Also, notice that $r = r'$. Thus, subgoal $s_2'.M_d|_r' = s_1'.M_d|_r'$ follows by substitution.

• This concludes our case.

7. Case creturn-silent-context:

We obtain the precondition $s_1.pcc \not\subseteq dom(\pi.M_c)$, which immediately contradicts conjunct $s_1.pcc \subseteq dom(\pi.M_c)$ of Proposition (9).

So, any goal is provable.

This concludes all cases for $s_1, s_1' \sim_\pi s_1, s_1'$, which concludes the proof of Lemma 143.

\[ \square \]

**Corollary 11** (Star silent actions on strongly-similar states satisfy simulation).

\[ \forall \pi, t_1, s_1, s_2, s_1', s_2', \ldots, s_1, s_2 \in \text{range}(\pi) \}

\[ t_1 \triangleright_\pi \text{exec} s_1 \}

\[ t_2 \triangleright_\pi \text{exec} s_2 \}

\[ s_1, s_1' \approx_\pi s_2, s_2' \}

\[ s_1, s_1' \text{exec} s_1', s_2', s_2' \}

\[ \exists s_2', s_2'. \}

\[ s_2, s_2' \approx_\pi s_1, s_1' \}

\[ s_1, s_1' \approx_\pi s_2, s_2' \}

\[ Proof. \text{ Follows from Lemma 143 and claim 14 and Corollary 2.} \]

\[ \square \]

**Lemma 144** (Strong state-similarity determines non-silent output actions and is weakened by them).

\[ \forall \pi, t_1, s_1, s_2, s_1', s_2' \}

\[ t_1 \triangleright_\pi \text{silent} s_1, s, r_1, na, M_d \}

\[ t_2 \triangleright_\pi \text{silent} s_2, s, r_2, na, M_d \}

\[ s_1, s \approx_\pi s_2, s \}

\[ s_1, s \text{exec} s_1', s_2', s_2' \}

\[ \lambda \in ! \}

\[ \implies \exists s_2'. \}

\[ s_2, s \approx_\pi s_2', s_2' \}

\[ s_1, s \approx_\pi s_2, s_2' \]
Proof. We fix arbitrary \( \tau, t_1, s_1, \varsigma, t_2, s_2, s_1', \varsigma' \), and assume the antecedent:

\[
\begin{align*}
& t_1 \models_\tau \text{exec} s_1 \land t_2 \models_\tau \text{exec} s_2 \\
& \land s_1, \text{pcc} \in \text{dom}(\tau, M_c) \\
& \land s_1, \varsigma \approx s_2, \varsigma \land s_1, \varsigma \vdash s_1', \varsigma' \land \lambda \in !
\end{align*}
\]

(12)

From conjunct \( s_1, \varsigma \approx s_2, \varsigma \) of Proposition (12) and by Definition 84, we have (after substituting \( s_1, \text{pcc} \in \text{dom}(\tau, M_c) \) of Proposition (12) in Definition 80):

\[
r = \bigcup_{\text{mid} \in \text{dom}(\tau, \text{imp})} \text{reachable_addresses}(\{s_1, \text{mstc}(\text{mid}), \tau, \text{imp}(\text{mid}), \text{ddc}\}, s_1, M_d)
\]

By substituting \( s_1, \text{pcc} = s_2, \text{pcc} \) of Proposition (13) in conjunct \( s_1, \text{pcc} \in \text{dom}(\tau, M_c) \) of Proposition (12), we get:

\[
s_2, \text{pcc} \in \text{dom}(\tau, M_c)
\]

(14)

But from conjuncts \( t_1 \models_\tau \text{exec} s_1 \land t_2 \models_\tau \text{exec} s_2 \) of Proposition (12), we know by rules valid-linking and exec-state (after inversion using Silent-state invariant) that:

\[
s_1, M_c = t_1, M_c \uplus \tau, M_c
\]

(15)

and

\[
s_2, M_c = t_2, M_c \uplus \tau, M_c
\]

(16)

respectively.

So, we obtain that \( s_1, M_c(s_1, \text{pcc}) = \tau, M_c(s_1, \text{pcc}) \) by Propositions (12) and (15);

thus \( \tau, M_c(s_1, \text{pcc}) = \tau, M_c(s_2, \text{pcc}) \) by \( s_1, \text{pcc} = s_2, \text{pcc} \) of Proposition (13);

thus \( \tau, M_c(s_2, \text{pcc}) = s_2, M_c(s_2, \text{pcc}) \) by Propositions (14) and (16);

thus by transitivity, we obtain:

\[
s_1, M_c(s_1, \text{pcc}) = s_2, M_c(s_2, \text{pcc})
\]

(17)

We then show our goal \( \exists s_2', s_2, \varsigma \in \pi s_1', \varsigma' \land s_1', \varsigma' \approx s_2', \varsigma' \). The second conjunct unfolds by Definition 84 into:

\[
r' = \rho(\tau)(s_1', \varsigma') = \rho(\tau)(s_2', \varsigma') \land s_1', \text{stk} \approx s_2', \text{stk} \land s_1', \text{mstc} \approx s_2', \text{mstc} \\
\land s_1', \text{imp} = s_2', \text{imp} \land s_1', \phi = s_2', \phi \land s_1', M_d|r' = s_2', M_d|r'
\]

The proof is by considering all the possible cases of the rule \( s_1, s_1' \approx s_2, s_2' \) subject to \( \lambda \in !: \)

1. Case cinvoke-compiled-to-context:

- In this case, we obtain the precondition \( s_1, M_c(s_1, \text{pcc}) = \text{Cinvoke mid fid } \tau \) from which by Proposition (17), we know \( s_2, M_c(s_2, \text{pcc}) = \text{Cinvoke mid fid } \tau. \)
- We also obtain the precondition \( s_1 \supseteq s_1' \), and we would like to conclude \( s_2 \supseteq s_2' \). So by rule cinvoke-aux, we want to show that all the preconditions on \( s_2 \) that are necessary for \( s_2 \supseteq s_2' \) are satisfied.

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• In particular, we have to verify that \((\mid, \fid) \in \dom(s_2, \phi)\), but this follows immediately from \((\mid, \fid) \in \dom(s_1, \phi)\) by conjunct \(s_1, \phi = s_2, \phi\) of Proposition (13).

• We also have to verify that \(\mid \in \dom(s_2, \imp)\), but this follows immediately from \(\mid \in \dom(s_1, \imp)\) by conjunct \(s_1, \imp = s_2, \imp\) of Proposition (13).

• We also have to verify that \(\mid \in \dom(s_2, \mstc)\), but this follows immediately by inverting conjunct \(\vdash_{\text{exec}} s_2\) of Proposition (12) using rule \text{exec-state} and by knowing \(\mid \in \dom(s_2, \imp)\) (the latter we just obtained).

• Finally, in order to show \(s_2 \models s_2\), we need to verify that \(\forall i \in [0, n\text{Args}]. \tau(i), s_2, M_d, s_2, \ddc, s_2, \stc, s_2, \pcc \downarrow v_i\). This follows by Lemma 133, since we already know that:

\[
\forall i \in [0, n\text{Args}]. \tau(i), s_1, M_d, s_1, \ddc, s_1, \stc, s_1, \pcc \downarrow v_i.
\]

• Having satisfied all the possibly-unsatisfiable preconditions of \text{cinvoke-aux}, we know \(\exists s_2\), such that \(s_2 \models s_2\).

• Conjuncts \(s_1, \imp = s_2, \imp \) and \(s_1, \phi = s_2, \phi\) of our goal follow by Lemma 2 and by substitution using the corresponding conjuncts of Proposition (13).

\[
\text{Conjunct } s_1, \mstc \approx s_2, \mstc \text{ follows immediately from } s_1, \mstc \approx s_2, \mstc \text{ by the pre-condition } \mid \notin \dom(\tau, \imp).
\]

• Conjunct \(s_1, \stk \approx s_2, \stk\) follows by instantiating Lemma 139 then Lemma 138.

• For proving conjunct \(\xi_1 = \xi_2\) of our goal, we have the following obligation:

\[
\text{reachable_addresses}_\text{closure}(\xi_1 \cup r_1, s_1', M_d) = \text{reachable_addresses}_\text{closure}(\xi_2 \cup r_2, s_2', M_d)
\]

where:

\[
\begin{align*}
r_1 &= \text{reachable_addresses}(\{\tau(i) \mid i \in [0, n\text{Args}] \land \tau(i) = (\delta, \_ , \_ , \_ )\}, s_1', M_d), \\
r_2 &= \text{reachable_addresses}(\{\tau(i) \mid i \in [0, n\text{Args}] \land \tau(i) = (\delta, \_ , \_ , \_ )\}, s_2', M_d).
\end{align*}
\]

(By Lemma 133, we were able to use the same values \(\tau\) for both \(s_1 \rightarrow s'_1\) and \(s_2 \rightarrow s'_2\).)

• By conjunct \(\xi_1 = \xi_2\) of Proposition (13), our subgoal becomes:

\[
\text{reachable_addresses}_\text{closure}(\xi_1 \cup r_1, s_1', M_d) = \text{reachable_addresses}_\text{closure}(\xi_2 \cup r_2, s_2', M_d)
\]

• Now, we argue that \(r_1 = r_2\).

We first notice that by Lemma 25, we have that:

\[
\forall i \in [0, n\text{Args}]. \tau(i) = (\delta, \sigma, e, \_ ) \implies |s, e| \subseteq \text{reachable_addresses}(\{s_1, \stc, s_1, \ddc\}, s_1, M_d).
\]

By rule \text{cinvoke-aux}, we would like to show that:

\[
\text{reachable_addresses}(\{\tau(i) \mid i \in [0, n\text{Args}] \land \tau(i) = (\delta, \_ , \_ , \_ )\}, s_1, M_d|\off + 1 + i \rightarrow v_i \forall i \in [0, n\text{Args}] |\off + 1 + n\text{Args} + i \rightarrow 0 \forall i \in [0, n\text{Local}]) = \text{reachable_addresses}(\{\tau(i) \mid i \in [0, n\text{Args}] \land \tau(i) = (\delta, \_ , \_ , \_ )\}, s_2, M_d|\off + 1 + i \rightarrow v_i \forall i \in [0, n\text{Args}] |\off + 2 + n\text{Args} + i \rightarrow 0 \forall i \in [0, n\text{Local}]).
\]

(Sketch) By relying on inverting our assumptions (twice) using rule \text{Silent-state invariant}, we should obtain facts that enable us to simply apply Lemma 21 \(n\text{Args} + n\text{Local}\) many times to each side of the goal, then we obtain the equivalent goal:

\[
\text{reachable_addresses}(\{\tau(i) \mid i \in [0, n\text{Args}] \land \tau(i) = (\delta, \_ , \_ , \_ )\}, s_1, M_d) = \text{reachable_addresses}(\{\tau(i) \mid i \in [0, n\text{Args}] \land \tau(i) = (\delta, \_ , \_ , \_ )\}, s_2, M_d).
\]

(Sketch) By completeness of reachable addresses (Lemma 25), and again by invariance to unreachable memory (Lemma 21), we can satisfy this goal from \(s_1, M_d|_r = s_2, M_d|_r\) of Proposition (13).

• Moreover, observe that \(\xi \cup r_1 \subseteq r\), and hence the same for \(\xi \cup r_2\).

Thus, our subgoal above follows by instantiating Lemma 29 using Proposition (13).

• For proving conjunct \(\rho_{\tau}(\xi_1, \xi') = \rho_{\tau}(\xi_2, \xi')\) of our goal, we conclude from rule \text{valid-linking} that \(s_1, \pcc \notin \dom(\tau, M_c)\) and \(s_2, \pcc \notin \dom(\tau, M_c)\).
This gives us by Definition 80 the following obligation:

\[
\bigcup_{\text{mid} \in \text{dom}(\tau.\text{imp})} \text{reachable_addresses}(\{s'_1.\text{mstc}(\text{mid}), \tau.\text{imp}(\text{mid}).\text{ddc}\}, s'_1.\text{M}_d) \setminus \varsigma' = \\
\bigcup_{\text{mid} \in \text{dom}(\tau.\text{imp})} \text{reachable_addresses}(\{s'_2.\text{mstc}(\text{mid}), \tau.\text{imp}(\text{mid}).\text{ddc}\}, s'_2.\text{M}_d) \setminus \varsigma'.
\]

- By conjunct \(s'_1.\text{mstc} \approx_{[\tau]} s'_2.\text{mstc}\) of our goal that we already obtained above, and by noticing the condition \(\text{mid} \in \text{dom}(\tau.\text{imp})\) on the expressions \(s'_1.\text{mstc}(\text{mid})\) and \(s'_2.\text{mstc}(\text{mid})\), our subgoal is equivalent to:

\[
\bigcup_{\text{mid} \in \text{dom}(\tau.\text{imp})} \text{reachable_addresses}(\{s'_1.\text{mstc}(\text{mid}), \tau.\text{imp}(\text{mid}).\text{ddc}\}, s'_1.\text{M}_d) \setminus \varsigma' = \\
\bigcup_{\text{mid} \in \text{dom}(\tau.\text{imp})} \text{reachable_addresses}(\{s'_2.\text{mstc}(\text{mid}), \tau.\text{imp}(\text{mid}).\text{ddc}\}, s'_2.\text{M}_d) \setminus \varsigma'.
\]

(Sketch) This should follow by easy substitutions after relying on the assumptions we get by inverting (twice) the antecedents using rule Silent-state invariant.

2. Case \text{creturn-to-context}: (Sketch) Similar to the previous case; except the subgoal about stack similarity relies on instantiating Lemma 140.

\[\square\]

Lemma 145 (Option simulation: preservation of stack similarity by a silent action).

\[
\forall \tau, t_1, s_1, s_1', t_2, s_2, s_1', s_1'. \\
t_1 \prec \tau \vdash_{\text{exec}} s_1 \land \\
t_2 \prec \tau \vdash_{\text{exec}} s_2 \land \\
s_1.\text{pcc} \cap \text{dom}(\tau.\text{M}_c) = \emptyset \land \\
s_1.\text{stk} \sim_{[\tau]} s_2.\text{stk} \land \\
s_1, s_1' \xrightarrow{\tau} s_1', s_1' \\
\implies s_1'.\text{stk} \sim_{[\tau]} s_2.\text{stk}
\]

Proof.
We assume the antecedents.

By unfolding the assumptions using Definition 83, we obtain \(f\) with:

We prove our goal by induction:

- **Case \text{trace-closure-refl}:**
  Here, the goal is immediate by assumption.

- **Case \text{trace-closure-trans}:**
  Here, we know:

  (S1-STAR-STEPS-S1”):
  \(s_1, s_1' \xrightarrow{\tau} s_1', s_1'\)

  (S1”-STEPS-S1”):
  \(s_1', s_1' \xrightarrow{\tau} s_1', s_1'\)

  And by the induction hypothesis, we know:

  (S1”-STK-SIM-S2-STK):
  \(s_1'.\text{stk} \approx_{[\tau]} s_2.\text{stk}\)
By instantiation of Corollary 7 (twice), we know:

\[ s'_1, \text{pcc} \cap \text{dom}(\bar{\pi}.M_c) = \emptyset \]

and

\[ s'_1.\text{pcc} \cap \text{dom}(\bar{\pi}.M_c) = \emptyset \]

To prove our goal, we distinguish the following cases of (S1”-STEPS-S1’):

- **Case assign-silent**, 
- **Case alloc-silent**, and 
- **Case jump-silent**:
  In these cases, picking the obtained \( f \) suffices to prove our goal, and the frame relatedness condition holds by assumption after substitution using \( s'_1.stk = s''_1.stk \).

- **Case cinvoke-silent-context**:
  Here, again we pick \( f' := f \).

We have \( s'_1.stk = s''_1.stk + [frame] \) where \( \text{frame.pcc} \notin \text{dom}(\bar{\pi}.M_c) \).

Thus, we obtain by the required condition on \( \text{dom}(f') \) from Definition 83 the subgoal:

\[ \text{length}(s'_1.stk) - 1 \notin \text{dom}(f') \]

That is immediate by the choice that \( f' = f \) (unfolding Definition 83).

The remaining conditions about \( f' \) from Definition 83 are also immediate by the choice that \( f' = f \).

- **Case creturn-silent-context**:
  Here, again we pick \( f' := f \).

The subgoals from Definition 83 about \( \text{dom}(f') \) and \( \text{range}(f') \) are immediate by noticing that:

\[ s''_1.stk = s'_1.stk + [frame] \] where \( \text{frame.pcc} \notin \text{dom}(\bar{\pi}.M_c) \).

The remaining conditions about \( f' \) from Definition 83 are immediate by the choice that \( f' = f \).

The remaining cases are impossible.

This concludes the proof of Lemma 145. \( \square \)

**Lemma 146** (Option simulation: preservation of \( mstc \) similarity by a silent action).

\[
\forall \bar{\pi}, t_1, s_1, \varsigma_1, t_2, s_2, \varsigma_2, s'_1, s'_1. \\
t_1 \not\vdash_{\text{exec}} s_1 \land \\
t_2 \not\vdash_{\text{exec}} s_2 \land \\
s_1, \text{pcc} \cap \text{dom}(\bar{\pi}.M_c) = \emptyset \land \\
s_1, \text{mstc} \approx_\bar{\pi} \mid s_2 \text{mstc} \land \\
s_1, s_1 \xrightarrow{\tau} s'_1, s'_1 \land \\
\implies \quad s'_1, \text{mstc} \approx_\bar{\pi} \mid s_2 \text{mstc}
\]

**Proof.**

We assume the antecedents.

By unfolding the assumptions using Definition 81, we obtain:

\[
\forall \text{mid}, \text{mid} \in \text{dom}(\bar{\pi}.\text{imp}) \implies s_1.\text{mstc}(\text{mid}) = s_2.\text{mstc}(\text{mid})
\]

We prove our goal by induction:
• Case **trace-closure-refl**:  
Here, the goal is immediate by assumption.

• Case **trace-closure-trans**:  
Here, we know:

(S1-STAR-STEPS-S1"):
\[ s_1, \varsigma_1 \xrightarrow{t} c s_1'' \]

(S1"-STEPS-S1"):
\[ s''_1, \varsigma_1' \xrightarrow{t} c s''_1' \]

And by the induction hypothesis, we know:

(S1"-MSTC-SIM-S2-STK):
\[ s''_1.mstc \approx s_2.mstc, \]

By instantiation of Corollary 7 (twice), we know:
\[ s''_1.pcc \cap \text{dom}(\tau.M_c) = \emptyset \]

and
\[ s'_1.pcc \cap \text{dom}(\tau.M_c) = \emptyset \]

To prove our goal (\( \forall \text{mid}. \text{mid} \in \text{dom}(\tau.imp) \implies s'_1.mstc(\text{mid}) = s_2.mstc(\text{mid}) \)), we distinguish the following cases of (S1"-STEPS-S1"):

– **Case assign-silent**,  
– **Case alloc-silent**, and  
– **Case jump-silent**:  
Here, our goal is immediate from the assumption after substitution using \( s'_1.mstc = s''_1.mstc \).

– **Case cinvoke-silent-context**:  
Here, by the preconditions and by inversion using cinvoke and cinvoke-aux we have:
\[ s'_1.mstc = s''_1.mstc[\text{mid} \mapsto \_] \]
where
\[ \text{mid} \notin \text{dom}(\tau.imp) \]
Thus, our goal follows from (S1"-MSTC-SIM-S2-STK).

– **Case creturn-silent-context**:  
Here, by the preconditions and by inversion using creturn, we have:
\[ s'_1.mstc = s''_1.mstc[\text{modID} \mapsto \_] \]
where
\[ \text{modID} = \text{top}(s''_1.stk).\text{mid} \]
It suffices for our goal to show:
\[ \text{modID} \notin \text{dom}(\tau.imp) \]
By rule exec-state, it suffices to show the following two subgoals:

* \( t_1 \vdash_{\text{exec}} s''_1 \)
Here, apply Corollary 2 obtaining the following subgoals:

• \( t_1 \vdash_{\text{exec}} s_1 \)  
  Immediate by assumption.
• \( s_1 \rightarrow^* s''_1 \)
Here, apply Claim 15 obtaining a subgoal that is immediate by (S1-STAR-STEPS-S1").
This follows from the obtained preconditions of rule \texttt{return-silent-context} and by inversion of the previous subgoal using \texttt{exec-state}.

The remaining cases are impossible.

This concludes the proof of Lemma 146.

\begin{lemma} [Option simulation: preservation of weak similarity by a silent action] \label{lem:option-simulation}
\begin{align*}
\forall \tau, t_1, s_1, s_1', t_2, s_2, s_2', \varsigma_1, \varsigma_1', M_{\text{border}}, M_c, \text{na}_{\text{border}}, r_{t_1}, r_{t_2}, \\
& s_1.pcc \cap \text{dom}(\tau, M_c) = \emptyset \land \\
& t_1 \not\vdash_{\text{silent}} s_1, \varsigma_1, s_1', \text{na}_{\text{border}}, M_{\text{border}} \land \\
& t_2 \not\vdash_{\text{silent}} s_2, \varsigma_2, s_2', \text{na}_{\text{border}}, M_{\text{border}} \land \\
& s_1, s_1' \sim_\tau [\varsigma_1], \text{dom}(M_{\text{border}}) \land \\
& s_2, s_2' \sim_\tau [\varsigma_2], \text{dom}(M_{\text{border}}) \\
\implies \\
& s_1', s_1' \sim_\tau [\varsigma_1], \text{dom}(M_{\text{border}}) \land \\
& s_2, s_2'
\end{align*}
\end{lemma}

\textit{Proof.}
We assume the antecedents.

By instantiating Lemma 152, we additionally obtain:
\[
t_1 \not\vdash_{\text{silent}} s_1', s_1', \text{na}_{\text{border}}, M_{\text{border}}
\]

By unfolding the assumptions using Definition 84, and by inversion using rule \texttt{Silent-state invariant}, we obtain:

\begin{itemize}
  \item \texttt{EXEC-1}
  \begin{align*}
  t_1 \not\vdash_{\text{exec}} s_1
  \end{align*}
  \item \texttt{EXEC-2}
  \begin{align*}
  t_2 \not\vdash_{\text{exec}} s_2
  \end{align*}
  \item \texttt{TAU-STEPS-1}
  \begin{align*}
  s_1, s_1' \vdash_{\text{tau}} s_1', s_1'
  \end{align*}
  \item \texttt{PCC-1-NOT-C}
  \begin{align*}
  s_1.pcc \cap \text{dom}(\tau, M_c) = \emptyset
  \end{align*}
  \item \texttt{PCC-2-NOT-C}
  \begin{align*}
  s_2.pcc \cap \text{dom}(\tau, M_c) = \emptyset
  \end{align*}
  \item \texttt{STK-SIM}
  \begin{align*}
  s_1.s\text{tk} \sim_\tau s_2.s\text{tk}
  \end{align*}
  \item \texttt{MSTC-SIM}
  \begin{align*}
  s_1.mstc \equiv_\tau s_2.mstc
  \end{align*}
  \item \texttt{VARSIGMA-EQ}
  \begin{align*}
  \varsigma_1 = \varsigma_2
  \end{align*}
  \item \texttt{PRIVATE-MEM-EQ}
  \begin{align*}
  s_1.M_d|\text{dom}(M_{\text{border}}) = s_2.M_d|\text{dom}(M_{\text{border}})
  \end{align*}
\end{itemize}
PRIVATE-MEM-S1-IS-MBORDER
\[ s_{1} \cdot M_{d} | \text{dom}(M_{\text{border}}) = M_{\text{border}} \]

PRIVATE-MEM-S1'-IS-MBORDER
\[ s'_{1} \cdot M_{d} | \text{dom}(M_{\text{border}}) = M_{\text{border}} \]

Our goal is \( s'_{1}, s_{1} \sim_{\cdot} \text{dom}(M_{\text{border}}) s_{2}, s_{2} \).

By unfolding it using Definition 84, we obtain the following subgoals:

1. \( s'_{1}.pcc \cap \text{dom}(\pi_{M_{c}}) = \emptyset \)
   - Follows by instantiating Corollary 7 using assumptions (EXEC-1) and (TAU-STEPS-1) then substitution using assumption (PCC-1-NOT-C).

2. \( s'_{1}.stk \sim_{\cdot} s_{2}.stk \)
   - Follows by applying Lemma 145 obtaining subgoals that are immediate by assumptions (EXEC-1), (EXEC-2), (EXEC-2), (STK-SIM), and (PCC-1-NOT-C).

3. \( s'_{1}.mstc \approx_{\cdot} s_{2}.mstc \)
   - Follows by applying Lemma 146 obtaining subgoals that are immediate by assumptions (EXEC-1), (EXEC-2), (EXEC-2), (STK-SIM), and (PCC-1-NOT-C).

4. \( s'_{1} = s_{2} \)
   - Follows by instantiating Claim 17 using assumption (TAU-STEPS-1) then substitution using assumption (VARSIGMA-EQ).

5. \( s'_{1}.M_{d} | \text{dom}(M_{\text{border}}) = s_{2}.M_{d} | \text{dom}(M_{\text{border}}) \)
   - Immediate by substitution using assumptions (PRIVATE-MEM-S1-IS-MBORDER) then (PRIVATE-MEM-S1'-IS-MBORDER) in assumption (PRIVATE-MEM-EQ).

This concludes the proof of Lemma 147.

Lemma 148 (Matching input actions retrieve back strong state-similarity).

\[ \forall \pi, t_{1}, s_{1}, \varsigma, t_{2}, s_{2}, s'_{1}, s'_{2}, M_{\text{border}}, na_{\text{border}}, r_{t_{1}}, r_{t_{2}}, \]
\[ s_{1}.pcc \cap \text{dom}(\pi_{M_{c}}) = \emptyset \land \]
\[ t_{1} \times \pi \vdash \text{silent} s_{1}, \varsigma, _, r_{t_{1}}, na_{\text{border}}, M_{\text{border}} \land \]
\[ t_{2} \times \pi \vdash \text{silent} s_{2}, \varsigma, _, r_{t_{2}}, na_{\text{border}}, M_{\text{border}} \land \]
\[ s_{1}, \varsigma \sim_{\cdot} \text{dom}(M_{\text{border}}) s_{2}, \varsigma \land \]
\[ s_{1}, \varsigma \xrightarrow{\lambda} s'_{1}, s'_{1} \land \]
\[ s_{2}, \varsigma \xrightarrow{\lambda} s'_{2}, s'_{2} \land \]
\[ \lambda \in \bullet \]
\[ \implies \]
\[ s'_{1}, s'_{1} \approx_{\cdot} s'_{2}, s'_{2} \]

Proof.
(Sketch)
After unfolding using Definition 84 and inversion using rule Silent-state invariant, we proceed by case distinction on the step \( s_{1}, \varsigma_{1} \xrightarrow{\lambda} s'_{1}, \varsigma'_{1} \).

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Definition 85 (Per-subject state-universal predicate). A predicate $P : \mathcal{V} \to \mathbb{B}$ holds universally for
all values of a program state \(s\) where \(t\) is the subject of \(s\) when:

\[
\text{per}\_\text{subject}\_\text{state}\_\text{universal}(P, s, t) \overset{\text{def}}{=} \\
\text{s.pcc} \subseteq \text{dom}(t.\mathcal{M}_c) \land \\
\forall a. a \in \bigcup_{\text{mid} \in \text{dom}(t.\text{imp})} \text{reachable}\_\text{addresses}(\{\text{s.mstc(mid), t.imp(mid).ddc}, \text{s.M}_d\}) \implies P(\text{s.M}_d(a)) \\
\land \\
P(s.\text{ddc}) \land P(s.\text{stc}) \land P(s.\text{pcc}) \land \\
\forall \text{mid} \in \text{dom}(t.\text{imp}). P(s.\text{imp(mid).pcc}) \land P(s.\text{imp(mid).ddc}) \land P(s.\text{mstc(mid)}) \land \\
\forall (cc, dc, \_ , \_ ) \in s.\text{stk}. \text{cc} \subseteq \text{dom}(t.\mathcal{M}_c) \implies P(cc) \land P(dc)
\]

**Lemma 149** (Predicates that are guaranteed to hold on the result of expression evaluation under the execution of a specific subject).

\[
\forall t, t_1, t_2, \mathcal{E}, s, v. \\
\mathcal{E}, s.\mathcal{M}_d, s.\text{ddc}, s.\text{stc}, s.\text{pcc} \downarrow v \land \\
t \in \{t_1, t_2\} \land \\
t_1 \prec t_2 \vdash_{\text{exec}} s \land \\
\text{per}\_\text{subject}\_\text{state}\_\text{universal}(P, s, t) \land \\
\text{offset}\_\text{oblivious}(P) \land \\
\text{z}\_\text{trivial}(P) \land \\
\text{subcap}\_\text{closed}(P) \\
\implies P(v)
\]

**Proof.** Similar to Lemma 44.

**Lemma 150** (Preservation of per-subject state universality of predicates).

\[
\forall P, t, t_{ctx}, \overline{c}, s, s', \nabla. \\
\text{s.nalloc} < 0 \land \\
t \in \{t_{ctx}, \overline{c}\} \land \\
t_{ctx} \prec \overline{c} \vdash_{\text{exec}} s \land \\
\text{per}\_\text{subject}\_\text{state}\_\text{universal}(P, s, t) \land \\
\text{allocation}\_\text{compatible}(P, s'.\text{nalloc} - 1) \land \\
\text{offset}\_\text{oblivious}(P) \land \\
\text{z}\_\text{trivial}(P) \land \\
\text{subcap}\_\text{closed}(P) \land \\
s, \varsigma \xrightarrow{\pi^*} s', \varsigma \\
\implies \text{per}\_\text{subject}\_\text{state}\_\text{universal}(P, s', t) \land s'.\text{nalloc} < 0
\]

**Proof.** Similar to Lemma 45.
**Definition 86** (Four-origin policy).

\[
four\_origin\_policy_{t,s,\varsigma,\alpha}(v) \overset{\text{def}}{=} \]

\[
v = (\delta, \sigma, e, _) \Rightarrow \exists mid \in \text{dom}(t.\text{imp}). [\sigma, e] \subseteq t.\text{imp}(mid).\text{ddc} \lor \exists mid \in \text{dom}(t.\text{imp}). [\sigma, e] \subseteq s.\text{mstc}(mid) \lor \exists \alpha' \in \varsigma, \text{idx} \in [0, |\alpha|]. [\sigma, e] \subseteq \text{mem}(\alpha(\text{idx}))(\alpha') \lor \exists i \in \text{allocation\_intervals}(?, \alpha). [\sigma, e] \subseteq i
\]

**Claim 34** (Border state invariant to silent state invariant - \( \tau \) executing).

\[
\begin{align*}
t_{ctx} \vdash_\text{\textit:border} \alpha, s, \varsigma \land 
caps = \{ v \mid four\_origin\_policy_{t_{ctx} \vdash_\tau s, \varsigma, \alpha}(v) \} \land 
rt = \bigcup_{mid \in \text{dom}(t_{ctx}.\text{imp})} \text{reachable\_addresses}\left(\{s.\text{mstc}(mid), t_{ctx}.\text{imp}(mid).\text{ddc}, s.M_d\}\right) 
\end{align*}
\]

\[
\Rightarrow 
\exists M_d, 
t_{ctx} \vdash_\text{\textit:silent} s, \varsigma, caps, rt, s.\text{nalloc}, M_d
\]

(Proof Sketch): Follows from Definition 86 after inversion of rule Border-state invariant.

**Claim 35** (Border state invariant to silent state invariant - \( t_{ctx} \) executing).

\[
\begin{align*}
t_{ctx} \vdash_\tau \text{\textit:border} \alpha, s, \varsigma \land 
caps = \{ v \mid four\_origin\_policy_{t_{ctx} \vdash_\tau s, \varsigma, \alpha}(v) \} \land 
rt = \bigcup_{mid \in \text{dom}(\tau.\text{imp})} \text{reachable\_addresses}\left(\{s.\text{mstc}(mid), \tau.\text{imp}(mid).\text{ddc}, s.M_d\}\right) 
\end{align*}
\]

\[
\Rightarrow 
\exists M_d, 
t_{ctx} \vdash_\tau \text{\textit:silent} s, \varsigma, caps, rt, s.\text{nalloc}, M_d
\]

Similar to Claim 34.
Lemma 151 (Possible origins of capability values at border states).

∀t_ctx, τ, α, s, ς, δ, σ, e.

\( t_{ctx} \times \tau \vdash_{\text{border}} \alpha, s, \varsigma \land \)

\( E, s, M_d, s, \text{ddc}, s, \text{stc}, s, \text{pcc} \downarrow (\delta, \sigma, e, \_ \_ \_ ) \land \)

\( I_{ctx} = \text{allocation\_intervals}(\_ \_ \_ , \alpha ) \land \)

\( I_{\tau} = \text{allocation\_intervals}(! \_ \_ \_ , \alpha ) \land \)

\( \implies \)

\( s, \text{pcc} \subseteq \text{dom}(\tau, M_c) \land \)

(\( \exists i \in I_{\tau}, [\sigma, e] \subseteq i \lor \)

\( \exists a' \in \varsigma, idx \in [0, |\alpha|], [\sigma, e] \subseteq \text{mem}(\alpha(idx))(a') \lor \)

\( \exists \midmid \in \text{dom}(\tau, \text{imp}), [\sigma, e] \subseteq \tau, \text{imp}(\midmid).\text{ddc} \lor \)

\( \exists \midmid \in \text{dom}(\tau, \text{imp}), [\sigma, e] \subseteq \tau, \text{mstc}(\midmid) \)

\( \lor \)

\( s, \text{pcc} \subseteq \text{dom}(t_{ctx}, M_c) \land \)

(\( \exists i \in I_{\tau}, [\sigma, e] \subseteq i \lor \)

\( \exists a' \in \varsigma, idx \in [0, |\alpha|], [\sigma, e] \subseteq \text{mem}(\alpha(idx))(a') \lor \)

\( \exists \midmid \in \text{dom}(t_{ctx}, \text{imp}), [\sigma, e] \subseteq t_{ctx}, \text{imp}(\midmid).\text{ddc} \lor \)

\( \exists \midmid \in \text{dom}(t_{ctx}, \text{imp}), [\sigma, e] \subseteq t_{ctx}, \text{mstc}(\midmid) \)

\( 207 \)

Proof.

• We assume the antecedents, and prove our lemma by induction on the evaluation of \( E \).

  – Case evalconst,

  – Case evalCapType,

  – Case evalCapStart,

  – Case evalCapEnd,

  – Case evalCapOff, and

  – Case evalBinOp:

    These cases are vacuous.

  – Case evalddc:

    Here, we distinguish the following two cases:

    * Case \( s, \text{pcc} \subseteq \text{dom}(\tau, M_c) \):

      In this case, we choose to prove the left disjunct of our goal.

      Further, we choose to prove the following disjunct:

      \( \exists \midmid \in \text{dom}(\tau, \text{imp}), [s, \text{ddc}, \sigma, s, \text{ddc}, e] \subseteq \tau, \text{imp}(\midmid).\text{ddc} \)

      Now this latter goal follows by inverting assumption \( t_{ctx} \times \tau \vdash_{\text{border}} \alpha, s, \varsigma \) using rule \text{Border-state invariant}, and then inverting its preconditions using rule \text{exec-state}.

    * Case \( s, \text{pcc} \subseteq \text{dom}(t_{ctx}, M_c) \):

      In this case, we choose to prove the right disjunct of our goal.

      Further, we choose to prove the following disjunct:

      \( \exists \midmid \in \text{dom}(t_{ctx}, \text{imp}), [s, \text{ddc}, \sigma, s, \text{ddc}, e] \subseteq t_{ctx}, \text{imp}(\midmid).\text{ddc} \)

      Now this latter goal follows by inverting assumption \( t_{ctx} \times \tau \vdash_{\text{border}} \alpha, s, \varsigma \) using rule \text{Border-state invariant}, and then inverting its preconditions using rule \text{exec-state}.
– Case evalstc:
Here, we distinguish the following two cases:

* Case \( \mathbb{stc} \subseteq \text{dom}(\mathbb{c}.M_c) \):
In this case, we choose to prove the left disjunct of our goal.
Further, we choose to prove the following disjunct:
\[ \exists \text{mid} \in \text{dom}(\mathbb{c}.\text{imp}). \ [s.\text{ddc}.\sigma, s.\text{ddc}.e] \subseteq \mathbb{c}.\text{mstc}(\text{mid}) \]
Now this latter goal follows by inverting assumption \( t_{ctx} \vdash \text{border } \alpha, \sigma, \varsigma \) using rule \text{Border-state invariant}, and then inverting its preconditions using rule \text{exec-state}.

* Case \( \mathbb{stc} \subseteq \text{dom}(t_{ctx}.M_c) \):
In this case, we choose to prove the right disjunct of our goal.
Further, we choose to prove the following disjunct:
\[ \exists \text{mid} \in \text{dom}(t_{ctx}.\text{imp}). \ [s.\text{ddc}.\sigma, s.\text{ddc}.e] \subseteq t_{ctx}.\text{mstc}(\text{mid}) \]
Now this latter goal follows by inverting assumption \( t_{ctx} \vdash \text{border } \alpha, \sigma, \varsigma \) using rule \text{Border-state invariant}, and then inverting its preconditions using rule \text{exec-state}.

– Case evalIncCap:
Here, \( E = \text{inc}(E_c, E_z) \), and we have the preconditions:

\((\text{Ec-eval)}):
\qquad E_c, s.M_d, s.\text{ddc}, s.\text{stc}, s.\mathbb{pcc} \Downarrow (x, \sigma, e, \text{off}), \) and  

\((\text{Ez-eval)}):
\qquad E_z, s.M_d, s.\text{ddc}, s.\text{stc}, s.\mathbb{pcc} \Downarrow z \)

We distinguish two cases:

* Case \( x = \delta \):
Here, our goal follows immediately from the induction hypothesis on (Ec-eval) after substitution.

* Case \( x \neq \delta \):
Here, our goal is vacuously true.

– Case evalLim:
Here, \( E = \text{inc}(E_c, E_z) \), and we have the preconditions:

\((\text{Ec-eval)}):
\qquad E_c, s.M_d, s.\text{ddc}, s.\text{stc}, s.\mathbb{pcc} \Downarrow (x, \sigma, e, \_), \) and

\((\text{CAP-BOUNDS-SUB)}):
\qquad [\sigma', e'] \subseteq [\sigma, e] \)

We distinguish two cases:

* Case \( x = \delta \):
Here, our goal follows immediately from the induction hypothesis on (Ec-eval) after applying transitivity of \( \subseteq \) using (CAP-BOUNDS-SUB).

* Case \( x \neq \delta \):
Here, our goal is vacuously true.

– Case evalDeref:
Here, \( E = \text{deref}(E_z) \).
We have the following preconditions:

\((\text{Ec-eval)}):
\qquad E_c, s.M_d, s.\text{ddc}, s.\text{stc}, s.\mathbb{pcc} \Downarrow (x, \sigma', e', \_), \)

\((\text{Ec-delta)}):
\qquad \vdash \delta (x, \sigma', e', \_), \) and

\((\text{Mem-deref)}):
\qquad s.M_d(\sigma' + \_\_\_\_\_\_\_\_) = (\delta, \sigma, e, \_\_\_\_\_\_\_\_\_\_) \)

We claim (Bounds-reachable):
\[ [\sigma', e'] \subseteq \text{reachable_addresses}(\{s.\text{stc}, s.\text{ddc}\}, s.M_d) \]
We apply Lemma 25 to this claim to obtain the following subgoals:
Now, we distinguish the following two cases:

- \( s.pcc = (\kappa, -1, -1) \),
- \( s.ddc = (\delta, -1, -1) \), and
- \( s.stc = (\delta, -1, -1) \).

Using (Bounds-reachable) and (Ec-delta)–unfolding Definition 2, we know (Addr-reachable):

\[
\ast \quad \ast s .
\]

All of these follow by inverting assumption \( t_{ctx} \not\vdash border \alpha, s, \varsigma \) using rule Border-state invariant, and then inverting its preconditions using rule exec-state.

\( \mathcal{E}_c, s.M_d, s.ddc, s.stc, s.pcc \downarrow (\delta, \sigma', e', off) \)

Immediate by (Ec-eval) and (Ec-delta).

Using (Bounds-reachable) and (Ec-delta)–unfolding Definition 2, we know (Addr-reachable):

\( \sigma' + off \in \text{reachable_addresses}(\{s.stc, s.ddc\}, s.M_d) \)

Now, we distinguish the following two cases:

- **Case** \( s.pcc \subseteq \text{dom}(\tau.M_c) \):
  
  We choose to prove the left disjunct of our goal.

  Here, we claim (Addr-reachable-all):

  \( \sigma' + off \in \bigcup_{mid \in \text{dom}(\tau.imp)} \text{reachable_addresses}(\{s.mstc(mid), \tau.imp(mid).ddc\}, s.M_d) \)

  We apply Lemma 18 to this claim obtaining the following subgoals:

  - \( \{s.stc, s.ddc\} \subseteq \bigcup_{mid \in \text{dom}(\tau.imp)} \{s.mstc(mid), \tau.\text{imp(mid).ddc}\} \)
    
    This follows by substituting the case condition in the preconditions obtained by inverting assumption \( t_{ctx} \not\vdash border \alpha, s, \varsigma \) using rule Border-state invariant, and then inverting its preconditions using rule exec-state.

  - \( \sigma' + off \in \text{reachable_addresses}(\{s.stc, s.ddc\}, s.M_d) \)
    
    This is immediate by (Addr-reachable).

  We now distinguish two cases:

  - **Case** \( \sigma' + off \in \varsigma \):
    
    Here, we choose to prove the following disjunct of (the necessary top-level left disjunct of) our goal:

    \( \exists a' \in \varsigma, idx \in [0, |\alpha|]. \ [\sigma, e] \subseteq \text{mem}(\alpha(idx))(a') \)

    We pick:

    \( a' := \sigma' + off \), and

    \( idx := |\alpha| - 1 \)

    Thus, it remains to show that:

    \( [\sigma, e] \subseteq \text{mem}(\alpha(|\alpha| - 1))(\sigma' + off) \)

    We apply the substitution:

    \( \text{mem}(\alpha(|\alpha| - 1)) = s.M_d \)

    obtaining the following two subgoals:

    1. \( \text{mem}(\alpha(|\alpha| - 1)) = s.M_d \)
      
      This is immediate by inverting assumption \( t_{ctx} \not\vdash border \alpha, s, \varsigma \) using rule Border-state invariant.

    2. \( [\sigma, e] \subseteq s.M_d(\sigma' + off) \)
      
      Here, we apply reflexivity of \( \subseteq \), so our goal is immediate by (Mem-deref).

  - **Case** \( \sigma' + off \notin \varsigma \):
    
    Here, by inverting assumption \( t_{ctx} \not\vdash border \alpha, s, \varsigma \) using rule Border-state invariant, we obtain the following preconditions:

    (Re-def):

    \( R_e = \bigcup_{mid \in \text{dom}(\tau.imp)} \text{reachable_addresses}(\{s.mstc(mid), \tau.\text{imp(mid).ddc}\}, s.M_d) \),

    and

    (All-privately-held-caps):

    \( \forall a \in R_e \setminus \varsigma \quad s.M_d(a) = (\delta, \sigma, e, _) \implies (\exists i \in I_e \quad [\sigma, e] \subseteq i \lor \)
\[\exists a' \in \varsigma, \text{id}x \in [0, \text{id}x]. \left[ (\sigma, e) \subseteq \text{mem}(\text{id}(\text{id}x))(a') \lor \exists \text{mid} \in \text{dom}(\text{imp}). (\sigma, e) \subseteq \text{imp}(\text{mid}).\text{ddc} \lor \exists \text{mid} \in \text{dom}(\text{imp}). (\sigma, e) \subseteq \text{mstc}(\text{mid}) \right] \]

We instantiate the latter (All-privately-held-caps) with \( a := \sigma' \) obtaining the following two subgoals:
1. \( \sigma' + \text{off} \in R_{c} \)
   By unfolding \( R_{c} \) using (Rc-def), this goal is immediate by (Addr-reachable-all).
2. \( \sigma' + \text{off} \notin \varsigma \)
   This is immediate by the case condition.

The instantiation immediately gives us our goal.

* **Case** \( \text{s.pcc} \subseteq \text{dom}(t_{\text{ctx}}.M_{c}) \):
   We choose to prove the right disjunct of our goal. The proof is analogous to the previous case. We omit it for brevity.

This concludes the proof of *case evalDeref*.

This concludes the proof of Lemma 151. \( \square \)

**Silent-state invariant**

**Lemma 152** (Preservation of the silent-state invariant).

\[
\forall t_{\text{ctx}}, \tau, s, \varsigma, \text{caps}_4\text{origin}, \text{border}, r_t, \text{border}, \text{na}_\text{border}, \text{M}_t, \text{na}_\text{border}, s', \nabla.
\]

\[
t_{\text{ctx}} \not\vdash_{\tau} \text{silent} s, \varsigma, \text{caps}_4\text{origin}, \text{border}, r_t, \text{border}, \text{na}_\text{border}, \text{M}_t, \text{na}_\text{border} \land
\]

\[
s, \varsigma \xleftarrow{s', \varsigma} \nabla s', \varsigma
\]

\[
\implies
t_{\text{ctx}} \not\vdash_{\tau} \text{silent} s', \varsigma, \text{caps}_4\text{origin}, \text{border}, r_t, \text{border}, \text{na}_\text{border}, \text{M}_t, \text{na}_\text{border}
\]

**Proof.**

- We assume the antecedents, and prove our goal by induction on the relation \( s, \varsigma \xleftarrow{s', \varsigma} \nabla s', \varsigma \)
- **Case** trace-closure-refl:
   Here, the goal is immediate by assumption.
- **Case** trace-closure-trans:
   Here, by assumption, we have \( s'' \) with:
   \[
s, \varsigma \xleftarrow{s'', \varsigma} \nabla s'', \varsigma, \text{and}
\]
   \[
s'', \varsigma \xleftarrow{s', \varsigma} \nabla s', \varsigma,
\]
   and the induction hypothesis
   \[
t_{\text{ctx}} \not\vdash_{\tau} \text{silent} s'', \varsigma, \text{caps}_4\text{origin}, \text{border}, r_t, \text{border}, \text{na}_\text{border}, \text{M}_t, \text{na}_\text{border}.
\]
   By inversion of the induction hypothesis using rule Silent-state invariant, we obtain the following assumptions:
   - **Valid linking**:
     \[
t_{\text{ctx}} \not\vdash_{\tau} \text{silent} s'', \varsigma, \text{caps}_4\text{origin}, \text{border}, r_t, \text{border}, \text{na}_\text{border}, \text{M}_t, \text{na}_\text{border}.
\]
   - **Compiled component**:
     \[\bar{\tau} \in \text{range}(\bar{\cdot})\]
Exec state:
\[ t_0 \vdash_{exec} s'' \]

Arbitrary \( t \):
\[ t \in \{ t_{ctx}, \tau \} \]

Arbitrary \( \bar{t} \):
\[ \bar{t} \in \{ t_{ctx}, \tau \} \setminus \{ t \} \]

\( t \) is executing:
\[ s'' \]
\[ pcc \subseteq \text{dom}(t.M_c) \]

Private memory of \( \bar{t} \) is untouched:
\[ \forall a \in \text{dom}(M_{\bar{t}, \text{border}}). M_{\bar{t}, \text{border}}(a) = s''.M_d(a) \]

Private memory was indeed private:
\[ ((-\infty, na_{\text{border}}) \cup r_t) \cap \text{dom}(M_{\bar{t}, \text{border}}) = \emptyset \]

Private memory is compatible with the history of sharing:
\[ \varsigma \cap \text{dom}(M_{\bar{t}, \text{border}}) = \emptyset \]

Reachable addresses of \( t \):
\[ R''_t = \bigcup_{\text{mid} \in \text{dom}(t.imp)} \text{reachable_addresses}\{s''.\text{mstc(mid)}, t.imp(mid).\text{ddc}, s''.M_d\} \]

New allocation is bounded by \( na_{\text{border}} \):
\[ s''.nalloc \leq na_{\text{border}} \]

Reachable addresses of \( t \) can grow only by allocation:
\[ R''_t \subseteq (r_t \cup [s''.nalloc, na_{\text{border}})) \]

The border capabilities contain capabilities on \( t \)'s static memory:
\[ \bigcup_{\text{mid} \in \text{dom}(t.imp)} \{t.\text{mstc(mid)}, t.imp(mid).\text{ddc}\} \subseteq \text{caps}_{\text{origin, border}} \]

Five-origin policy:
\[ \forall a \in R''_t, s''.M_d(a) = (\delta, \sigma, e, \_ ) \implies \exists \text{cap} \in \text{caps}_{\text{origin, border}}, [\sigma, e] \subseteq \text{cap} \lor [\sigma, e] \subseteq [s''.nalloc, na_{\text{border}}] \]

By applying rule Silent-state invariant to our goal, we obtain subgoals about \( s' \) that we refer to using the names given above to the corresponding assumptions:

- Subgoals “Valid linking”, “Compiled component”, “Arbitrary \( t' \)”, “Arbitrary \( \bar{t} \)”, “Private memory was indeed private”, “Private memory is compatible with the history of sharing”, and “The border capabilities contain capabilities on \( t \)'s static memory” are immediate.
- There is nothing to prove about the definition Reachable addresses of \( t \).
- To prove subgoal Exec state, we apply Corollary 2 obtaining the following subgoals:
  * \( t \vdash_{exec} s'' \)
    This is immediate by assumption Exec state.
* $s'' \rightarrow^* s'$

To prove this, we apply Claim 15 obtaining the following subgoal: $s'', \varsigma \xrightarrow{\tau^*} s', \varsigma'$
This is immediate by assumptions after applying rule trace-closure-refl.

- To prove the remaining subgoals, we distinguish the possible cases of assumption $s''', \varsigma \xrightarrow{\tau^*} s', \varsigma'$:
  * **Case assign-silent:**
    By inversion of the assumptions of assign-silent using rule assign, we obtain
    (S'-MEM):
    $s'.M_d = s'''.M_d[c \mapsto v]$,
    (v-EVAL'd-IN-t):
    $E_R, s'''.M_d, s'''.ddc, s'''.stc, s'''.pcc \vdash v$,
    (c-EVAL'd-IN-t):
    $E_L, s'''.M_d, s'''.ddc, s'''.stc, s'''.pcc \vdash c$,
    (c-IN-BOUNDS):
    $\vdash c$,
    (EQUAL-MSTC):
    $s'''.mstc = s'.mstc$, and
    (EQUAL-NALLOC):
    $s'''.nalloc = s'.nalloc$

    We first prove the goal **Reachable addresses of $t$ can grow only by allocation.**

    Assuming $R'_t = \bigcup_{mid \in \text{dom}(t.imp)} \text{reachable_addresses}(\{s'''.mstc(mid), t.imp(mid).ddc\}, s'.M_d)$,
    our goal is $R'_t \subseteq (r_t \cup (s'.nalloc, na_{border})]$.

    By the transitivity of $\subseteq$, it suffices to show that:
    $R'_t \subseteq r_t$.

    We prove our goal by applying transitivity of $\subseteq$ obtaining the following two subgoals:
    1. $R'_t \subseteq r_t$
       Immediate by assumption **Reachable addresses of $t$ can grow only by allocation.**
    2. $R'_t \subseteq R''_t$

    Here, we apply Lemma 38 obtaining the following subgoals:
    (a) $c.\sigma + c.off \in R''_t$

        Here, we apply Lemma 25 obtaining subgoals that are immediate by (c-EVAL'd-IN-t), (c-IN-BOUNDS), and by inversion of assumption **Exec state** using rule exec-state.
    (b) $v = (\delta, \sigma, c, _) \implies \bigcup_{mid \in \text{dom}(t.imp)} \{s'''.mstc(mid), t.imp(mid).ddc\}, s'''.M_d \models v$

        Assuming $v = (\delta, \sigma, c, _) \implies \bigcup_{mid \in \text{dom}(t.imp)} \{s'''.mstc(mid), t.imp(mid).ddc\}, s'''.M_d \models v$

        By applying Lemmas 6 and 18, we obtain the following two subgoals:
        i. $[\sigma, c] \subseteq \text{reachable_addresses}(\{s'''.stc, s'''.ddc\}, s'''.M_d)$

            Here, we apply Lemma 25 obtaining subgoals that are immediate by (v-EVAL'd-IN-t), and by inversion of assumption **Exec state** using rule exec-state.
        ii. $\exists mid \in \text{dom}(t.imp). s'''.mstc(mid) \models s'''.stc$
        iii. $\exists mid \in \text{dom}(t.imp). s'''.imp(mid).ddc \models s'''.ddc$

            These two subgoals are immediate by inverting assumption **Exec state** using rule exec-state and substituting in the preconditions using assumption $\text{t is executing}.$
    (c) (applying Lemma 6) $s'''.mstc \models s'.mstc$

    Immediate by (EQUAL-MSTC).
Next, we prove the goal **Five-origin policy.**

We fix an arbitrary \( a \in R_t' \), and assume \( s'.M_d(a) = (\delta, \sigma, e, \_). \)

Out goal is (after substitution using (EQUAL-NALLOC)):
\[
\exists \text{cap} \in \text{caps}_{\text{ Five-origin, border.}} \quad [\sigma, e] \subseteq \text{cap} \lor [\sigma, e] \subseteq [s''.\text{nalloc, na}\text{\_border})
\]

We distinguish the following two cases:

1. **Case** \( a = c.\sigma + c.\text{off} \):
   
   We instantiate Lemma 26 using (v-EVAL’d-IN-t) and using subgoal **Exec state**
   inverted by rule **exec-state** to obtain
   
   (3-ORIGINS):
   \[
   [\sigma, e] \subseteq s''.\text{ddc} \lor [\sigma, e] \subseteq s''.\text{stc} \lor
   \exists a_o. \ [\sigma, e] \subseteq s''.M_d(a_o) \land a_o \in \text{reachable\_addresses}\{s''.\text{ddc}, s''.\text{stc}\}, s''.M_d)
   \]
   
   We distinguish the following three cases of (3-ORIGINS):
   (a) **Case** \( [\sigma, e] \subseteq s''.\text{ddc} \), and
   (b) **Case** \( [\sigma, e] \subseteq s''.\text{stc} \)
   
   In these two cases, we apply the transitivity of \( \subseteq \) obtaining the subgoals
   \[
   [\sigma, e] \subseteq s''.\text{ddc} \quad \text{and} \quad [\sigma, e] \subseteq s''.\text{stc}
   \]
   respectively.
   
   Both of these subgoals are immediate by the assumption “\( t \) is executing”
   together with the assumption “The border capabilities contain capabilities
   on \( t's \) static memory”.
   (c) **Case** \( \exists a_o. \ [\sigma, e] \subseteq s''.M_d(a_o) \land a_o \in \text{reachable\_addresses}\{s''.\text{ddc}, s''.\text{stc}\}, s''.M_d) \):
   
   Here, we obtain \( a_o \), and use it to instantiate assumption **Five origin policy**
   thus immediately proving our goal.
   
   (The instantiation is possible by Lemma 18.)

2. **Case** \( a \neq c.\sigma + c.\text{off} \):
   
   Here, we apply assumption **Five-origin policy** obtaining the following subgoals:
   (a) \( s''.M_d(a) = (\delta, \sigma, e, \_) \)
   
   Immediate by (S’-MEM).
   (b) \( a \in R_t'' \)
   
   Follows from assumption \( a \in R_t' \) and \( R_t' \subseteq R_t'' \). The latter was proved in the
   previous goal.

Next, we prove the goal **t is executing.**

Immediate from the corresponding assumption by noticing that \( s''.\text{pcc} = s'.\text{pcc} \).

Next, we prove the goal **New allocation is bounded by na\_border.**

This is immediate from the corresponding assumption after substitution using (EQUAL-NALLOC).

Next, we prove the goal **Private memory of \( \_t \) is untouched.**

We pick an arbitrary \( a \in \text{dom}(M_{t,\text{\_border}}) \),

and our goal is to show that \( s'.M_d(a) = M_{t,\text{\_border}}(a) \).

By the corresponding assumption (i.e., assumption **Private memory of \( \_t \) is untouched**) about \( s'' \), it suffices by the transitivity of equality to show that:
\[
s'.M_d(a) = s''.M_d(a)
\]

By (S’-MEM), it thus suffices to show that:
\[
a \neq c.\sigma + c.\text{off} \]

For this, it suffices to show that \( \text{dom}(M_{t,\text{\_border}}) \cap R_t' = \emptyset \)

But since by the previously proven subgoal **Reachable addresses of \( t \) can grow only by allocation**, we know \( R_t' \subseteq (r_t \cup [s'.\text{nalloc, na}\_\text{\_border}}) \), then it suffices to show that
The latter is immediate by subgoal **Private memory was indeed private** using simple arithmetic and interval arithmetic identities. This concludes the proof of case **assign-silent**.

* Case alloc-silent:*
By inversion of the assumptions of alloc-silent using rule allocate, we obtain (c-EVAL's-IN-t):

\[
E_L, s''., M_d, s''., ddc, s''., stc, s''., pcc \downarrow c,
\]

(v-POSITIVE):
\[v \in \mathbb{Z}^+\]

(c-IN-BOUNDS):
\[\vdash \delta_i, c\]

(S'-MEM):
\[s'.M_d = s''., M_d\]

(S'-NALLOC):
\[s'.nalloc = s''., nalloc - v, \text{ and}\]

(EQUAL-MSTC):
\[s''., mstc = s'.mstc\]

We first prove the goal **Reachable addresses of** \(t\) **can grow only by allocation.**
Assuming \(R'_t = \bigcup_{\text{mid} \in \text{dom}(t.imp)} \text{reachable_addresses}(\{s'.mstc(mid), t.imp(mid).ddc\}, s'.M_d)\), our goal is \(R'_t \subseteq (r_t \cup (s'.nalloc, na_{border}))\).
By inversion of assumption **Exec state** using rule exec-state, and by rewriting using Lemma 18, we know that (**):
\[\forall a. a \in R''_t \implies a \geq s''., nalloc\]

Let \(M_{enlarged} = s''., M_d[i \mapsto 0 | i \in [s''.nalloc - v, s''.nalloc]]\)
And let \(R_{t, enlarged} = \bigcup_{\text{mid} \in \text{dom}(t.imp)} \text{reachable_addresses}(\{s'.mstc(mid), t.imp(mid).ddc\}, M_{enlarged})\)

We claim (DECOMPOSED-REACHABILITY):
\[R_{t, enlarged} = R''_t\]

We prove this claim by induction on \(k \in [s''.nalloc - v, s''.nalloc)\) where \(M_k = s''., M_d[i \mapsto 0 | i \in [s''.nalloc - v, k])\), and
\[R_{t, k} = \bigcup_{\text{mid} \in \text{dom}(t.imp)} \text{reachable_addresses}(\{s'.mstc(mid), t.imp(mid).ddc\}, M_k)\).

The base case is immediate by reflexivity after substitution using (EQUAL-MSTC).

In the inductive step, our goal is \(R_{t, k} = R''_t\).
We apply Lemma 21 (after substitution using (EQUAL-MSTC)) obtaining the subgoal:
\[k - 1 \notin R''_{t, k-1}\]
Using the induction hypothesis, we can instead prove:
\[k - 1 \notin R''_{t, k-1}\]
Because \(k < s''.nalloc\) by choice, then we know \(k - 1 < s''.nalloc\).
But then by instantiating the contrapositive of (**), using \(k - 1\), we immediately obtain our subgoal.
Now notice from (S'-MEM) and by the definition of partial maps that
(S'-MEM-DECOMPOSED):
\[ s'.M_d = M_{enlarged}[c \mapsto (\delta, s'',nalloc - v, s''.nalloc, 0)] \]

We pick an arbitrary \( a \in R'_1 \), and our goal is to show that \( a \in r_1 \cup [s''.nalloc, na_{border}) \).

By instantiating Lemma 40 using the rewriting (S'-MEM-DECOMPOSED), and using:
\[ M_d = M_{enlarged}, a_a = a, \hat{a} = c.\sigma + c.off, \sigma = s''.nalloc - v, e = s''.nalloc, \]
we know:
\[ a \in R_{t,enlarged} \quad \vee \quad a \in [s''.nalloc - v, s''.nalloc) \]
Thus, by rewriting using (DECOMPOSED-REACHABILITY) and using (S'-NALLOC), we know:
\[ a \in R''_t \quad \vee \quad a \in [s''.nalloc, s''.nalloc) \]
We now distinguish these two cases:
1. \( a \in R''_t \)
   Here, by the induction hypothesis, we know \( a \in r_1 \cup [s''.nalloc, na_{border}) \).
   But by (S'-NALLOC), we know \([s''.nalloc, na_{border}) \subseteq [s''.nalloc, na_{border}) \)
   Thus, using both and by the definition of \( \subseteq \), our goal is immediate.
2. \( a \in [s''.nalloc, s''.nalloc) \)
   Again, here by (S'-NALLOC), and the assumption New allocation is bounded by na_{border}, we know
\[ [s''.nalloc, s''.nalloc) \subseteq [s''.nalloc, na_{border}) \], which by the definition of \( \subseteq \) gives us our goal.

This concludes the proof of the goal Reachable addresses of \( t \) can grow only by allocation.

Next, we prove the goal Five-origin policy.
We fix an arbitrary \( a \in R'_1 \), and assume \( s'.M_d(a) = (\delta, \sigma, e, _) \).
Out goal is:
\( \exists cap \in caps_{foreign, border}. \quad [\sigma, e] \subseteq cap \quad \vee \quad [\sigma, e] \subseteq [s'.nalloc, na_{border}) \)
We distinguish the following three cases:
1. Case \( a = c.\sigma + c.off \):
   Here, we know \( \sigma = s'.nalloc, e = s''.nalloc. \)
   We prove the right disjunct of our goal.
   So it suffices to prove that
\[ [s'.nalloc, s''.nalloc) \subseteq [s'.nalloc, na_{border}) \]
   Thus, it suffices to prove that
\[ s''.nalloc \leq na_{border} \]
   This is immediate by assumption New allocation is bounded by na_{border}.
2. Case \( a \in [s'.nalloc, s''.nalloc) \):
   Here, the assumption \( s'.M_d(a) = (\delta, \sigma, e, _) \) is false. So our goal holds vacuously.
3. Case \( a \notin \{c.\sigma + c.off\} \cup [s'.nalloc, s''.nalloc) \):
   Here, we know by (S'-MEM) that \( s''.M_d(a) = s'.M_d(a) \)
   Thus, we know (*):
\[ s''.M_d(a) = (\delta, \sigma, e, _) \]
We instantiate Lemma 40 using \( C = \bigcup_{mid \in dom(t.imp)} \{s'.mstc(mid), t.imp.ddc\} \),
\[ M_d = M_{enlarged}, a_a = a, \hat{a} = c.\sigma + c.off, \sigma = s''.nalloc - v, e = s''.nalloc \]
to obtain:
\[ a \in R'_t \implies a \in reachable_addresses(C, M_{enlarged}) \quad \vee \quad a \in [s'.nalloc, s''.nalloc) \]
Thus, by instantiation using our assumption about \( a \), then by elimination using
our case condition, we conclude:
\[ a \in \text{reachable\_addresses}(C, M_{\text{enlarged}}) \]
By substitution using (DECOMPOSED-REACHABILITY), we obtain (**):
\[ a \in R''_t \]
(Notice that we reuse the claim (DECOMPOSED-REACHABILITY) that was defined in the proof of a previous subgoal. The same goes for the definition of \( M_{\text{enlarged}} \), etc.)
Using (*) and (**), we instantiate assumption **Five-origin policy** obtaining:
\[ \exists \text{cap} \in \text{caps}_{\text{origin\_border}}. [\sigma, e] \subseteq \text{cap} \land [\sigma, e] \subseteq [s''.\text{nalloc}, na_{\text{border}}] \]
We distinguish the following two cases:
(a) **Case** \( \exists \text{cap} \in \text{caps}_{\text{origin\_border}}. [\sigma, e] \subseteq \text{cap} \):
Here, the left disjunct of our goal is immediate.
(b) **Case** \([\sigma, e] \subseteq [s''.\text{nalloc}, na_{\text{border}}] \):
Here, we prove the right disjunct of our goal by applying the transitivity of \( \subseteq \) obtaining the subgoal \( s'.\text{nalloc} \leq s''.\text{nalloc} \) which is immediate by \( \text{(S'-NALLOC)} \) and the condition on \( v \) being positive.
This concludes the proof of subgoal **Five-origin policy**.

Next, we prove the goal \( t \) is **executing**.
Immediate from the corresponding assumption by noticing that \( s''.\text{pcc} \models s'.\text{pcc} \).

Next, we prove the goal **New allocation is bounded by \( na_{\text{border}} \)**.
This is immediate from the corresponding assumption and \( \text{(S'-NALLOC)} \).

Next, we prove the goal **Private memory of \( t \) is untouched**.
We pick an arbitrary \( a \in \text{dom}(M_{\text{t\_border}}) \),\nand our goal is to show that \( s'.M_d(a) = M_{\text{t\_border}}(a) \).
By the corresponding assumption (i.e., assumption **Private memory of \( t \) is untouched** about \( s'' \)), it suffices by the transitivity of equality to show that:
\[ s'.M_d(a) = s''.M_d(a) \]
By \( \text{(S'-MEM)} \), it thus suffices to show that:
\[ a \notin \{c.\sigma + c.\text{off}\} \cup [s'.\text{nalloc}, s''.\text{nalloc}] \]
Showing that \( a \notin \{c.\sigma + c.\text{off}\} \) is the same proof as in **case assign-silent**.
We show that \( a \notin [s'.\text{nalloc}, s''.\text{nalloc}] \).
For this, it suffices to show that:
\[ \text{dom}(M_{\text{t\_border}}) \cap [s'.\text{nalloc}, s''.\text{nalloc}] = \emptyset \]
By assumption **New allocation is bounded by \( na_{\text{border}} \)** about \( s''.\text{nalloc} \), it suffices to show that:
\[ \text{dom}(M_{\text{t\_border}}) \cap [s'.\text{nalloc}, na_{\text{border}}] = \emptyset \]
By interval identities, it suffices to show that:
\[ \text{dom}(M_{\text{t\_border}}) \cap (-\infty, na_{\text{border}}] = \emptyset \]
By set identities, this follows from assumption **Private memory was indeed private**.
This concludes the proof of subgoal **Private memory of \( t \) is untouched**.

* **Case jump-silent**:
From the assumptions of **jump-silent**, we distinguish the following two cases.
  · **Case jump0**:
  Here, we have the following assumptions:
  (JUMP-INSTR):
  \[ s''.M_c(s''.\text{pcc}) = \text{JumpIfZero} E_{\text{cond}} E_{\text{size}} \]
  (size-EVAL):
  \[ E_{\text{size}}, s''.M_d, s''.\text{ddc}, s''.\text{stc}, s''.\text{pcc} \Downarrow v \]
We first prove the goal **Reachable addresses of \( t \) can grow only by allocation.**
Assuming \( R'_t = \bigcup_{\text{mid} \in \text{dom}(t.\text{imp})} \text{reachable_addressess}(\{s'.\text{mstc}(\text{mid}), t.\text{imp}(\text{mid}).\text{ddc}\}, s'.\mathcal{M}_d) \)
our goal is \( R'_t \subseteq (r_t \cup (s'.\text{nalloc}, \text{na}_{\text{border}})) \).
After substitution using (S'-MEM), (S'-MSTC), and (S'-NALLOC), this goal is immediate by assumptions **Reachable addresses of \( t \) and Reachable addresses of \( t \) can grow only by allocation.**

Next, we prove the goal **Five-origin policy.**
We fix an arbitrary \( a \in R'_t \), and assume \( s'.\mathcal{M}_d(a) = (\delta, \sigma, e, _) \).
Our goal is:
\[
\exists cap \in \text{caps}_{\text{four\_origin}}, [\sigma, e] \subseteq cap \lor [\sigma, e] \subseteq [s'.\text{nalloc}, \text{na}_{\text{border}}] 
\]
After substitution using (S'-MEM), (S'-MSTC), and (S'-NALLOC), this goal is immediate by assumptions **Reachable addresses of \( t \) and Five-origin policy.**

Next, we prove the goal **\( t \) is executing.**
This is immediate by the corresponding assumption after noticing from (S'-PCC) that \( s'.\text{pcc} = s''.\text{pcc} \).

Next, we prove the goal **New allocation is bounded by \( \text{na}_{\text{border}}.**
This is immediate from the corresponding assumption after substitution using (S'-NALLOC).

Next, we prove the goal **Private memory of \( t \) is untouched.**
We pick an arbitrary \( a \in \text{dom}(\mathcal{M}_{t,\text{border}}) \),
and our goal is to show that \( s'.\mathcal{M}_d(a) = \mathcal{M}_{t,\text{border}}(a) \).
This is immediate from the corresponding assumption after substitution using (S'-MEM).

**Case jump1:**
Here, we have the following assumptions:
(S'-PCC):
\( s'.\text{pcc} = \text{inc}(s''.\text{pcc}, v) \)
(S'-MEM):
\( s'.\mathcal{M}_d = s''.\mathcal{M}_d \)
(S'-NALLOC):
\( s'.\text{nalloc} = s''.\text{nalloc} \)
(S'-MSTC):
\( s'.\text{mstc} = s''.\text{mstc} \)
We prove the goal **\( t \) is executing.**
From (S'-PCC) and by unfolding the definition of \( \text{inc} \), we immediately have that \( s'.\text{pcc} = s''.\text{pcc} \). So, our goal is immediate from the assumption **\( t \) is executing** about \( s''.\text{pcc} \).
All other goals are identical to the corresponding goals of **case jump0** above.
* Case cinvoke-silent-compiled:

By the assumptions of cinvoke-silent-compiled and by their inversion using rule cinvoke and then cinvoke-aux, we obtain:

(IN-BOUNDS-S'-PCC):
\[ \vdash \kappa_s'' . pcc \]

(S'-PCC):
\[ s'' . pcc \in \text{dom}(\varepsilon . M_c) \]

(S'-IMP-MID):
\[ mid \in \text{dom}(\varepsilon . \text{imp}) \]

(S'-PCC):
\[ s' . pcc = \text{inc}(s'' . \text{imp}(mid) . pcc, s'' . \text{imp}(mid) . \text{offs}(fid)) \]

(S'-DDC):
\[ s' . ddc = s'' . \text{imp}(mid) . ddc \]

(S'-STC):
\[ s' . stc = \text{inc}(s'' . \text{mstc}(mid), nArgs + nLocal) \]

(IN-BOUNDS-S'-STC):
\[ \vdash \delta_s' . stc \]

(STC-POINTER):
\[ s'' . \text{mstc}(mid) = (\delta, \sigma, e, \_ \_ ) \]

(S'-MEM):
\[ s' . M_d = s'' . M_d[\sigma + \text{off} + i \mapsto v_i \forall i \in [0, nArgs]][\sigma + \text{off} + nArgs + i \mapsto 0 \forall i \in [0, nLocal}] \]

(S'-NALLOC):
\[ s' . \text{nalloc} = s'' . \text{nalloc} \]

(S'-MSTC):
\[ s' . \text{mstc} = s'' . \text{mstc}[\text{mid} \mapsto (\delta, s'' . \text{mstc}(mid), \sigma, s'' . \text{mstc}(mid), \sigma, \_ \_ )] \]

We first prove the goal **Reachable addresses of \( t \) can grow only by allocation**.

Assuming \( R'_t = \bigcup_{\text{mid} \in \text{dom}(t . \text{imp})} \text{reachable_addresses}(\{s'' . \text{mstc}(mid), t . \text{imp}(mid) . ddc\}, s' . M_d) \), our goal is \( R'_t \subseteq (r_t \cup (s'' . \text{nalloc}, \text{naborder})) \).

By substitution using \( (S'\text{-NALLOC}) \), our goal becomes:
\[ R'_t \subseteq (r_t \cup (s'' . \text{nalloc}, \text{naborder})) \]

Thus, using assumption **Reachable addresses of \( t \) can grow only by allocation**, and by the transitivity of \( \subseteq \), it suffices to prove:
\[ R'_t \subseteq R''_t \]

Similarly to the proof of the corresponding goal in case alloc-silent, the proof proceeds by induction on the number of memory updates defining intermediate memories indexed by the updated address.

For updates at addresses in \( [\sigma + \text{off} + nArgs, \sigma + \text{off} + nArgs + nLocal] \), we apply Lemma 37 that immediately solves our goal.

For updates at addresses in \( [\sigma + \text{off}, \sigma + \text{off} + nArgs] \), we apply Lemma 38 that immediately solves our goal.

We omit the details because they are very similar to the proof of the same goal in case alloc-silent.

Next, we prove the goal **Five-origin policy**.

We fix an arbitrary \( a \in R'_t \), and assume \( s' . M_d(a) = (\delta, \sigma, e, \_ \_ ) \).

Our goal is:
\[ \exists \text{cap} \in \text{caps}_{\text{4origin, border}}. [\sigma, e] \subseteq \text{cap} \lor [\sigma, e] \subseteq [s'' . \text{nalloc}, \text{naborder}] \]

By substitution using \( (S'\text{-NALLOC}) \), our goal becomes:
∃cap ∈ caps{origin, border}. (σ, e) ⊆ cap ∨ (σ, e) ⊆ ([s''.nalloc, na.border])
By using the proposition \( R'_t \subseteq R''_t \) proved above, we know \( a \in R''_t \).
We then distinguish three cases:

1. **Case** \( a \in [σ + off + nArgs, σ + off + nArgs + nLocal] \):
   
   Here, from the contradiction to the assumption \( s'.M_d(a) = (δ, σ, e, _) \) obtained by instantiating (S’-MEM), we have our goal.

2. **Case** \( a \in [σ + off, σ + off + nArgs] \):
   
   This case is similar to the proof of the corresponding goal (Five-origin policy) of case **assign-silent**. We omit it for brevity.

3. **Case** \( a /∈ [σ + off, σ + off + nArgs + nLocal] \):
   
   Here, we know by instantiating (S’-MEM) that \( s''.M_d(a) = s'.M_d(a) \).
   
   Thus, our goal is immediate by instantiating assumption **Five-origin policy**.

Next, we prove the goal **New allocation is bounded by na.border**.
This is immediate from the corresponding assumption after substitution using (S’-NALLOC).

Next, we prove the goal **Private memory of \( \tau \) is untouched**.

We pick an arbitrary \( a \in \text{dom}(M_{\tau.border}) \), and our goal is to show that \( s'.M_d(a) = M_{\tau.border}(a) \).
By the corresponding assumption (i.e., assumption **Private memory of \( \tau \) is untouched**) about \( s'' \), it suffices by the transitivity of equality to show that:
\( s'.M_d(a) = s''.M_d(a) \)
By (S’-MEM), it thus suffices to show that:
\( a \notin [σ + off, σ + off + nArgs + nLocal] \)
For this, it suffices by set identities to show both that:
\( [σ + off, σ + off + nArgs + nLocal] \subseteq R'_t \)
and that:
\( \text{dom}(M_{\tau.border}) \cap R'_t = \emptyset \)

1. **Subgoal** \( [σ + off + nArgs + nLocal] \subseteq R'_t \):
   
   Using the proposition \( R'_t \subseteq R''_t \) proved in a previous goal and by the transitivity of \( \subseteq \), it suffices to show that:
\( [σ + off, σ + off + nArgs + nLocal] \subseteq R''_t \)

Using (IN-BOUNDS-S’-STC), (S’-STC), and (STC-POINTER), and by unfolding Definition 2, we conclude:
\( [σ + off, σ + off + nArgs + nLocal] \subseteq \text{mstc}(mid) \)
Thus, it suffices for our goal by the transitivity of \( \subseteq \) to show that:
\( [s''.\text{mstc}(mid).σ, s''.\text{mstc}(mid).e] \subseteq R''_t \)

By unfolding \( R''_t \) using assumption **Reachable addresses of \( \tau \)** (after substitution using \( t = \tau \)) that we proved in an earlier subgoal and instantiation using (S’-
IMP-MID)) then unfolding Definition 22, it suffices by easy set identities and by
additivity (Lemma 17) to show that:

\[ s''.\text{mstc}([mid], \sigma, s'.\text{mstc}(mid), e) \subseteq \text{access}_{s''} \cap s''.\text{mstc}(mid), \sigma, s'.\text{mstc}(mid), e] \]

The latter is immediate by expansiveness (Lemma 8).

2. Subgoal \( \text{dom}(\mathcal{M}_{t, \text{border}}) \cap R_i = \emptyset \):

Since by the previously proven subgoal Reachable addresses of \( t \) can grow
only by allocation, we know \( R_i \subseteq (r_i \cup [s'.\text{nalloc}, \text{na}_{\text{border}}]) \), then it suffices to show that
\( \text{dom}(\mathcal{M}_{t, \text{border}}) \cap (r_i \cup [s'.\text{nalloc}, \text{na}_{\text{border}}]) = \emptyset \)

The latter is immediate by subgoal Private memory was indeed private using
simple arithmetic and interval arithmetic identities.

* Case cinvoke-silent-context:
This is very similar to the previous case. We omit the proof for brevity.

* Case creturn-silent-compiled:
By the assumptions of creturn-silent-compiled and by their inversion using rule cre-
turn, we obtain:
(IN-BOUNDS-S'-PCC):
\[ \vdash s''.\text{pcc} \]
(S'-PCC):
\[ s''.\text{ppc} \subseteq \text{dom}(\tau, \mathcal{M}_c) \]
(S'-FFT):
\[ s'.\text{ppc} \subseteq \text{dom}(\tau, \mathcal{M}_c) \]
(S'-MEM):
\[ s'.\mathcal{M}_d = s''.\mathcal{M}_d \]
(S'-NALLOC):
\[ s'.\text{nalloc} = s''.\text{nalloc} \]
(S'-PCC-SAME-MID-STC):
\[ \exists \text{mid}', s'.\text{ppc} \subseteq s''.\text{IMP(mid')}.\text{ppc} \land s'.\text{stc} = \text{mstc(mid')} \]
(S'-MSTC):
\[ s'.\text{stc} = s''.\text{mstc}[\text{mid} \mapsto \text{inc}(s''.\text{mstc(mid)}, \_)] \]
(S'-DDC):
\[ s'.\text{stk}, (s'.\text{ddc}, s'.\text{ppc}, \_), \_ \sigma = \text{pop}(s''.\text{stk}) \]

We first prove the goal Reachable addresses of \( t \) can grow only by allocation.
Assuming \( R_i = \bigcup_{\text{mid} \in \text{dom}(t, \text{imp})} \text{reachable_addresses}\{s'.\text{mstc(mid), t.imp(mid), ddc}, s'.\mathcal{M}_d\}, \)
our goal is \( R_i \subseteq (r_i \cup [s'.\text{nalloc}, \text{na}_{\text{border}}]) \).
By substitution using (S'-NALLOC), our goal becomes:
\( R_i = \bigcup_{\text{mid} \in \text{dom}(t, \text{imp})} \text{reachable_addresses}\{s'.\text{mstc(mid), t.imp(mid), ddc}, s''.\mathcal{M}_d\} \)
But by substitution using (S'-MEM) in the definition of \( R_i \), we have:
\( R_i = \bigcup_{\text{mid} \in \text{dom}(t, \text{imp})} \text{reachable_addresses}\{s'.\text{mstc(mid), t.imp(mid), ddc}, s''.\mathcal{M}_d\} \)
By applying Lemma 18, and then using induction on the size of \( \{\text{mid} \mid \text{mid} \in \text{dom}(t, \text{imp})\} \),
we can show that \( R_i = R''_i \).
(The proof instantiates Lemma 6 using (S'-MSTC).)

Thus, by substitution using \( R_i = R''_i \) in our goal, it becomes immediate by the as-
sumption Reachable addresses of \( t \) can grow only by allocation.

Next, we prove the goal Five-origin policy.
We fix an arbitrary \( a \in R_i \), and assume \( s'.\mathcal{M}_d(a) = (\delta, \sigma, e, \_). \)
Our goal is:
\[ \exists \text{cap} \in \text{caps}_{\text{origin, border}}, [\sigma, e] \subseteq \text{cap} \lor [\sigma, e] \subseteq [s'.\text{nalloc}, \text{na}_{\text{border}}] \]
By substitution using (S’-NALLOC), our goal becomes:
\[ \exists \text{cap} \in \text{caps}_{\text{origin, border}}, \, [\sigma, e] \subseteq \text{cap} \lor [\sigma, e] \subseteq [s'', \text{nalloc, na}_{\text{border}}] \]
By using the proposition \( R'_t = R''_t \) proved above, we know \( a \in R''_t \).
But also using the (S’-MEM), we know \( s''.\mathcal{M}_d(a) = (\delta, \sigma, e, \_ \_ \_) \)
Now our goal is immediate by instantiating assumption \textbf{Five-origin policy} using \( a \).

Next, we prove the goal \textit{\( t \) is executing}.
By substitution in assumption \( t \) is executing using (S’-PCC) and (IN-BOUNDS-S’-PCC), our goal becomes:
\[ s'.\text{pcc} \subseteq \text{dom}(\tau, \mathcal{M}_t) \]
Immediate by (S’-PCC).

Next, we prove the goal \textbf{New allocation is bounded by na}_{\text{border}}.
This is immediate from the corresponding assumption after substitution using (S’-NALLOC).

Next, we prove the goal \textbf{Private memory of \( t \) is untouched}.
We pick an arbitrary \( a \in \text{dom}(\mathcal{M}_{\text{t, border}}) \),
and our goal is to show that \( s'.\mathcal{M}_d(a) = \mathcal{M}_{\text{t, border}}(a) \).
This is immediate from the corresponding assumption after substitution using (S’-MEM).

\* Case creturn-silent-context:
This case is similar to the previous case. We omit the proof for brevity.

\begin{lemma}
(Preservation of the border-state invariant \( \vdash_{\text{border}}\).)
\[ \forall t_{ctx}, \tau, \alpha, s, \varsigma, \lambda, s', \varsigma'. \]
\[ t_{ctx} \times \tau \vdash_{\text{border}} \alpha, s, \varsigma \land \]
\[ s, \varsigma \overset{\lambda}{\Delta}_{\tau, \mathcal{M}_t} s', \varsigma' \land \]
\[ \lambda \neq \checkmark \]
\[ \Rightarrow \]
\[ t_{ctx} \times \tau \vdash_{\text{border}} \alpha, s', \varsigma' \]
\end{lemma}

\begin{proof}

\begin{itemize}
  \item We fix arbitrary \( t_{ctx}, \tau, \alpha, s, \varsigma, \lambda, s', \varsigma' \), and assume the antecedents.
  \item By inversion of our assumptions using rule \texttt{trace-steps-lambda}, we obtain the following preconditions:

  \texttt{(STAR-TAU-STEPS)}:
  \[ s, \varsigma \overset{s''}{\Delta}_{\tau, \mathcal{M}_t} s'', \varsigma'' \]

  \[ \texttt{(NON-SILENT-STEP)}: \]
  \[ s'', \varsigma'' \overset{\lambda}{\Delta}_{\tau, \mathcal{M}_t} s', \varsigma' \land \lambda \neq \checkmark \]

  \item By inversion of the assumptions using rule \texttt{Border-state invariant}, we obtain the following preconditions:
\end{itemize}
\end{proof}

\[ \Box \]
We claim (EXEC-S):

To prove it, we apply Corollary 2 obtaining the following subgoals:

- Valid linking
  \[ t_{\text{ctx}} \times \tau = [t] \]

Compiled program
  \[ \tau \in \text{range}(\{t\}) \]

Exec invariant
  \[ t \vdash_{\text{exec}} s \]

Reachable addresses of the context
  \[ R_{\text{ctx}} = \bigcup_{\text{mid} \in \text{dom}(t_{\text{ctx}} . \text{imp})} \text{reachable_addresses}\{s.\text{mstc(mid)}, t_{\text{ctx}} . \text{imp(mid)}.\text{ddc}, s.\mathcal{M}_d\} \]

Reachable addresses of the compiled program
  \[ R_\sigma = \bigcup_{\text{mid} \in \text{dom}(\tau . \text{imp})} \text{reachable_addresses}\{s.\text{mstc(mid)}, \tau.\text{imp(mid)}.\text{ddc}, s.\mathcal{M}_d\} \]

Memory at the border is described by the trace label
  \[ \text{mem}(\alpha(|\alpha|-1)) = s.\mathcal{M}_d \]

All mutually reachable addresses were recorded as shared
  \[ R_{\text{ctx}} \cap R_\sigma \subseteq \varsigma \]

Allocation intervals of the context
  \[ I_{\text{ctx}} = \text{allocation_intervals}(?, \alpha) \]

Allocation intervals of the compiled program
  \[ I_\sigma = \text{allocation_intervals}(!, \alpha) \]

Four-origin policy for privately-held capabilities of the context
  \[ \forall a \in R_{\text{ctx}} \setminus \varsigma. s.\mathcal{M}_d(a) = (\delta, \sigma, e, \_ ) \implies \]
  \[ (\exists i \in I_{\text{ctx}}. [\sigma, e] \subseteq \varsigma) \lor \exists a' \in \varsigma. idx \in [0, |\alpha|]. [\sigma, e] \subseteq \text{mem}(\alpha(idx))(a') \lor \]
  \[ \exists \text{mid} \in \text{dom}(t_{\text{ctx}} . \text{imp}). [\sigma, e] \subseteq t_{\text{ctx}} . \text{imp(mid)}.\text{ddc} \lor \]
  \[ \exists \text{mid} \in \text{dom}(t_{\text{ctx}} . \text{imp}). [\sigma, e] \subseteq t_{\text{ctx}} . \text{mstc(mid)} \]

Four-origin policy for privately-held capabilities of the compiled program
  \[ \forall a \in R_\sigma \setminus \varsigma. s.\mathcal{M}_d(a) = (\delta, \sigma, e, \_ ) \implies \]
  \[ (\exists i \in I_\sigma. [\sigma, e] \subseteq \varsigma) \lor \exists a' \in \varsigma. idx \in [0, |\alpha|]. [\sigma, e] \subseteq \text{mem}(\alpha(idx))(a') \lor \]
  \[ \exists \text{mid} \in \text{dom}(\tau . \text{imp}). [\sigma, e] \subseteq \tau.\text{imp(mid)}.\text{ddc} \lor \]
  \[ \exists \text{mid} \in \text{dom}(\tau . \text{imp}). [\sigma, e] \subseteq \tau.\text{mstc(mid)} \]

- We apply rule Border-state invariant to our goal obtaining subgoals (about \(\alpha, \lambda, s', \) and \(\varsigma'\)) that are analogous to the preconditions above (about \(\alpha, s, \) and \(\varsigma\)). We skip the explicit stating of the subgoals for the sake of brevity, and re-use the names for the preconditions that are introduced above.

  We let:
  \[ R'_{\text{ctx}} = \bigcup_{\text{mid} \in \text{dom}(t_{\text{ctx}} . \text{imp})} \text{reachable_addresses}\{s'.\text{mstc(mid)}, t_{\text{ctx}} . \text{imp(mid)}.\text{ddc}, s'.\mathcal{M}_d\} \]
  \[ R'_\sigma = \bigcup_{\text{mid} \in \text{dom}(\tau . \text{imp})} \text{reachable_addresses}\{s'.\text{mstc(mid)}, \tau.\text{imp(mid)}.\text{ddc}, s'.\mathcal{M}_d\} \]
  \[ I'_{\text{ctx}} = \text{allocation_intervals}(?, \alpha), \text{ and} \]
  \[ I'_\sigma = \text{allocation_intervals}(!, \alpha) \]

- We claim (EXEC-S’):
  \[ t \vdash_{\text{exec}} s'' \]

To prove it, we apply Corollary 2 obtaining the following subgoals:

- \[ t \vdash_{\text{exec}} s \]
  
  This is immediate by assumption Exec invariant.
\[ s \rightarrow^* s'' \]

To prove this, we apply Claim 15 obtaining the following subgoal:

\[ s, \varsigma \xrightarrow{\tau \in \Gamma} s'', \varsigma'' \]

This is immediate by (STAR-TAU-STEPS).

- From our lemma assumption, we know by instantiating (conditionally on \( s, \varsigma \)) either Claim 34 or Claim 35, that:

\[ \exists caps, r_t, \mathcal{M}_d. \ t_{ctx} \Leftarrow \varsigma \vdash \text{silent } s, \varsigma, caps, r_t, s.nalloc, \mathcal{M}_d \]

- Thus, by instantiating Lemma 152 using (STAR-TAU-STEPS), we know that:

\[ (\text{SILENT-}s'') : \exists caps, r_t, \mathcal{M}_d. \ t_{ctx} \Leftarrow \varsigma \vdash \text{silent } s'', \varsigma, caps, r_t, s.nalloc, \mathcal{M}_d \]

- Goals Valid linking and Compiled program are immediate.

- The remaining goals are proved by distinguishing the following cases for (NON-SILENT-STEP):

  - Case cinvoke-context-to-compiled:

To prove the goal Exec invariant, we apply Lemma 53 obtaining the following subgoals:

\[ \ast \ s'' \Rightarrow s' \]

This is immediate by inversion of rule cinvoke-context-to-compiled.

\[ \ast \ t \vdash \text{exec } s'' \]

This is immediate by (EXEC-S').

The goal Memory at the border is described by the trace label, i.e., \( \text{mem}(\alpha \lambda(\alpha \lambda - 1)) = s'.\mathcal{M}_d | s' \) is immediate by definition of \( \lambda \) that we get by inversion of rule cinvoke-context-to-compiled.

To prove the goal Four-origin policy for privately-held capabilities of the context, we pick an arbitrary \( a \in R'_{ctx} \setminus s' \), and assume \( s'.\mathcal{M}_d(a) = (\delta, \sigma, e, \_). \)

Our goal is:

\[ \exists i \in I'_{ctx}. \ [\sigma, e] \subseteq i \ \lor \ \\
\exists d \in R'_{ctx}. \ [\sigma, e] \subseteq \text{mem}(\alpha \lambda(\alpha \lambda - 1)) \ \lor \ \\
\exists mid \in \text{dom}(t_{ctx}.\text{imp}). \ [\sigma, e] \subseteq t_{ctx}.\text{imp}(mid).\text{ddc} \ \lor \ \\
\exists mid \in \text{dom}(t_{ctx}.\text{imp}). \ [\sigma, e] \subseteq t_{ctx}.\text{mstc}'(mid) \]

We distinguish the following cases:

\[ \ast \ \text{Case } [\sigma, e] \subseteq (s'.\text{nalloc}, -1) : \]

\[ \ast \ \text{Case } [\sigma, e] \not\subseteq (s'.\text{nalloc}, -1) : \]

Here, we claim:

(NO-MIXED-STATIC-DYNAMIC-CAPABILITY):

\[ [\sigma, e] \cap (s'.\text{nalloc}, -1) = \emptyset \]

(Sketch) Then the proof follows in both cases from (SILENT-S'') by inversion of rule Silent-state invariant.

To prove the goal All mutually-reachable addresses were recorded as shared, we pick an arbitrary \( a \in R'_{ctx} \cap R''_{\_} \). The goal is to show that:

\[ a \in s' \]

By substitution from the preconditions of rule cinvoke-context-to-compiled, the goal becomes:

\[ a \in \text{reachable_addressesClosure}(s'' \cup r, s'.\mathcal{M}_d) \]

(where \( r = \text{reachable_addresses\{pcc(i) | i \in [0, nArg] \} s'' \mathcal{M}_d} \), and \( pcc = \{ i \mapsto v_i | \forall i \in [0, nArg]\} s''.\mathcal{M}_d, s''.\text{ddc}, s''.\text{stc}, s''.pcc \downarrow v_i \} \) )
By unfolding our goal using Definition 22, our goal becomes:
\[ a \in \bigcup_{k \in [0, s', M_d]} \text{access}_{k, s', M_d}(s'' \cup r) \]
After instantiating Claim 17 using (STAR-TAU-STEPS), our goal by substitution becomes:
\[ a \in \bigcup_{k \in [0, s', M_d]} \text{access}_{k, s', M_d}(s \cup r) \]
(Sketch) The proof of this is tedious, but should follow from the conditions on \( s'', M_d \) that we obtain by inversion of (SILENT-S") using rule Silent-state invariant.

The goal Four-origin policy for privately-held capabilities of the program is similar to the previous one.

- Case cinvoke-compiled-to-context:
- Case creturn-to-compiled:
- Case creturn-to-context:
  (Sketch): These cases are similar to the representative one above.

Back-Translation

Structure of the emulating context

Definition 87 (Main module of the emulating context).

\[
\text{mainModule}(\alpha) \overset{\text{def}}{=} \left( \text{"mainModule"}, \text{mainGlobalVars}(\alpha), \text{mainModuleFuncs} \right)
\]
where mainGlobalVars and mainModuleFuncs are as defined below (Definitions 100 and 103).

We first give some auxiliary definitions.

Definition 88 (Context module IDs of a trace).

\[
\text{contextModIDs}(\alpha) \overset{\text{def}}{=} \{ \text{mid} \mid \text{call}(\text{mid}, \text{fid})_!\_\_ \in \alpha \}
\]

Definition 89 (Context function IDs of a trace).

\[
\text{contextFunIDs}(\alpha) \overset{\text{def}}{=} \{ \text{"mid\_fid"} \mid \text{call}(\text{mid}, \text{fid})_!\_\_ \in \alpha \}
\]

Definition 90 (Number of arguments of a function inferred from either the trace \( \alpha_1 \) or the trace \( \alpha_2 \)).

\[
\forall \tau. \text{call}(\text{mid}, \text{fid})^{\tau}_{!\_\_ \_} \in \alpha_1 \lor \text{call}(\text{mid}, \text{fid})^{\tau}_{!\_\_ \_} \in \alpha_2 \implies nArgs("mid\_fid", \alpha_1, \alpha_2) = |\tau|
\]

Definition 91 (Memory of a trace label).

\[
\begin{align*}
\text{mem}(\tau) & \equiv \bot \\
\text{mem}(\checkmark) & \equiv \bot \\
\text{mem}(\text{ret} \_ M_{d, \_}) & \equiv M_d \\
\text{mem}(\text{call}(\_, \_ \_ M_{d, \_ \_})) & \equiv M_d
\end{align*}
\]
Definition 92 (Allocation status of a trace label).

\[
\begin{align*}
nalloc(\tau) & \overset{\text{def}}{=} \bot \\
nalloc(\checkmark) & \overset{\text{def}}{=} \bot \\
nalloc(\text{ret } \_ \_ \_ M_d, n) & \overset{\text{def}}{=} n \\
nalloc(\text{call}(\_, \_ \_ \_ M_d, n)) & \overset{\text{def}}{=} n
\end{align*}
\]

Definition 93 (Shared addresses throughout a trace prefix \(\alpha\)).

\[
\text{sharedAddresses}(\alpha) \overset{\text{def}}{=} \bigcup_i \text{dom}(\text{mem}(\alpha(i)))
\]

Definition 94 (Context addresses collected from a trace).

\[
\text{ctx_addresses}(\alpha) \overset{\text{def}}{=} \bigcup \{ \text{dom}(\text{mem}(\alpha(i))) \mid \alpha(i) \in ? \} \setminus \text{dom}(\text{mem}(\alpha(i-1)))
\]

Definition 95 (Data segment that the context shares (collected from a trace)).

\[
\text{shareable_data_segment_ctx}(\alpha) \overset{\text{def}}{=} [\min(\text{ctx_addresses}(\alpha) \cap [0, \infty)), \max(\text{ctx_addresses}(\alpha) \cap [0, \infty))] + 1
\]

Definition 96 (A trace compatible with a program’s data segment).

\[
\text{data_segment_compatible_trace}(\alpha, \Sigma, \Delta, \text{modIDs}) \overset{\text{def}}{=} \min(\text{shareable_data_segment_ctx}(\alpha)) > \max(\text{static_addresses}(\Sigma, \Delta, \text{modIDs}))
\]

Definition 97 (A trace satisfies monotonic sharing).

\[
\text{monotonic_sharing}(\alpha) \overset{\text{def}}{=} \forall i. \text{mem}(\alpha(i+1)) \supseteq \text{mem}(\alpha(i))
\]

Definition 98 (A trace satisfies no-deallocation).

\[
\text{no_dealloc}(\alpha) \overset{\text{def}}{=} \forall i. \text{nalloc}(\alpha(i+1)) \leq \text{nalloc}(\alpha(i))
\]

Definition 99 (Syntactically-sane trace).

\[
\text{syntactically_sane}(\alpha, \Sigma, \Delta, \text{modIDs}) \overset{\text{def}}{=} \alpha \in \text{Alt}\checkmark^* \land \text{no_dealloc}(\alpha) \land \text{monotonic_sharing}(\alpha) \land \text{data_segment_compatible_trace}(\alpha, \Sigma, \Delta, \text{modIDs})
\]
Definition 100 (Global variables of the module \texttt{mainModule}).

\[
\text{mainGlobalVars}(\alpha) \overset{\text{def}}{=} \\
\{ \text{"static\_universal\_array"}, \text{"current\_trace\_index"} \} \\
\cup \{ \text{"arg\_store\_tIdx\_fid\_arg"} | \text{tIdx} \in [0,|\alpha|) \land \\
\text{fid} \in \text{contextFunIDs}(\alpha) \land \\
\text{arg} \in [0, \text{nArgs}(\text{fid}, \alpha)) \} \\
\cup \{ \text{"snapshot\_tIdx\_addr"} | \text{tIdx} \in [0,|\alpha|) \land \\
\text{addr} \in \text{sharedAddresses}(\alpha) \\
\cup \{ \text{"own\_allocation\_ptr\_tIdx"} | \text{tIdx} \in [0,|\alpha|) \}
\]

Before we give Definition 103 of the functions defined by the \texttt{mainModule}, we explain intuitively what these functions are for. The purpose of the \texttt{mainModule} is to perform various bookkeeping tasks. All the bookkeeping data is stored in the global variables \texttt{mainGlobalVars} which are statically allocated (because we know upfront as a function of the trace \(\alpha\) what variables we need). Thus, for the bookkeeping, no extra memory allocation is performed. This is important because memory allocation is an observable event. And, we do not want the bookkeeping that our source context will perform to interfere with the events observable by the source program. Remember that intuitively our goal is that observable source-level events mimic the target-level observable events precisely.

The bookkeeping tasks are initiated whenever the \texttt{mainModule} is informed that a call/return to any of the context’s modules took place.

Definition 101 (The function \texttt{readAndIncrementTraceIdx}).

\[
\text{readAndIncrementTraceIdx} \overset{\text{def}}{=} \\
(\text{mainModule}, \\
\text{readAndIncrementTraceIdx}, \\
[\text{ptrRetVal}], \\
[\text{current\_trace\_index}], \\
[\text{addr}(\text{current\_trace\_index}) \to \text{current\_trace\_index} + 1, \\
\text{Return}])
\]

Definition 102 (The functions \texttt{saveArgs}).

\[
\text{saveArgs}(\text{fid}, \text{tIdx}, \alpha) \overset{\text{def}}{=} \\
(\text{mainModule}, \\
\text{saveArgs\_fid\_tIdx}, \\
[\text{argVal}_i | i \in [0, \text{nArgs}(\text{fid}, \alpha)], \\
[\text{addr}(\text{arg\_store\_tIdx\_fid\_arg}_i) \to \text{argVal}_i | i \in [0, \text{nArgs}(\text{fid}, \alpha)])]) \\
++ \\
\text{[Return]} )
\]
**Definition 103** (Functions of the module `mainModule`).

\[
\text{mainModuleFuncs}(\alpha) \overset{\text{def}}{=} \{\text{readAndIncrementTraceIdx}\} \cup \{\text{saveArgs}(\text{fid}, \alpha) \mid \text{fid} \in \text{contextFunIDs}(\alpha)\}
\]

**Definition 104** (Constructing dereferences from path).

\[
\text{construct}_{\text{-deref}} : \mathbb{Z} \to \mathcal{E} \to \mathcal{E}
\]

\[
\text{construct}_{\text{-deref}}([], \text{expr}) \overset{\text{def}}{=} \text{expr}
\]

\[
\text{construct}_{\text{-deref}}(\text{off} :: p, \text{expr}) \overset{\text{def}}{=} \text{construct}_{\text{-deref}}(p, \text{deref}(\text{expr}[\text{off}]))
\]

**Definition 105** (Constructing path to target address).

\[
\text{path} : (\{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z} \to \text{DataMemory} \to \mathbb{Z}
\]

\[
\text{path}_{\text{-depthlimited}} : (\{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z} \to \text{DataMemory} \to \mathbb{N} \to \mathbb{Z}
\]

\[
\text{find} : \forall \alpha, \beta. \overline{\alpha} \to (\alpha \to \text{Option} \beta) \to \text{Option} (\alpha \times \beta)
\]

\[
\text{find} [] \overset{\text{def}}{=} \text{None}
\]

\[
\text{find} (x :: xs) f \overset{\text{def}}{=} \text{case } f(x) \text{ of}
\]

\[
| \text{Some } y \to \text{Some } (x, y)
| \text{None } \to \text{find } xs f
\]

\[
\text{path}_{\text{-depthlimited}}((\delta, \sigma, e, _), a, M_d, -1) \overset{\text{def}}{=} []
\]

\[
\text{path}_{\text{-depthlimited}}((\delta, \sigma, e, _), a, M_d, k + 1) \overset{\text{def}}{=}
\]

\[
\text{if } a \in [\sigma, e] \text{ then } [a - \sigma]
\]

\[
\text{else let } f = \lambda x. \text{ case } M_d(x) \text{ of}
\]

\[
| (\delta, \sigma', e', _) \to \text{ let } p = \text{path}_{\text{-depthlimited}}((\delta, \sigma', e', _), a, M_d, k) \text{ in}
\]

\[
\text{ case } p \text{ of } | [] \to \text{ None } | _ \to \text{ Some } p
\]

\[
| _ \to \text{ None}
\]

\[
\text{in case find } [\sigma, e] \text{ f of}
\]

\[
| \text{None } \to []
| \text{ Some } (a', p) \to [a' - \sigma] ++ p
\]

\[
\text{path}((\delta, \sigma, e, _), a, M_d) \overset{\text{def}}{=} \text{path}_{\text{-depthlimited}}((\delta, \sigma, e, _), a, M_d, |M_d|)
\]

**Definition 106** (Construct address back-translation for addresses reachable from a capability argument).

\[
\text{cap}_{\text{-arg-reachable-map}} : (\{\delta\} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) \to \text{DataMemory} \to \text{VarID} \to (\mathbb{Z} \to \mathcal{E})
\]

\[
\text{cap}_{\text{-arg-reachable-map}}(dc, M_d, vid) \overset{\text{def}}{=} \bigcup_{a \in \text{reachable_addresses}(\{dc\}, M_d)} a \mapsto \text{construct}_{\text{-deref}}(\text{path}(dc, a, M_d), vid)
\]
Definition 107 (Construct address back-translation map from a call-/return to-context label).

\[
\begin{align*}
\text{args}_{\text{back-translate}} & : \lambda \to \mathbb{N} \to (\mathbb{Z} \to \mathcal{E}) \\
\text{args}_{\text{back-translate}}(\text{call}(\text{mid}, \text{fid}) \pi \mathcal{M}_d, n, \text{cur_idx}) & \equiv \bigcup \{\text{cap_arg_reachable_map}(v, \mathcal{M}_d, \text{arg_mid_fid}_i \text{cur_idx}) \mid i \in [1, \text{len}(\pi)] \land v = \pi(i)\}
\end{align*}
\]

Notice that Definition 107 provides a way for finding a valid capability for any reachable address (i.e., including also for every shared address). We assume that relying on this definition, we can define functions that using these capabilities read the shared locations and stores them in mainModule’s book-keeping variables.

Definition 108 (Diverging block of code).

\[
\text{diverge} \equiv [\text{JumpIfZero } 0 0]
\]

Definition 109 (Converging block of code).

\[
\text{converge} \equiv [\text{Exit}]
\]

Definition 110 (If-then-else in ImpMod).

\[
\text{ifnotzero-then-else} : \mathcal{E} \to \mathcal{Cmd} \to \mathcal{Cmd} \to \mathcal{Cmd}
\]

\[
\text{ifnotzero-then-else}(e_{\text{cond}}, \text{cmdsthen}, \text{cmdselse}) \equiv \\
[\text{JumpIfZero } e_{\text{cond}} \mid \text{cmdsthen} \mid 2] \\
\quad \text{++ } \text{cmdsthen} \\
\quad \text{++ } [\text{JumpIfZero } 0 \mid \text{cmdselse} \mid 1] \\
\quad \text{++ } \text{cmdselse}
\]

Definition 111 (Switch-block for integers in ImpMod).

\[
\text{switch} : \mathcal{E} \to \mathbb{Z} \to \mathcal{Cmd} \to \mathcal{Cmd}
\]

\[
\text{switch}(\text{\_}, [\bullet], [\bullet]) \equiv [\bullet]
\]

\[
\text{switch}(e_{\text{cond}}, z :: zl, \text{cmdsl} :: \text{cmdsl\_per\_val}) \equiv \\
\text{ifnotzero-then-else}(e_{\text{cond}} - z, \text{switch}(e_{\text{cond}}, zl, \text{cmdsl\_per\_val}), \text{cmdsl})
\]

Definition 112 (Upcoming commands at an execution state).

\[
\text{upcoming}_\text{commands} \subseteq \text{ProgState} \times \mathcal{Cmd}
\]

\[
\text{upcoming}_\text{commands}(s, \text{cmd}) \iff \\
\quad s.pc = (\text{fid}, n, \text{\_}) \land \forall i \in [0, \text{cmd}] . \text{commands}(s.Fd(\text{fid}))(n + i) = \overline{\text{cmd}}(i)
\]
Lemma 154 (If-then-else construction is correct).

\[ \forall s, \Sigma, \Delta, \beta, MVar, Fd_{\text{cond}}, \text{cmds}_{\text{then}}, \text{cmds}_{\text{else}}, \text{cmds}. \]

upcoming\_commands(s, if\ not\ zero\_then\_else(e_{\text{cond}}, \text{cmds}_{\text{then}}, \text{cmds}_{\text{else}}) ++ \text{cmds})

\[ \implies (e_{\text{cond}}, \Sigma, \Delta, \beta, MVar, Fd, s.\text{Mem}, s.\Phi, s.\phi \downarrow 0 \implies \Sigma; \Delta; \beta; MVar; Fd \vdash s \rightarrow s' \wedge \text{upcoming\_commands}(s', \text{cmds}_{\text{else}}++\text{cmds}) \wedge \]

upcoming\_commands(s', \text{cmds}_{\text{then}}))

\[ \]

Proof. Follows from Definitions 110 and 112 and rules Jump-zero and Jump-non-zero.

Lemma 155 (Switch construction is correct).

\[ \forall i, s, \Sigma; \Delta; \beta; MVar, Fd, e_{\text{cond}}, \text{zlist}, \text{cmdslist}. \]

|\text{zlist}| = |\text{cmdslist}| \wedge \text{upcoming\_commands}(s, \text{switch}(e_{\text{cond}}, \text{zlist}, \text{cmdslist})) \wedge \]

e_{\text{cond}}, \Sigma, \Delta, \beta, MVar, Fd, s.\text{Mem}, s.\Phi, s.\phi \downarrow v \wedge v \neq 0 \implies \Sigma; \Delta; \beta; MVar; Fd \vdash s \rightarrow s' \wedge \text{upcoming\_commands}(s', \text{cmds}_{\text{then}}))

\[ \]

Proof. Prove it by nested induction on zlist and on i after unfolding Definition 111 and then inversion using rule Evaluate-expr-binop. Follows from Lemma 154.

Lemma 156 (A converge block leads to a terminal state).

\[ \forall s. \text{upcoming\_commands}(s, \text{converge}) \implies \exists t. s \rightarrow^* t \wedge \vdash t \]

Proof. Follows by Definition 40 of a terminal state “\(\vdash t\)”, after unfolding Definitions 108 and 112, and taking \(t\) to be \(s\).

Lemma 157 (A diverge block does not lead to a terminal state).

\[ \forall s. \text{upcoming\_commands}(s, \text{diverge}) \implies \not\exists t. s \rightarrow^* t \wedge \vdash t \]

Proof. Follows by unfolding Definitions 108 and 112, then simulating execution and noticing from case Jump-zero that the following holds by induction on \(s \rightarrow^* s'\):

\[ \forall s'. s \rightarrow^* s' \implies \text{upcoming\_commands}(s', [\text{JumpIfZero 0 0}]) \]

Thus, by Definition 40, we get our thesis.
Lemma 158 (Effect of calling \texttt{readAndIncrementTraceIdx}).

\[
\forall K_{\text{mod}}, K_{\text{fun}}, \overline{\text{mods}}, \Sigma; \Delta; \beta; \text{MVar}; Fd, s, \alpha, v, \text{vid}.
\]

\begin{align*}
\text{emulating\_modules}(\alpha) = \overline{\text{mods}} & \wedge \\
K_{\text{mod}}, K_{\text{fun}}; \overline{\text{mods}} \not\in \Sigma; \Delta; \beta; \text{MVar}; Fd \vdash \text{exec} s & \wedge \\
s.\text{Mem}(\Delta(\text{mainModule}).1 + \beta(\text{current\_trace\_index}, \bot, \text{mainModule}).1) = v & \wedge v \in \mathbb{Z} & \wedge \\
\text{upcoming\_commands}(s, \text{\texttt{Call readAndIncrementTraceIdx addr(vid)}} \texttt{ + cmads}) & \wedge \\
vid \notin \text{localIds}(\text{Fd}(\text{pc, fid})) & \wedge \\
\Sigma(\text{mainModule}).1 + s.\Phi(\text{mainModule}) + 1 < \Sigma(\text{mainModule}).2 & \wedge \\
\text{addr}(vid), \Sigma; \Delta; \beta; \text{MVar}; Fd \downarrow (\delta, \sigma, e, \text{off}) & \wedge \\
[\sigma, e) \cap \Sigma(\text{moduleID}(\text{Fd}(\text{pc, fid}))) = \emptyset & \wedge \sigma \leq \sigma + \text{off} < e & \wedge \\
\text{moduleID(}\text{Fd}(\text{pc, fid})) \neq \text{mainModule} & \implies \\
\exists s'. s, \Sigma; \Delta; \beta; \text{MVar}; Fd \vdash s \rightarrow^4 s' & \wedge \\
s'.\text{Mem} = s.\text{Mem} & \\
[\Sigma(\text{mainModule}).1 + s.\Phi(\text{mainModule}) + 1 & \\
+ \beta(\text{ptrRetVal, readAndIncrementTraceIdx, mainModule}) \mapsto \bot] & \\
[\Delta(\text{mainModule}).1 + \beta(\text{current\_trace\_index}, \bot, \text{mainModule}).1] \mapsto v + 1] & \\
[\Delta(\text{moduleID}(\text{Fd}(\text{pc, fid}))).1 + \beta(\text{vid, } \bot, \text{moduleID}(\text{Fd}(\text{pc, fid})))) \mapsto v] & \wedge \\
s'.\Phi = s.\Phi & \\
\text{upcoming\_commands}(s', \text{cmads})
\end{align*}

Proof.

- We first show \(\exists s_1. s \rightarrow s_1\).
  - We apply rule \texttt{Call} obtaining the following subgoals:
    \begin{itemize}
      \item \texttt{commands(Fd(pc, fid))(pc.n) = Call fidi} \mapsto \varepsilon
          Immediate by unfolding Definition \ref{def:call} instantiating \(\text{fidi} = \text{readAndIncrementTraceIdx}\).
      \item Assuming \(\text{modID} = \text{moduleID(Fd(fidi))}\), and \(\text{frameSize} = \text{frameSize(Fd(fidi))}\),
          we prove:
          \(\Sigma(\text{modID}).1 + \Phi(\text{modID}) + \text{frameSize} < \Sigma(\text{modID}).2\)
          By Definition \ref{def:frameSize}, we know \(\text{frameSize(Fd(readAndIncrementTraceIdx))} = 1\).
          Thus, after substitution in the goal, it becomes immediate by assumptions.
      \item \texttt{addr(vid), }\Sigma, \Delta, \beta, \text{MVar, Fd, Mem, }\Phi, \text{pc} \downarrow (\delta, \sigma, e, \text{off})
      \item \(\sigma, e) \cap \Sigma(\text{curModID}) = \emptyset\)
          These two goals are immediate by assumption.
    \end{itemize}
  - And we know by unfolding the assumptions using Definition \ref{def:readAndIncrementTraceIdx} then Definitions \ref{def:modules}, \ref{def:frameSize} and \ref{def:traceIdx} that we obtain \(s_1\) with
    \begin{itemize}
      \item \texttt{(S1-UPCOMING-CMDS)}:
        \texttt{upcoming\_commands(s1, [}
      \item \texttt{Assign ptrRetVal current\_trace\_index},
      \item \texttt{Assign addr(current\_trace\_index) current\_trace\_index + 1},
      \item \texttt{Return}
        \texttt{])}
      \item \texttt{(S1-PC)}:
        \texttt{s1.pc = (readAndIncrementTraceIdx, 0)}
    \end{itemize}

\begin{align*}
\end{align*}
(S1-STK):
\[ s_1.stk = [s.pc] \leftrightarrow s.stk \]

(S1-PHI):
\[ s_1.\Phi = s.\Phi[mainModule \mapsto s.\Phi(mainModule) + 1] \]

(S1-MEM):
\[ s_1.Mem = s.Mem[\Sigma(mainModule).1 + s_1.\Phi(mainModule) + \beta(ptrRetVal, readAndIncrementTraceIdx, mainModule).1 \mapsto (\delta, \sigma, e, off)] \]

- So, now we show \( \exists s_2. s_1 \rightarrow s_2 \)
  
  - We apply rule Assign-to-var-or-arr to obtain the following subgoals:
    
    * commands(Fd(s_1.pc.fid))(s_1.pc.n) = Assign e_1 e_r
      Immediate by (S1-PC) and (S1-UPCOMING-CMDS) after unfolding using Definition 112.
    
    * ptrRetVal, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow (\delta, \sigma, e, off)
      We apply rule Evaluate-expr-var then Evaluate-expr-addr-local obtaining the following subgoals:
        
        - ptrRetVal \in localIDs(Fd(readAndIncrementTraceIdx)) \cup args(Fd(readAndIncrementTraceIdx))
          Immediate by Definition 101.
        
        - \Sigma(mainModule).1 + \Phi(mainModule) + \sigma_p < \Sigma(mainModule).1 + \Phi(mainModule) + e_p
          These are immediate by inversion of the assumptions using rules Exec-state-src, and Well-formed program and parameters.
        
        - s_1.Mem(\Sigma(mainModule).1 + s_1.\Phi(mainModule) + \sigma_p) = (\delta, \sigma, e, off)
          Immediate by (S1-MEM).
    
    * current_trace_index, \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow v
      We apply rule Evaluate-expr-var, and the generated subgoals are immediate by assumptions.
    
    - \forall s', e'. v = (\delta, s', e', _) \Rightarrow _
      Vacuously true by assumptions.
    
    - \sigma \leq \sigma + off < e
      Immediate by assumptions.
  
  - And we know that \( s_2 \) satisfies
    
    (S2-MEM):
    
    \[ s_2.Mem = s_1.Mem[\sigma + off \mapsto v], \text{ and} \]
    
    (S2-PC):
    
    \[ s_2.pc = (readAndIncrementTraceIdx, 1) \]
  
- Next, we show \( \exists s_3. s_2 \rightarrow s_3 \)
  
  - We apply rule Assign-to-var-or-arr to obtain the following subgoals:
    
    * commands(Fd(s_2.pc.fid))(s_2.pc.n) = Assign e_1 e_r
      Immediate by (S2-PC), (S1-PC) and (S1-UPCOMING-CMDS) after unfolding using Definition 112.
    
    * addr(current_trace_index), \Sigma, \Delta, \beta, MVar, Fd, Mem, \Phi, pc \downarrow (\delta, \sigma, e, off, c)
      We apply rule Evaluate-expr-addr-module and obtain the following subgoals:
        
        - current_trace_index \notin localIDs(Fd(readAndIncrementTraceIdx)) \cup args(Fd(readAndIncrementTraceIdx))
          Immediate by Definition 101.
current_trace_index ∈ MVar(mainModule)
Immediate by Definitions 87 and 100.
β(current_trace_index, ⊥, mainModule) = [σ', e']
Immediate by inversion of the assumptions using rules Exec-state-src, and Well-formed program and parameters.

We obtain the following substitutions:
σc = Δ(mainModule).1 + e, e′c = Δ(mainModule).1 + e′, offc = 0
* current_trace_index + 1, Σ; Δ; β; MVar; Fd ⇓ v + 1
We apply rule Evaluate-expr-binop then rules Evaluate-expr-const in parallel with (rule Evaluate-expr-var then Evaluate-expr-addr-module).
All subgoals are immediate by assumptions and Definitions 87 and 100.

And we know that s_3 satisfies
(S3-MEM):
s_3.Mem = s_2.Mem[σc + offc ↦ v + 1], and
(S3-PC):
s_3.pc = (readAndIncrementTraceIdx, 2)

And finally, we show ∃s_4, s_3 ↦ s_4

We apply rule Return to obtain the following subgoals:
* s_3.stk ≠ nil
This is immediate by (S1-STK), and observing that s_3.stk = s_2.stk = s_1.stk.

By (S3-PC), Definition 101, and rule Return, we know
(S4-PHI):
s_4.Φ = s_3.Φ[mainModule ↦ s_3.Φ(mainModule) − 1], and
(S4-PC):
s_4.pc = inc(top(s_3.stk))

Thus, we know s ↦ s_4.

We now show:
s_4.Mem = s.Mem[Σ(mainModule).1 + s.Φ(mainModule) + 1 ↦ _]
[Δ(mainModule).1 + β(current_trace_index, ⊥, mainModule).1 ↦ v + 1]
[Δ(moduleID(Fd(pc.fid))).1 + β(vid, ⊥, moduleID(Fd(pc.fid))).1 ↦ v]
This follows by (S1-MEM), (S2-MEM), (S3-MEM) and by noticing that s_4.Mem = s_3.Mem by rule Return.

But, it remains to show that the update locations are distinct:
Δ(moduleID(Fd(pc.fid))).1 + β(vid, ⊥, moduleID(Fd(pc.fid))).1 ≠ Δ(mainModule).1 + β(current_trace_index, ⊥, mainModule).1 ≠ Σ(mainModule).1 + s.Φ(mainModule) + 1
This follows from assumption moduleID(Fd(pc.fid)) ≠ mainModule and by the disjointness preconditions given by inversion of the assumptions using Exec-state-src and Well-formed program and parameters.
• Then, we show:
  \[ s_4.\Phi = s.\Phi \]
  This follows from (S1-P HI) and (S4-P HI) together with observing that \( s_3.\Phi = s_2.\Phi = s_1.\Phi \).

• Finally, we show \texttt{upcoming\_commands}(s_4, \texttt{cmds})
  Immediate by substitution from (S4-PC), (S1-STK), \( s_3.stk = s_2.stk = s_1.stk \), and assumption \texttt{upcoming\_commands}(s, \texttt{Call readAndIncrementTraceIdx addr(vid)} \texttt{++ cmds}) after unfolding it using Definition 112.
  
  This concludes the proof of Lemma 158. \qed

\textbf{Definition 113} (Independent set of assignments). A set of assignment commands is independent if all assigned addresses are distinct.

\textbf{Lemma 159} (Effect of calling \texttt{saveArgs}).

\texttt{emulating\_modules}(\alpha) = \overline{\texttt{mods}} \land

\[ \forall K_{mod}, K_{fun}, \overline{\texttt{mods}}, \Sigma; \Delta; \beta; \texttt{MVar}; \texttt{Fd}, s, \alpha, \texttt{fid}, \texttt{tIdx}, n, \texttt{argNames}, \texttt{argVals} \]

\texttt{upcoming\_commands}(s, \texttt{cmds})

\texttt{Sigma}(\texttt{mainModule}) \cdot 1 + s.\Phi(\texttt{mainModule}) + n < \Sigma(\texttt{mainModule}) \cdot 2 \land

\texttt{moduleID}(\texttt{Fd}(s.\texttt{pc.fid})) \neq \texttt{mainModule}

\[ \implies \]

\[ \exists s'. \Sigma; \Delta; \beta; \texttt{MVar}; \texttt{Fd} \vdash s \rightarrow^{n+2} s' \land \]

\texttt{Cmds} = \texttt{Cmds}

\[ \Delta(\texttt{mainModule}) \cdot 1 + \beta(\texttt{arg\_store\_tIdx\_fid\_i}, \perp, \texttt{mainModule}) \cdot 1 \implies \texttt{argVals}(i) | i \in [0, n] \]

\[ \Sigma(\texttt{mainModule}) \cdot 1 + s.\Phi(\texttt{mainModule}) + \beta(\texttt{argNames}(i), \texttt{saveArgs\_fid\_tIdx}, \texttt{mainModule}) \]

\[ \implies \texttt{argVals}(i) | i \in [0, n] \land \]

\[ s'.\Phi = s.\Phi \land \]

\texttt{upcoming\_commands}(s', \texttt{cmds})

\textit{Proof.}

• We prove \( \exists s_{-1}. s \rightarrow s_{-1} \).
  
  - We choose \( s_{-1} \) such that:
    (S-MINUS-1-PC):
    \( s_{-1}.pc = (\texttt{saveArgs\_fid\_tIdx}, 0) \)
    (S-MINUS-1-STK):
    \( s_{-1}.stk = s.stk \uplus [s.pc], \)

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Next, our goal is:

\[ \Delta(mainModule).1 + \beta(\text{argVal}_i, \text{saveArg}_\text{fid}_\text{tIdx}, \text{mainModule}).1 \mapsto \text{argVals}(i) \mid i \in [0, n], \]

(s-Minus-1-phi):

\[ s_{-1}.\Phi = s.\Phi[\text{mainModule} \mapsto s.\Phi(\text{mainModule}) + n], \]

And (s-Minus-1-Upcoming-Cmds):

upcoming\_commands(s_{-1},
    [ Assign addr(\text{arg\_store\_tIdx\_fid\_i}) \text{argVal}_i \mid i \in [0, n] ]
    ++
    [Return] )

We apply rule Call to obtain the following subgoals:

* commands(Fd(pc.fid))(pc.n) = Call fcd.bat \( \overline{e} \)
  
  Immediate by unfolding Definition 112 instantiating fcd.bat = saveArg\_fid\_tIdx.

* Assuming modID = moduleID(Fd(fcd.bat)), and frameSize = frameSize(Fd(fcd.bat)),
  
  we prove:
  
  \( \Sigma(\text{modID}).1 + \Phi(\text{modID}) + \text{frameSize} < \Sigma(\text{modID}).2 \)
  
  By Definition 102, we know frameSize(Fd(saveArg\_fid\_tIdx)) = n.
  
  Thus, after substitution in the goal, it becomes immediate by assumptions.

* \( \forall i \in [0, n], \text{argNames}(i), \Sigma, \Delta, \beta, \text{MVar}, \text{Fd}, \text{s\_Mem}, s.\Phi, s.\text{pc} \downarrow \text{argVals}(i) \)
  
  Here, we fix an arbitrary \( i \), and we apply rule Evaluate-expr-var then Evaluate-expr-addr-local obtaining the following subgoals:
  
  - \( \text{argNames}(i) \in \text{args}(Fd(s.\text{pc}.\text{fid})) \)
    
    Immediate by assumptions.
  
  - \( \beta(\text{argNames}(i), s.\text{pc}.\text{fid}, \text{moduleID}(Fd(s.\text{pc}.\text{fid}))) = [\sigma, e] \)
    
    Immediate by assumptions.
  
  - \( \phi = \Sigma(\text{moduleID}(Fd(s.\text{pc}.\text{fid}))).1 + \Phi(\text{moduleID}(Fd(s.\text{pc}.\text{fid}))) \)
    
    This subgoal is immediate by the fact that the given keys exist in the maps \( \Sigma, \Phi \), and \( \beta \) which is immediate by inverting the assumptions using Exec-state-src then Well-formed program and parameters.
  
  - \( \sigma < e \)
    
    Follows by inversion of the assumptions using Well-formed program and parameters.
  
  - \( s.\text{Mem}(\sigma + \phi) = \text{argVals}(i) \)
    
    Follows by assumptions.
  
  - \( \forall i \in [0, n], \text{argVals}(i) = (\delta, \_ , \_ , \_ ) \mapsto [\text{argVals}(i).\sigma, \text{argVals}(i).e] \cap \Sigma(\text{curModID}) = \emptyset \)
    
    Immediate by assumptions.
  
  * The remaining subgoals are immediate by (S-Minus-1-STK), (S-Minus-1-MEM), and (S-Minus-1-PHI). Also, (S-Minus-1-Upcoming-Cmds) becomes a proof obligation after substitution, and it follows immediately by Definition 102.

- Next, our goal is:
  
  \( \exists s_{n-1}, s_{-1} \rightarrow^n s_{n-1} \land \)
  
  \( s_{-1}.\Phi = s_{-1}.\Phi \land \)
  
  \( s_{n-1}.\text{Mem} = s_{-1}.\text{Mem} \)
  
  \( \Delta(\text{mainModule}).1 + \beta(\text{arg\_store\_tIdx\_fid\_i}, \perp, \text{mainModule}).1 \mapsto \text{argVals}(i) \mid i \in [0, n] \land \)
  
  \( s_{n-1}.\text{stk} = s_{-1}.\text{stk} \land \)
  
  upcoming\_commands(s_{n-1}, [\text{Return}])

We distinguish the following two cases for \( n \):

\[ 234 \]
– Case $n = 0$
  Here, our goal is immediate by choosing $s_{-1}$, and by the reflexivity of $\rightarrow^0$.

– Case $n > 0$
  
  First, we prove the following by induction on $k$:
  
  $k \in [0, n) \implies$

  $\exists s_k, s_{k-1}$.

  $s_{k-1} \rightarrow s_k$ \wedge

  $s_k\text{.Mem} = s_{k-1}\text{.Mem}$

  $[\Delta(\text{mainModule})]_{1 + \beta(\text{arg\_store\_tIdx\_fid\_k, \bot, mainModule})} \rightarrow \text{argVals}(k)] \wedge$

  $s_k\text{.}\Phi = s_{-1}\text{.}\Phi$ \wedge

  $s_k\text{.stk} = s_{-1}\text{.stk}$

  upcoming\_commands($s_k$),

  $[\text{Assign}\ addr(\text{arg\_store\_tIdx\_fid\_i})\ argVal\_i\ |\ i \in [k+1, n]]$

  $++$

  $[\text{Return}\ ]$

  • Base case ($k = 0$):
    
    We choose the state $s_{-1}$ that is given above in the proof of $s \rightarrow s_{-1}$.

    We choose $s_0$ such that:

    (S0-STK):

    $s_0\text{.stk} = s_{-1}\text{.stk}$,

    (S0-MEM):

    $s_0\text{.Mem} = s_{-1}\text{.Mem}$

    $[\Delta(\text{mainModule})]_{1 + \beta(\text{arg\_store\_tIdx\_fid\_0, \bot, mainModule})} \rightarrow \text{argVals}(0)]$,\n
    (S0-PHI):

    $s_0\text{.}\Phi = s_{-1}\text{.}\Phi$

    Now we prove that $s_{-1} \rightarrow s_0$ and

    upcoming\_commands($s_0$),

    $[\text{Assign}\ addr(\text{arg\_store\_tIdx\_fid\_i})\ argVal\_i\ |\ i \in [1, n]]$

    $++$

    $[\text{Return}\ ]$

    Using (S-MINUS-1-UPCOMING-CMDS), and Definition 112 we know:

    upcoming\_commands($s_{-1}$, $[\text{Assign}\ addr(\text{arg\_store\_tIdx\_fid\_0})\ argVal\_0]$)

    Thus, we apply rule Assign-to-var-or-arr to our goal obtaining the following subgoals:

    1. $\text{addr(\text{arg\_store\_tIdx\_fid\_0}), \Sigma, \Delta, \beta, MVar, Fd, s_{-1}\text{.Mem}, s_{-1}\text{.}\Phi, s_{-1}\text{.pc} \downarrow (\delta, \sigma_0, e_0, \text{off}_0}$

       Here, we apply rule Evaluate-expr-addr-module all of whose subgoals follow by simplification after unfolding the lemma assumptions using Definitions 87, 100, 103 and 122, inversion of the lemma assumptions using Well-formed program and parameters, and substitution using (S-MINUS-1-PC).

    2. $\text{argVal\_0, \Sigma, \Delta, \beta, MVar, Fd, s_{-1}\text{.Mem}, s_{-1}\text{.}\Phi, s_{-1}\text{.pc} \downarrow \text{argVals}(0)$

       Here, we apply rules Evaluate-expr-var then Evaluate-expr-addr-local obtaining the following subgoals:

       (a) $\text{argVal\_0} \in \text{args(\text{saveArgs\_fid\_tIdx})}$

          Immediate by Definition 102 and the assumptions about $n$ after unfolding the assumptions using Definitions 87 and 122.

       (b) $s_{-1}\text{.Mem}(\Sigma(\text{mainModule})]_{1 + s_{-1}\text{.}\Phi(\text{mainModule}) + \beta(\text{argVal\_0, saveArgs\_fid\_tIdx, mainModule})} = \text{argVals}(0)$

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Immediate by \( (S{-}1{-}MEM) \).

(c) The remaining subgoals follow from Well-formed program and parameters by unfolding the assumptions using first \textit{Exec-state-src}.

3. \( \sigma_0 < e_0 \)

Follows from unfolding the assumptions using \textit{Exec-state-src} then Well-formed program and parameters.

4. \[ \text{argVals}(0) = (\delta, \_ , \_ , \_ ) \implies \left[ \text{argVals}(0).\sigma, \text{argVals}(0).e \right] \cap \Sigma(\text{mainModule}) = \emptyset \]

Assume the contrary (for contradiction)

\( \text{ARGVAL0-IS-STACK-CAPABILITY} \):
\[ \text{argVals}(0) = (\delta, \_ , \_ , \_ ) \land \left[ \text{argVals}(0).\sigma, \text{argVals}(0).e \right] \cap \Sigma(\text{mainModule}) \neq \emptyset \]

Now, by inversion of the assumptions using \textit{Exec-state-src}, we know by instantiating the precondition “\textbf{Stack addresses only live on the stack}” using (ARGVAL0-IS-STACK-CAPABILITY) that

\( \text{CONTRADICTORY-LOCATION-FOR-ARGVAL0} \):
\[ \forall a \ s. \text{Mem}(a) = \text{argVals}(0) \implies a \in \Sigma(\text{mainModule}) \]

Now, we instantiate (CONTRADICTORY-LOCATION-FOR-ARGVAL0) using the assumption

\[ \text{Mem}(\Sigma(\text{moduleID}(\text{Fd}(s.pc.fid)))) + s.\Phi(\text{moduleID}(\text{Fd}(s.pc.fid))) \]

\[ + \beta(\text{argNames}(0), s.pc.fid, \text{moduleID}(\text{Fd}(s.pc.fid))).1) = \text{argVals}(0) \]

to conclude that:

\[ \Sigma(\text{moduleID}(\text{Fd}(s.pc.fid))).1 + s.\Phi(\text{moduleID}(\text{Fd}(s.pc.fid))) \]

\[ + \beta(\text{argNames}(0), s.pc.fid, \text{moduleID}(\text{Fd}(s.pc.fid))).1 \in \Sigma(\text{mainModule}) \]

We can derive a contradiction from this last statement using the preconditions of Well-formed program and parameters together with the lemma assumption

\[ \text{moduleID}(\text{Fd}(s.pc.fid)) \neq \text{mainModule} \]

5. The remaining subgoals that justify the choice of \((S0{-}MEM), (S0{-}STK), \) and \((S0{-}PHI)\) are immediate.

- Inductive case \((0 < k < n)\): The induction step is very similar to the base case. We avoid repetition.

This concludes the inductive proof.

* We instantiate the inductive statement obtained above with \(k = n - 1\) obtaining our goal.

This concludes the proof for case \(n > 0\).

- Now, it remains to show that:

\[ \exists s'. \ s_{n-1} \rightarrow s' \land \]
\[ s'.\text{Mem} = s_{n-1}.\text{Mem} \land \]
\[ s'.\Phi = s_{n-1}.\Phi[\text{mainModule} \mapsto s_{n-1}.\Phi(\text{mainModule}) - n] \]

Here, we apply rule Return obtaining the following subgoals:

- \( s_{n-1}.\text{stk} \neq \text{nil} \), and
- \text{upcoming \_ commands}(s', \text{cmds})

These follow from \((S{-}MINUS{-}1{-}STK), \) and \((S{-}N{-}1{-}STK)\) together with our lemma assumption about the upcoming commands of \(s\) after unfolding Definition 112.

This concludes the proof of Lemma 159.

\( \square \)
Definition 114 (Logged memory correct).

\[
\text{logged\_mem\_correct}(s)_{\alpha,i,\Delta,\beta} \overset{\text{def}}{=} \forall j, a. \\
j < i \land \\
a \in \text{dom}(\text{mem}(\alpha(j))) \\
\rightarrow \\
s.\text{Mem}(\Delta(\text{mainModule}).1 + \beta(\text{snapshot\_j\_a, \bot, mainModule})) = \text{mem}(\alpha(j))(a)
\]

Definition 115 (Arguments saved correctly).

\[
\text{arguments\_saved\_correctly}(s)_{\alpha,i,\Delta,\beta} \overset{\text{def}}{=} \forall j, \text{argIdx}, \text{fid}. \\
j < i \land \\
\alpha(j) = \text{call}(_{\text{fid}})_{\text{argIdx}} \land \\
\text{argIdx} \in [0, \text{len}(\tau)) \\
\rightarrow \\\ns.\text{Mem}(\Delta(\text{mainModule}).1 + \beta(\text{arg\_store\_j\_fid\_argIdx, \bot, mainModule})) = \tau(\text{argIdx})
\]

Definition 116 (Allocation pointers saved).

\[
\text{allocation\_pointers\_saved}(s)_{\alpha,i,\Delta,\beta} \overset{\text{def}}{=} \forall j. \\
j < i \land \\
\alpha(j) \in \ast \\
\rightarrow \\\ns.\text{Mem}(\Delta(\text{mainModule}).1 + \beta(\text{own\_allocation\_ptr\_j, \bot, mainModule})) = (\delta, \text{nalloc}(\alpha(j)) + 1, \text{nalloc}(\alpha(j - 1)), 0)
\]

Claim 36 (There is a source function that does allocations according to allocation_pointers_saved).

\[
\exists \text{cmd}. \\
\text{upcoming\_commands}(s, [\text{cmd}]) \land \\
\text{allocation\_pointers\_saved}(s)_{\alpha,i,\Delta,\beta} \land \\
s \rightarrow s' \\
\rightarrow \text{allocation\_pointers\_saved}(s')_{\alpha,i+1,\Delta,\beta}
\]

Definition 117 (Emulate call or return or exit command of i-th output action).

\[
\text{emulate\_ith\_action\_last\_cmd}(\alpha,i) \overset{\text{def}}{=} \\
[\text{Call} \text{fid} [\text{emulate\_value}(\tau(i), \alpha(:i)) \mid i \in [0, \text{len}(\tau)) ] ] \\
\text{where } \alpha(i) = \text{call}(_{\text{fid}})_{\text{argIdx}} \land \\
[\text{Return}] \\
\text{where } \alpha(i) = \text{ret} \\
[\text{Exit}] \\
\text{where } \alpha(i) = \checkmark
\]

(Notice that the existence of a function \text{emulate\_value}(\tau(i), \alpha(:i)) relies on Definition 106.)
Definition 118 (Emulate i-th output action).

\[
\text{emulate}_i(\alpha, \text{mid}, \text{fid}) \equiv \\
\text{Call readAndIncrementTraceIdx addr(current_trace_index)} \\
\text{Call saveArgs_fid}_i \text{argNamesList}(\alpha, \text{fid}) \\
\text{Call saveSnapshot}_i - 1, \\
\text{Call doAllocations}_i, \\
\text{Call mimicMemory}_i \\
\]

++

\[
\text{emulate}_i(\alpha, \text{last_cmd}) \\
\]

Definition 119 (Responses for suffix).

\[
\text{emulate}_i(\alpha, \text{mid}, \text{fid}) \equiv \\
\text{switch(} \\
\text{current_trace_index}_{\text{mid}}, \\
[i, i + 2, i + 4, \cdots, |\alpha|], \\
\text{[emulate}_i(\alpha, j, \text{mid}, \text{fid}) ++ \text{emulate}_i(\alpha, j + 2, \text{mid}, \text{fid}) | j \in [i, i + 2, i + 4, \cdots, |\alpha|]]} \\
\]

Lemma 160 (Adequacy of emulate_responses_for_suffix).

\[
(\text{C}_{\text{emul}}, \Delta_{\text{emul}}, \Sigma_{\text{emul}}, \beta_{\text{emul}}, K_{\text{modeul}}, K_{\text{funemul}}) = \text{emulate}(\alpha, p, \Delta, \Sigma, \beta) \land \\
p' = C_{\text{emul}} \ltimes p \land \\
(\Delta', \Sigma', \beta', K_{\text{mod}}, K_{\text{fun}}') = \\
(\Delta \uplus \Delta_{\text{emul}}, \Sigma \uplus \Sigma_{\text{emul}}, \beta \uplus \beta_{\text{emul}}, K_{\text{modeul}} \uplus K_{\text{modeul}}, K_{\text{funemul}} \uplus K_{\text{funemul}}) \land \\
p' \vdash_{\text{exec}} s \land \\
\text{upcoming Commands}(s, \text{emulate}_i(\alpha, \text{mid}, \text{fid}))(s', s_{\text{pc}}) \land \\
\Rightarrow \exists s'. s_{\text{pc}} \alpha(i) s_{\text{pc}}' s_{\text{pc}}'
\]

Proof.

After unfolding Definition 119 and Definition 118, the goal follows by successively instantiating Lemma 158 then Lemma 159, and Claim 36, together with unproved assumptions about the existence of functions saveSnapshot, and mimicMemory which rely on Definition 106.

Definition 120 (Emulating function).

\[
\text{emulate}_i(\alpha, \text{mid}, \text{fid}) \equiv \\
( \\
\text{mid}, \\
\text{fid}, \\
[\text{argVal}_i | i \in [0, nArgs(fid, \alpha)]], \\
[\text{emulate}_i(\alpha, 0, \text{mid}, \text{fid})] \\
)
Definition 121 (Emulating module).

\[
\text{emulating\_module}(\alpha, \text{mid}) \overset{\text{def}}{=} \\
( \\
\text{mid}, \\
[current\_trace\_index\_mid], \\
\{\text{emulating\_function}(\alpha, \text{mid}, \text{fid}) | \alpha(i) = \text{call}(\text{mid}, \text{fid}) \} \\
)
\]

Definition 122 (Emulating modules).

\[
\text{emulating\_modules}(\alpha) \overset{\text{def}}{=} [\text{mainModule}(\alpha)]++[\text{emulating\_module}(\alpha, \text{mid}) | \text{mid} \in \text{contextModIDs}(\alpha)]
\]

Definition 123 (The emulating context).

\[
\text{emulate}(\alpha, p, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}) \overset{\text{def}}{=} \\
(\text{emulating\_modules}(\alpha), \\
\text{data\_segment\_map\_extension}(p, \text{emulating\_modules}(\alpha), \Delta), \\
\text{stack\_map\_extension}(p, \text{emulating\_modules}(\alpha), \Sigma), \\
\text{variable\_bounds\_extension}(p, \text{emulating\_modules}(\alpha), \beta), \\
\text{Kmod\_extension}(p, \text{emulating\_modules}(\alpha), K_{\text{mod}}), \\
\text{Kfun\_extension}(p, \text{emulating\_modules}(\alpha), K_{\text{fun}}))
\]

Lemma 161 (The emulating context is linkable and loadable).

\[
(C_{emul}, \Delta_{emul}, \Sigma_{emul}, \beta_{emul}, K_{\text{modemul}}, K_{\text{funemul}}) = \text{emulate}(\alpha, p, \Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}) \wedge \\
\mathcal{C} \times [p]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}} = [t'] \wedge \\
\text{initial\_state}(t' + \omega, \text{main\_module}(t')), 0 \mathcal{A} [p]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}}, \forall s', \varsigma' \\
\implies \\
\exists m. \\
C_{emul}[p]_{\Delta, \Sigma} = m \wedge \\
\text{wfp\_params}(m, \\
\Delta \cup \Delta_{emul}, \Sigma \cup \Sigma_{emul}, \beta \cup \beta_{emul}, K_{\text{mod}} \cup K_{\text{modemul}}, K_{\text{fun}} \cup K_{\text{funemul}}) \wedge \\
\text{main\_module}(m) \neq \text{None}
\]

Proof.
(Sketch) By inverting the assumption using rule valid-linking, and unfolding it using Definition 123 then Definitions 88 and 89, we are able to instantiate rule Valid-linking-src satisfying our goal after instantiating Lemma 92 using our assumption.

Then, subgoal wfp_params follows by applying rule Well-formed program and parameters where all the generated subgoals follow by unfolding Definition 123 recursively (assuming there are suitable definitions for extending the linking and loading information, i.e., suitable definitions for data_segment_map_extension, stack_map_extension, variable_bounds_extension, Kmod_extension, and Kfun_extension).
Definition 124 (Emulate invariants).

\[ \text{emulate}_{\text{invariants}}(s)_{\alpha,i,p,\Delta,\Sigma,\beta} \overset{\text{def}}{=} \]  
\[(\forall pc \in s.stk, s'. s'.pc = pc \implies \exists j. j \leq i \wedge \text{upcoming}\_\text{commands}(s', \text{emulate}\_\text{responses}\_\text{for}\_\text{suffix}(\alpha, j, \text{moduleID}(\text{fd}\_\text{map}(p)(pc.\text{fid})), pc.\text{fid}))) \land \]  
\[(\alpha(i) \in ? \implies \exists j. j \leq i \wedge \text{upcoming}\_\text{commands}(s, \text{emulate}\_\text{responses}\_\text{for}\_\text{suffix}(\alpha, j, \text{moduleID}(\text{fd}\_\text{map}(p)(s.pc.\text{fid})), s.pc.\text{fid}))) \land \]  
\[\text{logged}_\text{mem}\_\text{correct}(s)_{\alpha,i,\Delta,\beta} \land \]  
\[\text{arguments}\_\text{saved}\_\text{correctly}(s)_{\alpha,i,\Delta,\beta} \land \]  
\[\text{allocation}\_\text{points}\_\text{saved}(s)_{\alpha,i,\Delta,\beta} \]

Lemma 162 (Initial state of emulate satisfies emulate\_invariants).

\[(C_{\text{emul}}, \Delta_{\text{emul}}, \Sigma_{\text{emul}}, \beta_{\text{emul}}, K_{\text{modemul}}, K_{\text{funemul}}) = \text{emulate}(\alpha, p, \Delta, \Sigma, \beta) \land \]  
p' = C_{\text{emul}} \times p \land \]  
\[(\Delta', \Sigma', \beta', K_{\text{mod}}', K_{\text{fun}}') = \]  
\[(\Delta \uplus \Delta_{\text{emul}}, \Sigma \uplus \Sigma_{\text{emul}}, \beta \uplus \beta_{\text{emul}}, K_{\text{mod}} \uplus K_{\text{modemul}}, K_{\text{fun}} \uplus K_{\text{funemul}}) \land \]  
s_{\text{emul}} = \text{initial}\_\text{state}(p', \Delta', \Sigma', \text{main}\_\text{module}(p')) \implies \]  
\[\text{emulate}\_\text{invariants}(s_{\text{emul}})_{\alpha,0,p,\Delta',\Sigma',\beta'} \]

Proof.
By unfolding Definition 124, we have the following subgoals:

- Vacuous subgoal because \( s.stk = nil \)
- Assuming \( \alpha(i) \in ? \), show:
  \[ \text{upcoming}\_\text{commands}(s_{\text{emul}}, \text{emulate}\_\text{responses}\_\text{for}\_\text{suffix}(\alpha, i, \text{moduleID}(\text{fd}\_\text{map}(p)(s_{\text{emul}}.pc.\text{fid})), s_{\text{emul}}.pc.\text{fid})) \]
  Follows from unfolding Definition 123 then Definition 122 then Definition 121.
- \[ \text{logged}_\text{mem}\_\text{correct}(s_{\text{emul}})_{\alpha, -} \]
  Immediate after unfolding Definition 114 by noticing that \( \alpha(-1) = \bot \).
- \[ \text{arguments}\_\text{saved}\_\text{correctly}(s_{\text{emul}})_{\alpha, -} \]
  Immediate after unfolding Definition 115 by noticing that \( \alpha(-1) = \bot \).
- \[ \text{allocation}\_\text{points}\_\text{saved}(s_{\text{emul}})_{\alpha, -} \]
  Immediate after unfolding Definition 116 by noticing that \( \alpha(-1) = \bot \).
\[ \square \]

Lemma 163 (Adequacy of emulate\_invariants).

\[ \begin{align*} \mathcal{C}_{\text{emul}} \times p & \vdash_{\text{exec}} s_{\text{emul}} \land \quad \alpha(i) \in ? \land \quad \text{emulate}_{\text{invariants}}(s_{\text{emul}})_{\alpha,i,p,\Delta,\Sigma,\beta} \\
\implies \quad \exists s'_{\text{emul}}. s_{\text{emul}} \text{. } \alpha(i) \overset{\text{[p]}}{\rightarrow} s'_{\text{emul}} \text{. } \end{align*} \]
Proof. After unfolding the assumption using Definition 124, the goal follows from Lemma 160.

\textbf{Lemma 164} (Preservation of emulate\_invariants).

\[ C_{emul} \times p \vdash execut_{emul} \land \]
\[ \text{emulate\_invariants}(s_{emul})_{\alpha,i,p,\Delta,\Sigma,\beta} \]
\[ s_{emul} \xrightarrow{\alpha(i)} [p] \quad s'_{emul} \]
\[ \implies \]
\[ \text{emulate\_invariants}(s_{emul})_{\alpha,i+1,p,\Delta,\Sigma,\beta} \]

Proof. (Sketch) After unfolding Definition 119 then instantiating Lemma 155, this should follow from Lemma 158 then Lemma 159, and Claim 36, together with unproved assumptions about the existence of functions saveSnapshot, and mimicMemory which rely on Definition 106.

6.4 Trace-Indexed Cross-Language (TrICL) simulation relation

\textbf{Definition 125} (Trace-Indexed Cross-Language (TrICL) simulation relation).

\[ \text{TrICL}(s_{emul}, s_{compiled}, s_{given}, \varsigma)_{\alpha,i,p,C_{emul},\Delta,\Sigma,\beta} \]
\[ \equiv \]
\[ \text{exec}_{emul} \land \]
\[ s_{emul} \equiv C_{emul} \times p \quad s_{compiled} \land \]
\[ (\alpha(i) \in \cdot \implies s_{compiled}, \varsigma \equiv [p], s_{given}, \varsigma) \land \]
\[ (\alpha(i) \in \cdot \implies s_{compiled}, \varsigma \sim [p], s_{given}, \varsigma) \]

\textbf{Lemma 165} (TrICL satisfies the alternating simulation condition).

\[ \alpha \in \text{Alt} \land \]
\[ \text{TrICL}(s_{emul}, s_{compiled}, s_{given}, \varsigma)_{\alpha,i,p,C_{emul},\Delta,\Sigma,\beta} \]
\[ \implies \]
\[ s_{emul} \vdash execut_{emul} \land \]
\[ s_{given} \vdash [p] \quad s_{given}, \varsigma \land \]
\[ s_{given}, \varsigma \xrightarrow{\alpha(i)} \quad [p] \quad s'_{given}, \varsigma' \]
\[ \implies \]
\[ \exists s'_{compiled}, s'_{emul}, s'_{given}, \varsigma' \]
\[ s_{given}, \varsigma \xrightarrow{\alpha(i)} [p] \quad s'_{compiled}, s'_{given}, \varsigma' \land \]
\[ s_{emul}, \varsigma \xrightarrow{\alpha(i)} [p] \quad s'_{emul}, s'_{given}, \varsigma' \land \]
\[ \text{TrICL}(s'_{emul}, s'_{compiled}, s'_{given}, \varsigma')_{\alpha,i+1,p,C_{emul},\Delta,\Sigma,\beta} \]

Proof. By \( \alpha \in \text{Alt} \) (unfolding Definition 69), it suffices to distinguish the following two cases:

- Case \( \alpha(i) \in \cdot \).
  
  By unfolding the assumption using Definition 125, we have:
  
  (EMUL-INVAR): emulate\_invariants\( (s_{emul})_{\alpha,i,p,\Delta,\Sigma,\beta} \)
(COMPILER-REL): \( s_{emul} \cong_{C_{emul}} p s_{compiled} \)

(STRONG-SIM): \( s_{compiled}, \varsigma \approx [p] s_{given}, \varsigma \)

Here, we can instantiate Lemma 144 (Weakening of strong similarity) using (STRONG-SIM) and the given step to obtain:

\[(G1): s_{compiled}, \varsigma \xrightarrow{\alpha(i)} [p] s'_{compiled}, \varsigma' \]

and

\[(G2): s'_{compiled}, \varsigma' \sim [p], \alpha, i+1 s'_{given}, \varsigma' \]

But then using (G1), and (COMPILER-REL), we can instantiate Lemma 126 (lifted compiler backward-simulation) to obtain:

\[(G3): s_{emul}, \varsigma \xrightarrow{\alpha(i)} [p] s'_{emul}, \varsigma' \]

and

\[(G4): s'_{emul} \cong_{C_{emul}} p s'_{compiled} \]

But then using (G3) and (EMUL-INVAR), we can instantiate Lemma 164 (preservation of the emulate invariants) to obtain:

\[(G5): emulate\_invariants(s_{emul})_{\alpha, i+1, p, \Delta, \Sigma, \beta} \]

After (G1), (G2), (G3), (G4), and (G5), no subgoals remain, so this concludes this case.

**Case \( \alpha(i) \in ? \):**

By unfolding the assumption using Definition 125, we have:

(EMUL-INVAR): \( emulate\_invariants(s_{emul})_{\alpha, i, p, \Delta, \Sigma, \beta} \)

(COMPILER-REL): \( s_{emul} \cong_{C_{emul}} p s_{compiled} \)

(WEAK-SIM): \( s_{compiled}, \varsigma \sim [p], \alpha, i s_{given}, \varsigma \)

Here, we can instantiate Lemma 163 (adequacy of the emulate invariants) using (EMUL-INVAR) to obtain:

\[(G1): s_{emul}, \varsigma \xrightarrow{\alpha(i)} [p] s'_{emul}, \varsigma' \]

(Notice that \( \alpha(i) \) determines \( \varsigma' \))

Then, we can instantiate Lemma 164 (preservation of the emulate invariants) using (G1) above to obtain:

\[(G2): emulate\_invariants(s_{emul})_{\alpha, i+1, p, \Delta, \Sigma, \beta} \]

Also, using the same emulating step (G1), together with (COMPILER-REL), we can instantiate Lemma 125 (lifted compiler forward-simulation) to obtain:

\[(G3): s_{compiled}, \varsigma \xrightarrow{\alpha(i)} [p] s'_{compiled}, \varsigma' \]

and

\[(G4): s'_{emul} \cong_{C_{emul}} p s'_{compiled} \]

But then using the last step (G3), the given step (from the assumption), and (WEAK-SIM) we can instantiate the strengthening lemma (Lemma 148) to obtain:

\[(G5): s'_{compiled}, \varsigma' \cong [p] s'_{given}, \varsigma' \]

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After (G1), (G2), (G3), (G4), and (G5), no subgoals remain, so this concludes this case.

This concludes the proof of Lemma 165.

Lemma 166 (Initial states are TrICL-related).

\[ \alpha \in Tr,\Sigma,\Delta,\beta, K_{mod}, K_{fun} \land \\
(\mathbb{C}_{emul}, \Delta_{emul}, \Sigma_{emul}, \beta_{emul}, K_{modemul}, K_{funemul}) = emulate(\alpha, p, \Delta, \Sigma, \beta) \land \\
p' = \mathbb{C}_{emul} \triangleleft p \land \\
(\Delta', \Sigma', \beta', K'_{mod}, K'_{fun}) = \\
(\Delta \cup \Delta_{emul}, \Sigma \cup \Sigma_{emul}, \beta \cup \beta_{emul}, K_{mod} \cup K_{modemul}, K_{fun} \cup K_{funemul}) \land \\
s_{emul} = initial_state(p', \Delta', \Sigma', \beta', \text{main module}(p')) \land \\
s_{compiled} = initial_state([p] \Delta, \Sigma, \beta, \text{main module}(p')) \land \\
s_{given} = initial_state([C_{given} \times [p] \Delta, \Sigma, \beta, K_{mod}, K_{fun}, \text{main module}(p')) \\
\implies \\
TrICL(s_{emul}, s_{compiled}, s_{given}, \emptyset) \alpha, 0, p, \mathbb{C}_{emul}, \Delta', \Sigma', \beta') \\
\]

Proof.
By unfolding Definition 125, we have the following subgoals:

- emulate invariants:
  Follows by instantiating Lemma 162.

- \[ s_{emul} \triangleleft \mathbb{C}_{emul} \times p \ s_{compiled} \]:
  Follows by instantiating Lemma 100.

- Assuming \( \alpha(0) \in \cdot \), show \( s_{compiled}, \Sigma \triangleleft[p] s_{given}, \Sigma \):
  Here, know by relying on Lemma 161, and by distinguishing the cases for \( \alpha(i) \) that:
  \( s_{given}, \text{pcc} \not\subseteq \text{dom}(C_{given}.M_c) \).
  Thus, our goal follows by Lemma 131.

- Assuming \( \alpha(0) \in ? \), show \( s_{compiled}, \Sigma \nearrow[p], \alpha, i \ s_{given}, \Sigma \):
  Here, know by relying on Lemma 161 and by distinguishing the cases for \( \alpha(i) \) that:
  \( s_{given}, \text{pcc} \subseteq \text{dom}(C_{given}.M_c) \)
  Thus, our goal follows by Lemma 132.

Lemma 167 (TrICL-related states are co-terminal).

\[ TrICL(s_{emul}, s_{compiled}, s_{given}, \emptyset) \alpha, 0, p, \mathbb{C}_{emul}, \Delta', \Sigma', \beta') \]

Proof.
Follows from Lemma 103, and by unfolding Definition 125 then Definition 117.

Lemma 168 (No trace is added by compilation).

\[ \alpha \in Tr,\Sigma,\Delta,\beta(p) \iff \alpha \in Tr,\Sigma,\Delta,\beta(K_{mod}, K_{fun}) \]
Proof.
By assumption (unfolding Definition 72), we have (*):

\[ \exists C_{\text{given}}, t' : \text{TargetSetup}, s'_i : \text{TargetState}, \delta' : 2^\varphi. \]
\[ C_{\text{given}} \models [p]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}} = [t'] \land \]
\[ \text{initial}\_\text{state}(t' + \omega, \text{main}\_\text{module}(t')), \emptyset \xrightarrow{\alpha}[p]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}}, \forall s', \delta' \]

And our goal (unfolding Definition 76) is:

\[ \exists C, m, s', \delta', \Delta_C, \Sigma_C, \beta_C. \]
\[ \Delta' = \Delta \cup \Delta_C \land \Sigma' = \Sigma \cup \Sigma_C \land \beta' = \beta \cup \beta_C \land \]
\[ C\{p\}_{\Delta', \Sigma'} = m \land \]
\[ \Sigma'; \Delta' + \omega; \beta'; \text{mvar}(m); \text{fd}\_\text{map}(m) \vdash \text{initial}\_\text{state}(m, \Delta' + \omega, \Sigma', \text{main}\_\text{module}(m)), \emptyset \xrightarrow{\alpha}[p], \forall s', \delta' \]

We pick for our goal the following instantiation:

\[ C := C_{\text{emul}}, \Delta_C := \Delta_{\text{emul}}, \Sigma_C := \Sigma_{\text{emul}}, \beta_C := \beta_{\text{emul}} \]

where (**):

\[ (C_{\text{emul}}, \Delta_{\text{emul}}, \Sigma_{\text{emul}}, \beta_{\text{emul}}, K_{\text{modemul}}, K_{\text{funemul}}) = \text{emulate}(\alpha, p, \Delta, \Sigma, \beta) \]

By instantiating Lemma 161 using (*) and (**), we know \( m \) exists, and that (WF-PARAMS):

\[ \text{C}_{\text{emul}}[p]_{\Delta, \Sigma} = m \land \]
\[ \text{wfp\_params}(m, \Delta \cup \Delta_{\text{emul}}, \Sigma \cup \Sigma_{\text{emul}}, \beta \cup \beta_{\text{emul}}, K_{\text{mod}} \cup K_{\text{modemul}}, K_{\text{fun}} \cup K_{\text{funemul}}) \]

Using (WF-PARAMS), we obtain by instantiating rule Module-list-translation a compiled program:

\[ p'_{\text{compiled}} = [p']_{\Delta \cup \Delta_{\text{emul}}, \Sigma \cup \Sigma_{\text{emul}}, \beta \cup \beta_{\text{emul}}, K_{\text{mod}} \cup K_{\text{modemul}}, K_{\text{fun}} \cup K_{\text{funemul}}} \]

Now, by instantiating Lemma 166 using our assumption and (WF-PARAMS) and (**), we have (INIT-TrICL):

\[ \text{TrICL}(\text{initial}\_\text{state}(p', \Delta', \Sigma', \text{main}\_\text{module}(p'))), \]
\[ \text{initial}\_\text{state}(p'_{\text{compiled}}, \text{main}\_\text{module}(p'))), \]
\[ \text{initial}\_\text{state}(C_{\text{given}} \models [p]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}, \text{main}\_\text{module}(p'))), \emptyset \xrightarrow{\alpha,0,p,C_{\text{emul}}} \]

By Lemma 109, and Lemma 167, it suffices to show the following for the alternating prefix \( \alpha|\varphi \):

\[ \forall i \in [0, |\alpha|], \exists s'_{\text{emul}}, s'_{\text{compiled}}, s'_{\text{given}}, s_i. \]
\[ \Sigma'; \Delta'; \delta'; \text{mvar}(p'); \text{fd}\_\text{map}(p') \vdash \text{initial}\_\text{state}(p', \Delta', \Sigma', \text{main}\_\text{module}(p')), \emptyset \xrightarrow{\alpha(0)\ldots\alpha(i)}[p], \forall s'_{\text{emul}}, s_i \land \]
\[ \text{initial}\_\text{state}(p'_{\text{compiled}}, \text{main}\_\text{module}(p')), \emptyset \xrightarrow{\alpha(0)\ldots\alpha(i)}[p]_{\Delta, \Sigma, \beta, K_{\text{mod}}, K_{\text{fun}}}, \forall s'_{\text{compiled}}, s_i \land \]
\[ \text{TrICL}(s'_{\text{emul}}, s'_{\text{compiled}}, s'_{\text{given}}, s_i)_{\alpha,i,p,C_{\text{emul}}} \]

We are able to show the above sufficient subgoal by proving an inductive version of Lemma 165 (relying on Lemma 153):
The base case follows from (INIT-TrICL) and instantiation of Lemma 165.

The inductive case follows by instantiation of Claim 9 using (*) then Lemma 165, followed by instantiation of the following:
- Claim 20 and rule trace-steps-alternating-src for the source trace, and
- Claim 8 and rule trace-steps-alternating for the compiled trace.

This concludes the proof of Lemma 168.
7 Corollaries for free

7.1 Completeness of the trace semantics of CHERIExpress

**Corollary 12** (Completeness of target trace equivalence for contextual equivalence of compiled components).

\[ [p_1] \Delta, \Sigma, \beta, K_{mod}, K_{fun} \xymatrix{\Rightarrow_{\omega, \nabla} \ar[r] & [p_2] \Delta, \Sigma, \beta, K_{mod}, K_{fun}} \iff [p_1] \Delta, \Sigma, \beta, K_{mod}, K_{fun} \simeq_{\omega, \nabla} [p_2] \Delta, \Sigma, \beta, K_{mod}, K_{fun} \]

**Proof.** Follows from the cycle in Figure 13 (i.e., the contrapositive of our goal is immediate by instantiating Lemma 118 then Lemma 113 then Lemma 116).

7.2 Soundness of the trace semantics of ImpMod

**Corollary 13** (Soundness of source traces).

\[ \forall \bar{m}_1, \bar{m}_2, \bar{\Delta}, \beta_1, \beta_2, \bar{\Sigma}, \nabla, \Delta, \Sigma. \]

\[ \text{dom}(\bar{\Sigma}) = \{\text{moduleID}(m) \mid m \in \bar{m}_1\} = \{\text{moduleID}(m) \mid m \in \bar{m}_2\} \land \]

\[ \text{dom}(\bar{\Delta}) = \{\text{moduleID}(m) \mid m \in \bar{m}_1\} = \{\text{moduleID}(m) \mid m \in \bar{m}_2\} \land \]

\[ \beta_1, \bar{m}_1 \xymatrix{\not\Rightarrow_{\omega, \nabla, \Delta, \Sigma} \ar[r] & \Delta, \beta_2, \bar{m}_2} \]

\[ \iff \Delta, \beta_1, \bar{m}_1 \simeq_{\omega, \nabla, \Delta, \Sigma} \Delta, \beta_2, \bar{m}_2 \]

**Proof.** Follows from the cycle in Figure 13 (i.e., the contrapositive of our goal is immediate by instantiating Lemma 116—after compiling both programs, then Lemma 110, then Lemma 118).

References


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