Typed Operational Reasoning
Homework #2: Normalization for Simply-Typed λ-Calculus

Instructor: Derek Dreyer

Assigned: Thursday, 30 October 2008
Due: Thursday, 6 November 2008

1 Termination in the Presence of Sums (5 points)

Consider extending the "natural" language from Homework #1 with sum types and the corresponding standard introduction and elimination forms:

$$\begin{array}{lrcl} \text{Types} & \tau & ::= & \cdots \mid \tau_1 + \tau_2 \\ \text{Terms} & e & ::= & \cdots \mid \texttt{inj}_i(e) \mid \texttt{case}\ e\ \texttt{of}\ \texttt{inj}_1(x) \Rightarrow e_1\ \texttt{|}\ \texttt{inj}_2(x) \Rightarrow e_2 \\ \text{Values} & v & ::= & \cdots \mid \texttt{inj}_i(v) \end{array}$$

The typing and small-step evaluation rules for sums (also standard) are as follows:

$$\frac{\Gamma \vdash e : \tau_i \quad i \in \{1,2\}}{\Gamma \vdash \texttt{inj}_i(e) : \tau_1 + \tau_2} \qquad \frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x{:}\tau_1 \vdash e_1 : \tau \quad \Gamma, x{:}\tau_2 \vdash e_2 : \tau}{\Gamma \vdash \texttt{case}\ e\ \texttt{of}\ \texttt{inj}_1(x) \Rightarrow e_1\ \texttt{|}\ \texttt{inj}_2(x) \Rightarrow e_2 : \tau}$$

$$\frac{e \leadsto e'}{\texttt{inj}_i(e) \leadsto \texttt{inj}_i(e')} \qquad \frac{e \leadsto e'}{\begin{array}{l}\texttt{case}\ e\ \texttt{of}\ \texttt{inj}_1(x) \Rightarrow e_1\ \texttt{|}\ \texttt{inj}_2(x) \Rightarrow e_2 \leadsto \\ \quad \texttt{case}\ e'\ \texttt{of}\ \texttt{inj}_1(x) \Rightarrow e_1\ \texttt{|}\ \texttt{inj}_2(x) \Rightarrow e_2\end{array}}$$

$$\frac{}{\texttt{case}\ \texttt{inj}_i(v)\ \texttt{of}\ \texttt{inj}_1(x) \Rightarrow e_1\ \texttt{|}\ \texttt{inj}_2(x) \Rightarrow e_2 \leadsto e_i[v/x]}$$

Problem: Extend the logical relations proof that well-typed terms terminate (under call-by-value reduction) to include the new constructs for sums. Just give the new cases of the proof. You should not need to modify any of the old cases (*i.e.,* the ones I showed in class). For uniformity of solutions, base your proof on the following formulation of the logical relation (you will need to add a definition for $\mathcal{V}[\![\tau_1 + \tau_2]\!]$):

$$\mathcal{C}[\![\tau]\!] \stackrel{\text{def}}{=} \{e : \tau \mid \exists v.\ e \leadsto^* v \wedge v \in \mathcal{V}[\![\tau]\!]\}$$

$$\begin{array}{rcl} \mathcal{V}[\![\texttt{nat}]\!] & \stackrel{\text{def}}{=} & \{n\} \\ \mathcal{V}[\![\tau_1 \to \tau_2]\!] & \stackrel{\text{def}}{=} & \{v : \tau_1 \to \tau_2 \mid \forall v' \in \mathcal{V}[\![\tau_1]\!].\ v(v') \in \mathcal{C}[\![\tau_2]\!]\} \end{array}$$

For the remaining problems, assume you are working in the "unnatural" language of Homework #1, but with full reduction, not just CBV reduction.

2 SN Closed Under Head Expansion (5 points)

Prove that SN, the set of strongly normalizing terms, is closed under head expansion, *i.e.,*

$$\text{If } \mathcal{E}\{e'[e/x]\} \in \text{SN and } e \in \text{SN, then } \mathcal{E}\{(\lambda x.e')(e)\} \in \text{SN}.$$

where $\mathcal{E}$ is an elimination context, as defined in class.

3 First Law of Thermodynamics (5 points)

Recall the logical relation employed in Tait's method for proving strong normalization:

$$\begin{aligned} \mathcal{C}[\![\texttt{nat}]\!] &\stackrel{\text{def}}{=} \text{SN} \\ \mathcal{C}[\![\tau_1 \to \tau_2]\!] &\stackrel{\text{def}}{=} \{e \mid \forall e' \in \mathcal{C}[\![\tau_1]\!].\ e(e') \in \mathcal{C}[\![\tau_2]\!]\} \end{aligned}$$

Suppose we change the definition to include an explicit requirement that $\mathcal{C}[\![\tau_1 \to \tau_2]\!] \subseteq \text{SN}$:

$$\begin{aligned} \mathcal{C}[\![\texttt{nat}]\!] &\stackrel{\text{def}}{=} \text{SN} \\ \mathcal{C}[\![\tau_1 \to \tau_2]\!] &\stackrel{\text{def}}{=} \{e \mid \boxed{e \in \text{SN} \wedge} \ \forall e' \in \mathcal{C}[\![\tau_1]\!].\ e(e') \in \mathcal{C}[\![\tau_2]\!]\} \end{aligned}$$

This new version of the logical relation avoids the need for (the first part of) the "Main Lemma" (*i.e.,* the part that says $\mathcal{C}[\![\tau]\!] \subseteq \text{SN}$), because that property is now completely obvious. However, the work that was required to prove that lemma before has not just disappeared — rather, it has been pushed elsewhere in the proof. In other words, by changing the definition of $\mathcal{C}[\![\tau]\!]$, we have made one part of the proof easier but other parts harder. Please explain precisely how/where the strong normalization proof changes when we use the new definition of $\mathcal{C}[\![\tau]\!]$. (You don't need to show the whole proof, just point out the differences from the proof given in class.)

4 Strong Normalization in the Presence of η-Reduction (5 points)

Suppose we extend the full reduction relation with a rule for η-reduction:

$$\frac{x \notin \text{FV}(e)}{\lambda x.(e\,x) \leadsto e}$$

Prove that strong normalization holds under full $\beta\eta$-reduction. (As in the previous problem, you don't need to show the whole proof. Just show any new work that has to be done above and beyond the proof of strong normalization for full β-reduction.)