Strong Normalization
for Simply Typed Lambda Calculus

Beta Ziliani
(Based on lectures by Derek Dreyer)

July 9, 2012

1 Calculus

We consider the simply typed \( \lambda \)-calculus:

- **Types** \( \phi ::= b \mid \phi_1 \rightarrow \phi_2 \)
- **Terms** \( M ::= x \mid \lambda x. M_1 \mid M_1 M_2 \)
- **Elim. Context** \( \mathcal{E} ::= \bullet \mid \mathcal{E} M \)

**Reduction**

\[
\frac{\lambda x. M \rightarrow \lambda x. M'}{M \rightarrow M'} \quad \frac{M_1 M_2 \rightarrow M_1' M_2'}{M_2 \rightarrow M_2'} \\
\frac{M_1 \rightarrow M'_1}{M_1 M_2 \rightarrow M_1' M_2'} \quad \frac{M_1 \rightarrow M'_1}{(\lambda x. M) N \rightarrow M[N/x]}
\]

2 Strong Normalization

We now wish to prove that every well-typed term is strongly normalizing, i.e., it cannot reduce indefinitely. We show below a sequence a proof attempts, starting from the most obvious one. The last of these attempts succeeds. The proof given here is not complete; in particular, it contains three unproved lemmas (called Lemmas 1, 2 and 3) here. However, all these Lemmas hold.

**Definition 1.** A value is a term that cannot be reduced, i.e., it does not contain any \( \beta \)-redex.

**Definition 2.** A term \( M \) is said to be strong normalizing if there are no infinite reduction from it. Put another way, every reduction must end in a value.

**Definition 3.** \( SN \) is the set of strongly normalizing terms.

We are going to first try to prove that every well-typed term is strongly normalizing by trying induction on the term. We will encounter a problem with this naive proof, which will force us to re-state the theorem.
**Theorem 1.** $\Gamma \vdash M : \varphi \implies M \in SN$

By induction on $M$

- Case $M = x$. A variable cannot be reduced, therefore it is in $SN$.
- Case $M = \lambda x.N$: By hypothesis, we know that
  
  $$\Gamma, x : \varphi_1 \vdash N : \varphi_2$$

  $$\Gamma \vdash \lambda x. N : \varphi_1 \rightarrow \varphi_2$$

  Applying IH, on $N$, $N \in SN$, therefore $M \in SN$.

- Case $M = M_1 M_2$:

  $$\Gamma \vdash M_1 : \varphi_1 \rightarrow \varphi_2 \quad \Gamma \vdash M_2 : \varphi_1$$

  $$\Gamma \vdash M_1 M_2 : \varphi_2$$

**Problem!** We can apply the IH and conclude that $M_1$ and $M_2$ are in $SN$, but of course this nothing says about what happen to $M_1 M_2$ since it may contain a new $\beta$-redex. For instance, $M_1$ may reduce to $\lambda x. N_1$.

The idea to fix the proof is to construct for each $\varphi$ a set $L[\varphi]$ included in $SN$, and state the theorem as

**Theorem 2.** $\Gamma \vdash M : \varphi \implies M \in L[\varphi]$

We construct $L$ for each type in such a way that it will help us solve easily the application case of the proof:

$$L[b] = SN$$

$$L[\varphi_1 \rightarrow \varphi_2] = \{ M \mid \forall N \in L[\varphi_1], M N \in L[\varphi_2] \}$$

Before proving the theorem, we need to show that really this definition works for our purpose, that is,

**Lemma 1.** $L[\varphi] \subseteq SN$

(Proof: Omitted)

As for Theorem 1, we try to prove Theorem 2 by induction on $M$. We show first the application case to see that we really have it solved:

- Case $M = M_1 M_2$:

  $$\Gamma \vdash M_1 : \varphi_1 \rightarrow \varphi_2 \quad \Gamma \vdash M_2 : \varphi_1$$

  $$\Gamma \vdash M_1 M_2 : \varphi_2$$

By IH, $M_1 \in L[\varphi_1 \rightarrow \varphi_2]$ and $M_2 \in L[\varphi_1]$. By definition of $L[\varphi_1 \rightarrow \varphi_2]$, then

$$M_1 M_2 \in L[\varphi_2]$$

and this is precisely what we need to show.
• Case $M = \lambda x.N$: to show

$\lambda x.N \in L[\varphi_1 \rightarrow \varphi_2] = \{M | \forall N' \in L[\varphi_1], M N' \in L[\varphi_2]\}$

Suppose $N' \in L[\varphi_1]$, then we have to show

$(\lambda x.N) N' \in L[\varphi_2]$

By IH we know $N \in L[\varphi_2]$. But from this we cannot conclude what we need. Again, we have to generalize the theorem to make this case go through.

**Definition 4.** We say $\gamma$ is a substitution if it is a map from variables to terms.

**Definition 5.** We extend the definition of logical relation to contexts:

$L[\Gamma] = \{\gamma | \text{dom } \Gamma = \text{dom } \gamma \land \forall x : \varphi \in \Gamma, \gamma x \in L[\varphi]\}$

We extend the theorem to consider a substitution $\gamma$.

**Theorem 3** (Fundamental Theoren of Logical Relations).

$\Gamma \vdash M : \varphi \land \gamma \in L[\Gamma] \implies \gamma M \in L[\varphi]$

*By induction on $M$.*

• Case $M = M_1 M_2$:

$\Gamma \vdash M_1 : \varphi_1 \rightarrow \varphi_2, \Gamma \vdash M_2 : \varphi_1$  

$\Gamma \vdash M_1 M_2 : \varphi_2$

By IH, $\gamma M_1 \in L[\varphi_1 \rightarrow \varphi_2]$ and $\gamma M_2 \in L[\varphi_1]$. By definition of $L[\varphi_1 \rightarrow \varphi_2]$, then

$\gamma M_1 \gamma M_2 \in L[\varphi_2]$

By an easy lemma not shown here, $(\gamma M_1) (\gamma M_2) = \gamma (M_1 M_2)$ therefore

$\gamma (M_1 M_2) \in L[\varphi_2]$

and this is precisely what we need to show.

• Case $M = \lambda x.N$: to show

$\gamma(\lambda x.N) \in L[\varphi_1 \rightarrow \varphi_2]$

Again, it is easy to show that $\gamma(\lambda x.N) = \lambda x.\gamma N$ (under $\alpha$-conversion to avoid name clashes).

Suppose $N' \in L[\varphi_1]$, then t.s.

$(\lambda x.\gamma N) N' \in L[\varphi_2]$

Let $\gamma' = \gamma, x \mapsto N'$. It is easy to see that $\gamma' \in L[\Gamma, x : \varphi_1]$.

By IH $\gamma' N \in L[\varphi_2]$. By definition of substitution and $\gamma'$,

$\gamma' N = (\gamma N)[N'/x] \in L[\varphi_2]$

We conclude by stating and applying a new lemma:
Lemma 2. \([L[\varphi] \text{ is closed under } \beta\text{-expansion}] \) If \(M[N/x] \in L[\varphi] \), then 
\((\lambda x. M) N \in L[\varphi] \)

The actual lemma is slightly different, but it will not be proved here.

• Case \(M = x\). To show \(\gamma x \in L[\varphi]\). This follows immediately by definition of \(L[\gamma]\).

Our original goal was to prove that

Theorem 4. \(\Gamma \vdash M : \varphi \implies M \in SN\)

but in our theorem we have to find a substitution \(\gamma\) in \(L[\Gamma]\). We instantiate the theorem with \(\gamma = \text{id}\), i.e., the identity substitution. But in order to do that we need to prove that the identity substitution is in \(L[\Gamma]\). We do that as a corollary of the following lemma:

Lemma 3. \(x \in L[\varphi]\)

As before, the actual lemma is slightly different, but it is not going to be shown here.