Strong Normalization
for Simply Typed Lambda Calculus

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(Based on lectures by Derek Dreyer)

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1 Calculus

We consider the simply typed λ-calculus:

Types	ϕ	$::=$	$b \mid \phi_1 \to \phi_2$
Terms	M	$::=$	$x \mid \lambda x.M_1 \mid M_1\ M_2$
Elim. Context	$\mathcal{E}$	$::=$	$\bullet \mid \mathcal{E}\, M$

Reduction

$$\frac{\lambda x.M \hookrightarrow \lambda x.M'}{M \hookrightarrow M'} \qquad \frac{M_1 M_2 \hookrightarrow M_1' M_2}{M_1 \hookrightarrow M_1'}$$

$$\frac{M_1 M_2 \hookrightarrow M_1 M_2'}{M_2 \hookrightarrow M_2'} \qquad (\lambda x.M)\ N \ \hookrightarrow\ M[N/x]$$

2 Strong Normalization

We now wish to prove that every well-typed term is strongly normalizing, i.e., it cannot reduce indefinitely. We show below a sequence a proof attempts, starting from the most obvious one. The last of these attempts succeeds. The proof given here is not complete; in particular, it contains three unproved lemmas (called Lemmas 1, 2 and 3) here. However, all these Lemmas hold.

Definition 1. *A value is a term that cannot be reduced,* i.e., *it does not contain any β-redex.*

Definition 2. *A term M is said to be* strong normalizing *if there are no infinite reduction from it. Put another way, every reduction must end in a* value*.*

Definition 3. *$\mathcal{SN}$ is the set of strongly normalizing terms.*

We are going to first try to prove that every well-typed term is strongly normalizing by trying induction on the term. We will encounter a problem with this naive proof, which will force us to re-state the theorem.

Theorem 1. $\Gamma \vdash M : \varphi \implies M \in \mathcal{SN}$

By induction on M

- Case $M = x$. A variable cannot be reduced, therefore it is in $\mathcal{SN}$.
- Case $M = \lambda x.N$: By hypothesis, we know that

$$\frac{\Gamma, x : \varphi_1 \vdash N : \varphi_2}{\Gamma \vdash \lambda x.N : \varphi_1 \to \varphi_2}$$

  Applying IH, on N, $N \in \mathcal{SN}$, therefore $M \in \mathcal{SN}$.

- Case $M = M_1\ M_2$:

$$\frac{\Gamma \vdash M_1 : \varphi_1 \to \varphi_2 \quad \Gamma \vdash M_2 : \varphi_1}{\Gamma \vdash M_1\ M_2 : \varphi_2}$$

  Problem! We can apply the IH and conclude that M_1 and M_2 are in $\mathcal{SN}$, but of course this nothing says about what happen to $M_1\ M_2$ since it may contain a new β-redex. For instance, M_1 may reduce to $\lambda x.N_1$.

The idea to fix the proof is to construct for each φ a set $L[\varphi]$ included in $\mathcal{SN}$, and state the theorem as

Theorem 2. $\Gamma \vdash M : \varphi \implies M \in L[\varphi]$

We construct L for each type in such a way that it will help us solve easily the application case of the proof:

$$\begin{aligned} L[b] &= \mathcal{SN} \\ L[\varphi_1 \to \varphi_2] &= \{M \mid \forall N \in L[\varphi_1], M\ N \in L[\varphi_2]\} \end{aligned}$$

Before proving the theorem, we need to show that really this definition works for our purpose, that is,

Lemma 1. $L[\varphi] \subseteq \mathcal{SN}$

(Proof: Omitted)

As for Theorem 1, we try to prove Theorem 2 by induction on M. We show first the application case to see that we really have it solved:

- Case $M = M_1\ M_2$:

$$\frac{\Gamma \vdash M_1 : \varphi_1 \to \varphi_2 \quad \Gamma \vdash M_2 : \varphi_1}{\Gamma \vdash M_1\ M_2 : \varphi_2}$$

  By IH, $M_1 \in L[\varphi_1 \to \varphi_2]$ and $M_2 \in L[\varphi_1]$. By definition of $L[\varphi_1 \to \varphi_2]$, then

$$M_1\ M_2 \in L[\varphi_2]$$

  and this is precisely what we need to show.

- Case $M = \lambda x.N$: to show

$$\lambda x.N \in L[\varphi_1 \to \varphi_2] = \{M | \forall N' \in L[\varphi_1], M\ N' \in L[\varphi_2]\}$$

Suppose $N' \in L[\varphi_1]$, then we have to show

$$(\lambda x.N)\ N' \in L[\varphi_2]$$

By IH we know $N \in L[\varphi_2]$. But from this we cannot conclude what we need. Again, we have to generalize the theorem to make this case go through.

Definition 4. *We say γ is a substitution if it is a map from variables to terms.*

Definition 5. *We extend the definition of logical relation to contexts:*

$$L[\Gamma] = \{\gamma \mid \mathsf{dom}\ \Gamma = \mathsf{dom}\ \gamma \wedge \forall x : \varphi \in \Gamma, \gamma\ x \in L[\varphi]\}$$

We extend the theorem to consider a substitution γ.

Theorem 3 (Fundamental Theoren of Logical Relations)**.**

$$\Gamma \vdash M : \varphi \wedge \gamma \in L[\Gamma] \implies \gamma M \in L[\varphi]$$

By induction on M.

- Case $M = M_1\ M_2$:

$$\frac{\Gamma \vdash M_1 : \varphi_1 \to \varphi_2 \quad \Gamma \vdash M_2 : \varphi_1}{\Gamma \vdash M_1\ M_2 : \varphi_2}$$

By IH, $\gamma M_1 \in L[\varphi_1 \to \varphi_2]$ and $\gamma M_2 \in L[\varphi_1]$. By definition of $L[\varphi_1 \to \varphi_2]$, then

$$\gamma M_1\ \gamma M_2 \in L[\varphi_2]$$

By an easy lemma not shown here, $(\gamma M_1)\ (\gamma M_2) = \gamma(M_1\ M_2)$ therefore

$$\gamma(M_1\ M_2) \in L[\varphi_2]$$

and this is precisely what we need to show.

- Case $M = \lambda x.N$: to show

$$\gamma(\lambda x.N) \in L[\varphi_1 \to \varphi_2]$$

Again, it is easy to show that $\gamma(\lambda x.N) = \lambda x.\gamma N$ (under α-conversion to avoid name clashes).

Suppose $N' \in L[\varphi_1]$, then t.s.

$$(\lambda x.\gamma N)\ N' \in L[\varphi_2]$$

Let $\gamma' = \gamma, x \mapsto N'$. It is easy to see that $\gamma' \in L[\Gamma, x : \varphi_1]$.

By IH $\gamma' N \in L[\varphi_2]$. By definition of substitution and γ',

$$\gamma' N = (\gamma N)[N'/x] \in L[\varphi_2]$$

We conclude by stating and applying a new lemma:

Lemma 2. *[$L[\varphi]$ is closed under β-expansion] If $M[N/x] \in L[\varphi]$, then $(\lambda x.M)\ N \in L[\varphi]$*

The actual lemma is slightly different, but it will not be proved here.

- Case $M = x$. To show $\gamma x \in L[\varphi]$. This follows immediately by definition of $L[\gamma]$.

□

Our original goal was to prove that

Theorem 4. $\Gamma \vdash M : \varphi \implies M \in \mathcal{SN}$

but in our theorem we have to find a substitution γ in $L[\Gamma]$. We instantiate the theorem with $\gamma = \mathsf{id}$, *i.e.,* the identity substitution. But in order to do that we need to prove that the identity substitution is in $L[\Gamma]$. We do that as a corollary of the following lemma:

Lemma 3. $x \in L[\varphi]$

As before, the actual lemma is slightly different, but it is not going to be shown here.