Proof Theory Seminar
Assignment 3

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Out: May 16, 2012
Due: May 29, 2012

Please submit your solutions on the due date before class. Both typeset and handwritten solutions are acceptable, either in hardcopy or by email to the TA.

Problem 1: Classical Proofs

Give classical proofs of the following sequents. $\# \ \varphi$ means $\cdot; \cdot \ \# \ \varphi \ \mathsf{false}$ and $\varphi_1 \to \varphi_2$ denotes $(\sim \varphi_1) \vee \varphi_2$.

1. $\# \ (\varphi_1 \wedge \varphi_2) \to \sim ((\sim \varphi_1) \vee (\sim \varphi_2))$
2. $\# \ \exists x.(\sim \varphi) \to \sim \forall x.\varphi$

Problem 2: Glivenko's Theorem

Consider the following statement of (one direction of) Glivenko's theorem from class.

Theorem 1. *If $\mathcal{D} :: \mathbb{T} \ \# \ \mathbb{F}$, then there exists $\mathcal{E} :: \neg\neg\mathbb{T}, \neg\mathbb{F} \Rightarrow \bot \ \mathsf{conc}$.*

1. In propositional logic this theorem can be proved by induction on $\mathcal{D}$. Show the case of the proof where $\mathcal{D}$ ends in the rule ($\sim$T). Prove any lemmas you use, other than standard metatheoretic properties of the sequent calculus.
2. Find a counterexample to Theorem 1 in first-order logic: Find a classically provable sequent $\cdot; \mathbb{T} \ \# \ \mathbb{F}$ such that there is no intuitionistic proof of $\cdot; \neg\neg\mathbb{T}, \neg\mathbb{F} \Rightarrow \bot \ \mathsf{conc}$. (If you cannot find a counterexample directly, try to prove Theorem 1 for first-order logic and see where you fail. This is an exploratory problem that will not count towards your final grade, so don't worry if you cannot solve it. An answer will be provided after the assignment due date.)

Problem 3: Admissibility of Cut with Quantifiers

In intuitionistic first-order logic, admissibility of cut is stated as follows:

Theorem 2 (Admissibility of cut)**.** *If $\mathcal{D} :: \Sigma; \Gamma \Rightarrow \varphi \ \mathsf{conc}$ and $\mathcal{E} :: \Sigma; \Gamma, \varphi \ \mathsf{hyp} \Rightarrow \psi \ \mathsf{conc}$, then there exists $\mathcal{F} :: \Sigma; \Gamma \Rightarrow \psi \ \mathsf{conc}$.*

Show the case of the proof of this theorem where the first derivation $\mathcal{D}$ ends in the rule ($\exists$R) and $\mathcal{E}$ ends in the rule ($\exists$L) and the cut is on the principal formula of the rule ($\exists$L). (Hint: You will have to use instantiation.)