A Modal Deconstruction of Access Control Logics

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Abstract. We present a translation from a logic of access control with a "says" operator to the classical modal logic S4. We prove that the translation is sound and complete. We also show that it extends to logics with boolean combinations of principals and with a "speaks for" relation. While a straightforward definition of this relation requires second-order quantifiers, we use our translation for obtaining alternative, quantifier-free presentations. We also derive decidability and complexity results for the logics of access control.

1 Introduction

In computer systems, access control checks restrict the operations that users, machines, and other principals can execute on objects such as files [27]. These checks are governed by access control policies—often by the combination of several policies at different layers and from different entities. In practice, the principals, the objects, the formulations of policies, and their implementations can be quite varied. The resulting gaps, inconsistencies, and obscurity endanger security.

In response to these concerns, specialized logics have been proposed as frameworks for describing, analyzing, and enforcing access control policies (e.g., [2, 3, 6, 10, 19, 20, 29, 30]). A number of research projects have applied these logics for designing or explaining various languages and systems (e.g., [4, 6–10, 13, 14, 16, 18, 26, 29, 35]). On the other hand, there have been only few, limited efforts to study the logics themselves (e.g., [2, 3, 19, 20]). Accordingly, the decidability, expressiveness, and semantics of these logics are largely unexplored.

Our objective in the present paper is to fill this gap. Specifically, we study a class of access control logics via sound and complete translations to the classical modal logic S4.

- Relying on the theory of S4 (e.g., [24, 25]), we obtain Kripke semantics for the logics. In the quantifier-free case, we also establish the decidability of the logics and their PSPACE complexity. The translations also open the possibility of re-using existing decision procedures for S4.
- Translating several logics to S4 enables us to compare their expressiveness. In particular, while a straightforward definition of the "speaks for" relation [26, 28] requires second-order quantifiers, we use our translations for obtaining

alternative, quantifier-free presentations. We prove that these quantifier-free presentations yield the same consequences as the second-order definition.
– The translations also suggest a logic with a boolean structure on principals. Although propositional, this new logic is rich and quite expressive. Previous logics with similar constructs allowed conjunctions and disjunctions of principals (but not negations); the present logic goes beyond them in ways that we consider both elegant and useful.

Access control logics (those studied here and most of those in the literature) include formulas of the form A `says` s, where A is a principal and s is a formula. Intuitively, A `says` s means that A asserts (or supports) s. For example, the administrator admin of a domain might certify that Alice is an authorized user; this assertion may be represented as admin `says` `auth_user`(Alice). In addition, many logics support the use of the "speaks for" relation: $\mathsf{A} \Rightarrow \mathsf{B}$ means that A speaks for B, that is, A `says` s implies B `says` s for every s. For example, when $\mathsf{Key}_{\mathsf{Alice}}$ represents the public key of Alice, one may write $\mathsf{Key}_{\mathsf{Alice}} \Rightarrow \mathsf{Alice}$. When a server S acts on Alice's behalf impersonating her, one may also write $\mathsf{S} \Rightarrow \mathsf{Alice}$. Despite these similarities, logics differ in their axioms. A 2003 survey discusses some of the options [1]. Recently, several works [2, 19, 20, 29] have basically relied upon the rules of lax logic and the computational lambda calculus [11, 17, 33] for the operator `says`. This approach has several benefits, for example validating the "handoff axiom" [2, 26]; a detailed discussion of its features is beyond the scope of this paper. We follow this approach in the logics that we consider.

The first of these logics, called ICL, extends propositional intuitionistic logic with the operator `says` which behaves as a principal-indexed lax modality (Section 2). ICL can be viewed as an indexed version of CL [11], hence its name, and also as the common propositional fragment of CDD [2, Section 8] and other systems [20, 29]. An extension of ICL, called $\mathrm{ICL}^{\Rightarrow}$, allows formulas of the form $\mathsf{A} \Rightarrow \mathsf{B}$ (Section 3). Another extension, called $\mathrm{ICL}^{\mathcal{B}}$, allows compound principals formed with boolean connectives (Section 4). Our translations and the resulting theorems apply to each of these logics. In addition, we show that $\mathsf{A} \Rightarrow \mathsf{B}$ can be encoded using either compound principals or a second-order universal quantifier (Sections 5 and 6). We conclude with a discussion of directions for further work (Section 7).

Related Work. Our translations are partly based on a translation from intuitionistic logic to S4 that goes back to Gödel [22]. Moreover, ICL can be seen as a rather direct generalization of lax logic. Nevertheless, our translation from ICL (and, as a special case, from lax logic) to S4 appears to be new.

Partly following Curry [15], Fairtlough and Mendler suggested interpreting lax logic in intuitionistic logic by mapping $\bigcirc s$ to $C \vee s$ or to $C \supset s$, where $\bigcirc$ is a lax modality and C is a fixed proposition [17]. These interpretations are sound but not complete. Composing them with a translation from intuitionistic logic to S4, one can map $\bigcirc s$ to $\Box((\Box C) \vee s)$ or to $\Box((\Box C) \supset s)$. A similar translation from lax logic to S4 follows from our definitions, as a special case; however, our translation does not put a $\Box$ on C, and it is sound and complete.

Other interpretations of lax logic have targeted multimodal logics or intuitionistic S4 [5, 11, 17, 34]. Our translations seem simpler; in particular, they target classical S4. Semantically, those interpretations lead to Kripke models with at least two accessibility relations, while we need only one.

Fairtlough and Mendler also deduced the decidability of lax logic from a subformula property [17]. Further, Howe developed a PSPACE decision procedure for lax logic [23]. It seems possible to extend Howe's approach to obtain an alternative proof of decidability for ICL. We do not know whether it would also apply to richer logics such as $\text{ICL}^{\Rightarrow}$ and $\text{ICL}^{\mathcal{B}}$, for which we have not established a subformula property.

Going beyond basic lax logic, not much is known about the theory of logics with compound principals or with a "speaks for" relation (such as $\text{ICL}^{\Rightarrow}$ and $\text{ICL}^{\mathcal{B}}$). Some of the early work on the subject started to explore semantics and decidability results [3]. Although sometimes helpful, the semantics were not sound and complete, and the decidability results applied only to fragments needed for certain access-control decisions. More recent systems like RT and SecPAL (where the "can act as" relation resembles $\Rightarrow$) include decision procedures for useful classes of formulas similar to Horn clauses [10, 31, 32].

2 ICL: A Basic Logic of Access Control

We start with a basic access control logic ICL that includes the operator says but not $\Rightarrow$. Although minimal in its constructs, the logic is reasonably expressive. We describe a translation from ICL to classical S4. From this translation we derive a Kripke semantics and a decidability result.

2.1 The Logic

Formulas in ICL may be atomic propositions (p, q, etc.) or constructed from standard connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\top$ (true), and $\bot$ (false), and the operator says.

$$s \ ::= \ p \mid s_1 \wedge s_2 \mid s_1 \vee s_2 \mid s_1 \supset s_2 \mid \top \mid \bot \mid \mathsf{A}\ \mathsf{says}\ s$$

The letters A, B, etc., denote principals, which are atomic and distinct from atomic propositions. They may be simple bit-string representations of names; in Section 4, we generalize principals to a richer algebra.

ICL inherits all the inference rules of intuitionistic propositional logic, which we elide here. For each principal A, the formula $\mathsf{A}\ \mathsf{says}\ s$ satisfies the following axioms:

$$\begin{array}{ll} \vdash s \supset (\mathsf{A}\ \mathsf{says}\ s) & \text{(unit)} \\ \vdash (\mathsf{A}\ \mathsf{says}\ (s \supset t)) \supset (\mathsf{A}\ \mathsf{says}\ s) \supset (\mathsf{A}\ \mathsf{says}\ t) & \text{(cuc)} \\ \vdash (\mathsf{A}\ \mathsf{says}\ \mathsf{A}\ \mathsf{says}\ s) \supset \mathsf{A}\ \mathsf{says}\ s & \text{(idem)} \end{array}$$

These mean that $\mathsf{A}\ \mathsf{says} \cdot$ is a lax modality [17]. We describe them briefly, referring the reader to the literature on lax logic and computational lambda calculus for more details and applications.

- (unit) states that every true formula s is supported by every principal A. (The converse is not true: principals may make false statements.)
- (cuc) allows us to reason with A's statements. It says that whenever A states $s \supset t$ and s, it also states t. Thus A's statements are closed under logical consequence.
- (idem) collapses applications of A says $\cdot$. In the context of (unit), (idem) implies that A says $\cdot$ is idempotent.

Example 1 We illustrate the use of ICL through a simple example. Consider a file-access scenario with an administrating principal admin, a user Bob, one file file1, and the following policy:

1. If admin says that file1 should be deleted, then this must be the case.
2. admin trusts Bob to decide whether file1 should be deleted.
3. Bob wants to delete file1.

Intuitively, from these facts we should be able to conclude that file1 should be deleted. We describe a logical presentation of this example in ICL. Suppose that the proposition deletefile1 means that file1 should be deleted. The three facts above can be written:

1. (admin says deletefile1) $\supset$ deletefile1
2. admin says ((Bob says deletefile1) $\supset$ deletefile1)
3. Bob says deletefile1

Using (unit) and (cuc), (1)–(3) imply deletefile1.

2.2 Translation from ICL to S4

Next we describe a central technical result of our work: a sound and complete translation from ICL to S4. Before describing the translation, we briefly sketch S4. More details may be found in standard references (e.g., [24]); S4 has been studied thoroughly over the years.

S4. S4 is an extension of classical logic with one modality $\Box$, and the rules:

From $\vdash s$ infer $\vdash \Box s$.
$\vdash \Box(s \supset t) \supset \Box s \supset \Box t$
$\vdash \Box s \supset s$
$\vdash \Box s \supset \Box\Box s$

Translation. Our translation $\ulcorner\cdot\urcorner$ from ICL to S4 is summarized in Figure 1. It is defined by induction on the structure of formulas. For atomic formulas and non-modal connectives, the translation is a slight simplification of Gödel's translation from intuitionistic logic to S4 [22]. (In Gödel's words, the basic idea is to "put a box around everything"; we simplify the translation by putting

$$
\begin{array}{rcl}
\ulcorner p \urcorner & = & \Box\, p \\
\ulcorner s \wedge t \urcorner & = & \ulcorner s \urcorner \wedge \ulcorner t \urcorner \\
\ulcorner s \vee t \urcorner & = & \ulcorner s \urcorner \vee \ulcorner t \urcorner \\
\ulcorner s \supset t \urcorner & = & \Box\, (\ulcorner s \urcorner \supset \ulcorner t \urcorner) \\
\ulcorner \top \urcorner & = & \top \\
\ulcorner \bot \urcorner & = & \bot \\
\ulcorner \mathsf{A}\ \mathsf{says}\ s \urcorner & = & \Box\, (\mathsf{A} \vee \ulcorner s \urcorner)
\end{array}
$$

Fig. 1. Translation from ICL to S4

boxes only around atomic formulas and implications.) The core of our work is the translation of A says s.

$$\ulcorner \mathsf{A}\ \mathsf{says}\ s \urcorner \;=\; \Box\, (\mathsf{A} \vee \ulcorner s \urcorner)$$

We interpret the principal A as an atomic formula in S4 and assume that such atomic formulas are distinct from the usual atomic formulas p, q, etc.. Informally, if we read $\Box$ as "in all possible worlds" and the atomic formula A as "principal A is unhappy", then $\Box\, (\mathsf{A} \vee \ulcorner s \urcorner)$ means that $\ulcorner s \urcorner$ holds in all possible worlds in which A is happy.

Alternatively, but equivalently, we could set: $\ulcorner \mathsf{A}\ \mathsf{says}\ s \urcorner \;=\; \Box\, (\mathsf{A} \supset \ulcorner s \urcorner)$. Since the target of the translation is a classical logic, the difference between $\Box\, (\mathsf{A} \vee \ulcorner s \urcorner)$ and $\Box\, (\mathsf{A} \supset \ulcorner s \urcorner)$ is only superficial. We prefer $\Box\, (\mathsf{A} \vee \ulcorner s \urcorner)$ because it leads to a more memorable interpretation of $\Rightarrow$ in Section 3.

This simple translation is correct in the sense that it is both sound and complete:

Theorem 1 (Soundness and Completeness) *For every ICL formula s, $\vdash s$ in ICL if and only if $\vdash \ulcorner s \urcorner$ in S4.*

Proof. See Appendix B.

2.3 Decidability and Kripke Models for ICL

Decidability is a desirable property in an access control logic: it allows the possibility of completely automated tools for analyzing policies. In the case of ICL, Theorem 1 implies PSPACE decidability since the same complexity bound is known for S4 [25]. This bound is the best we could expect, since PSPACE-hardness holds for plain intuitionistic propositional logic.

Corollary 1 (Decidability) *There is a polynomial space procedure that decides whether a given ICL formula is provable or not.*

Kripke models are attractive for access control logics from several perspectives. First, they provide a semantic grounding of the logics. They are also useful as mathematical objects, for instance for showing that certain formulas are not derivable. We use Theorem 1 and standard models of S4 to derive Kripke models for ICL.

Definition 1 (Kripke Models) A Kripke model for ICL is a tuple $\langle W, \leq, \rho, \theta \rangle$ where

- W is a set, whose elements are called *worlds* (denoted using the letter w and its decorated variants).
- $\leq$ is a binary relation on W called the *accessibility relation*. When $w \leq w'$, we say that w' is accessible from w. We assume that $\leq$ is reflexive and transitive. We often write $\geq$ for $(\leq)^{-1}$.
- ρ is a mapping from atomic formulas of ICL to $\mathcal{P}(W)$ (the power set of W), called the *assignment*. Intuitively, $\rho(p)$ is the set of worlds in which p holds. We assume that ρ is hereditary with respect to $\leq$, that is, if $w \in \rho(p)$, then for all w' such that $w' \geq w$, $w' \in \rho(p)$.
- θ is a mapping from principals of ICL to $\mathcal{P}(W)$, called the *view map*. When $w \in \theta(\mathsf{A})$, we say that w is *invisible* to A, else it is *visible* to A.

Definition 2 (Satisfaction) Given an ICL formula s and a Kripke model $\mathcal{K} = \langle W, \leq, \rho, \theta \rangle$, we define the satisfaction relation at a particular world ($w \models s$) by induction on s.

- $w \models p$ iff $w \in \rho(p)$
- $w \models s \wedge t$ iff $w \models s$ and $w \models t$
- $w \models s \vee t$ iff $w \models s$ or $w \models t$
- $w \models s \supset t$ iff for each $w' \geq w$, $w' \models s$ implies $w' \models t$
- $w \models \top$ for every w
- not($w \models \bot$) for every w
- $w \models \mathsf{A}\ \mathtt{says}\ s$ iff for every $w' \geq w$, either $w' \in \theta(\mathsf{A})$ or $w' \models s$

Thus, this definition implies that a world satisfies $\mathsf{A}\ \mathtt{says}\ s$ iff every reachable world that is visible to A satisfies s. For other constructs, the definition of satisfaction mirrors that in standard Kripke models of intuitionistic logic.

We say that $\mathcal{K} = \langle W, \leq, \rho, \theta \rangle \models s$ if $w \models s$ for every $w \in W$. A formula s is *valid* (written $\models s$) if $\mathcal{K} \models s$ for every Kripke model $\mathcal{K}$. The following result shows that provability in ICL coincides with validity.

Corollary 2 *For every ICL formula s, $\vdash s$ if and only if $\models s$.*

Proof. See Appendix B.

3 ICL$^{\Rightarrow}$: A Logic with A Primitive "Speaks For" Relation

In this section we extend the logic ICL to include a primitive "speaks for" relation. We call the new logic ICL$^{\Rightarrow}$. We also extend the results of Section 2 to ICL$^{\Rightarrow}$.

3.1 The Logic

ICL$^{\Rightarrow}$ extends ICL with formulas of the form $\mathsf{A} \Rightarrow \mathsf{B}$ and with the following axioms for these formulas:

$\vdash \mathsf{A} \Rightarrow \mathsf{A}$	(refl)
$\vdash (\mathsf{A} \Rightarrow \mathsf{B}) \supset (\mathsf{B} \Rightarrow \mathsf{C}) \supset (\mathsf{A} \Rightarrow \mathsf{C})$	(trans)
$\vdash (\mathsf{A} \Rightarrow \mathsf{B}) \supset (\mathsf{A}\ \mathsf{says}\ s) \supset (\mathsf{B}\ \mathsf{says}\ s)$	(speaking-for)
$\vdash (\mathsf{B}\ \mathsf{says}\ (\mathsf{A} \Rightarrow \mathsf{B})) \supset (\mathsf{A} \Rightarrow \mathsf{B})$	(handoff)

– (refl) and (trans) state that $\Rightarrow$ is reflexive and transitive.
– (speaking-for) states that if $\mathsf{A} \Rightarrow \mathsf{B}$ and $\mathsf{A}\ \mathsf{says}\ s$, then $\mathsf{B}\ \mathsf{says}\ s$.
– (handoff) states that whenever B says that A speaks for B, then A does indeed speak for B. This axiom allows every principal to decide which principals speak on its behalf [26].

Example 2 We modify Example 1: instead of having Bob says deletefile1 directly, Bob delegates his authority to Alice (fact 3), who wants to delete file1 (fact 4).

1. (admin says deletefile1) $\supset$ deletefile1
2. admin says ((Bob says deletefile1) $\supset$ deletefile1)
3. Bob says Alice $\Rightarrow$ Bob
4. Alice says deletefile1

Using (handoff) and (speaking-for), we can again derive deletefile1.

3.2 Translation from ICL$^{\Rightarrow}$ to S4

We extend to ICL$^{\Rightarrow}$ the translation from ICL to S4 by adding the clause:

$$\ulcorner \mathsf{A} \Rightarrow \mathsf{B} \urcorner = \Box(\mathsf{A} \supset \mathsf{B})$$

As above, A and B are interpreted as atomic formulas in S4, and these atomic formulas are assumed distinct from the atomic propositions of ICL$^{\Rightarrow}$. We have:

Theorem 2 (Soundness and Completeness) *For every ICL$^{\Rightarrow}$ formula s, $\vdash s$ in ICL$^{\Rightarrow}$ if and only if $\vdash \ulcorner s \urcorner$ in S4.*

Proof. See Appendix C.

3.3 Decidability and Kripke Models for ICL$^{\Rightarrow}$

Much as for ICL, Theorem 2 yields a decidability result:

Corollary 3 (Decidability) *There is a polynomial space procedure that decides whether a given ICL$^{\Rightarrow}$ formula is provable or not.*

It also leads to Kripke models for ICL$^{\Rightarrow}$. These are the same as those for ICL (Definition 1), with the satisfaction relation for $\mathsf{A} \Rightarrow \mathsf{B}$ at world w given by the clause:

$w \models \mathsf{A} \Rightarrow \mathsf{B}$ iff for every $w' \geq w$, $w' \in \theta(\mathsf{A})$ implies $w' \in \theta(\mathsf{B})$.

These models are sound and complete in the sense of Corollary 2.

4 $\mathrm{ICL}^{\mathcal{B}}$: A Logic with Boolean Principals

Principals in ICL and $\mathrm{ICL}^{\Rightarrow}$ are atomic and cannot be composed in any logically meaningful way. Early on it was observed that the use of compound principals can help in expressing policies [3, 26]. For example, the conjunction of two principals may be employed for representing joint statements, with the property

$$(\mathsf{A} \wedge \mathsf{B}) \,\mathtt{says}\, s \equiv (\mathsf{A} \,\mathtt{says}\, s) \wedge (\mathsf{B} \,\mathtt{says}\, s)$$

Disjunctions also arise, though they are more complex. Going further, we describe and study a systematic extension $\mathrm{ICL}^{\mathcal{B}}$ of ICL that allows arbitrary Boolean combinations of principals with the connectives $\wedge$, $\vee$, $\supset$, $\top$, and $\bot$. (However, we do not include operators such as "quoting" and "for".) We extend the results of Section 2 to $\mathrm{ICL}^{\mathcal{B}}$.

4.1 The Logic

The formulas of $\mathrm{ICL}^{\mathcal{B}}$ are the same as those of ICL, except that principals may contain Boolean connectives. We use the letters a, b, ... for denoting atomic principals (distinct from atomic propositions), and A, B, ... for denoting arbitrary principals.

$$\mathsf{A}, \mathsf{B} \ ::= \ a \mid \mathsf{A} \wedge \mathsf{B} \mid \mathsf{A} \vee \mathsf{B} \mid \mathsf{A} \supset \mathsf{B} \mid \top \mid \bot$$

We write $\neg\mathsf{A}$ for $(\mathsf{A} \supset \bot)$. We equip the set of principals with a notion of equality by letting $\mathsf{A} \equiv \mathsf{B}$ if A and B are provably equivalent when viewed as formulas in classical logic. With these definitions, the set of principals becomes a Boolean algebra.

$\mathrm{ICL}^{\mathcal{B}}$ inherits all the inference rules of ICL, and also includes the following additional rules:

$\vdash (\bot \,\mathtt{says}\, s) \supset s$	(trust)
If $\mathsf{A} \equiv \top$ then $\vdash \mathsf{A} \,\mathtt{says}\, \bot$.	(untrust)
$\vdash ((\mathsf{A} \supset \mathsf{B}) \,\mathtt{says}\, s) \supset (\mathsf{A} \,\mathtt{says}\, s) \supset (\mathsf{B} \,\mathtt{says}\, s)$	(cuc')

- (trust) states that $\bot$ is a truth teller.
- (untrust) states that any principal equivalent to $\top$ says false; it can be seen as a variant of the necessitation rule of modal logics.
- Similarly, (cuc') is the analogue of (cuc) for principals. It states that $\mathsf{A} \,\mathtt{says}\, s$ and $(\mathsf{A} \supset \mathsf{B}) \,\mathtt{says}\, s$ imply $\mathsf{B} \,\mathtt{says}\, s$.

We define $\mathrm{ICL}^{\mathcal{B}}$ as an extension of ICL, rather than $\mathrm{ICL}^{\Rightarrow}$, because we do not need built-in formulas of the form $\mathsf{A} \Rightarrow \mathsf{B}$. The "speaks for" relation is definable in $\mathrm{ICL}^{\mathcal{B}}$. As we show in Section 5, $\mathsf{A} \Rightarrow \mathsf{B}$ can be seen as an abbreviation for $(\mathsf{A} \supset \mathsf{B}) \,\mathtt{says}\, \bot$.

We can explain the intuitive meaning of $\mathsf{A} \,\mathtt{says}\, s$ when principal A is compound, as follows:

- $(\mathsf{A} \wedge \mathsf{B}) \,\mathtt{says}\, s$ is the same as $(\mathsf{A} \,\mathtt{says}\, s) \wedge (\mathsf{B} \,\mathtt{says}\, s)$.

- $(\mathsf{A} \vee \mathsf{B})$ says s means that, by combining the statements of A and B, we can conclude s. In particular, if A says $(s \supset t)$ and B says s then $(\mathsf{A} \vee \mathsf{B})$ says t. Disjunctions can be used in modeling groups in access control.
- $(\mathsf{A} \supset \mathsf{B})$ says s means that A speaks for B on s and on its consequences. We can show that if $(\mathsf{A} \supset \mathsf{B})$ says s and $s \supset s'$, then $(\mathsf{A}$ says $s') \supset (\mathsf{B}$ says $s')$. In the special case where s is $\perp$, we obtain the usual $\Rightarrow$ relation.
- $\top$ says s is provable for every formula s (including $\perp$). In access control terms, $\top$ may be seen as a completely untrustworthy principal.
- $\perp$ says s implies that s is true. Thus, $\perp$ is a completely trustworthy principal.

Example 3 The following policy is analogous to that of Example 1:

1. $(\mathsf{admin} \supset \perp)$ says $\mathsf{deletefile1}$
2. admin says $(\mathsf{Bob} \supset \mathsf{admin})$ says $\mathsf{deletefile1}$
3. Bob says $\mathsf{deletefile1}$

The first statement means that admin is trusted on deletefile1 and its consequences. The second statement means that admin further delegates this authority to Bob.

From (3) and (unit) it follows that admin says Bob says $\mathsf{deletefile1}$. From (2), (cuc), and (cuc') we get $(\mathsf{admin}$ says Bob says $\mathsf{deletefile1}) \supset (\mathsf{admin}$ says admin says $\mathsf{deletefile1})$. Hence we have admin says admin says $\mathsf{deletefile1}$. Using (idem), we obtain admin says $\mathsf{deletefile1}$. From (1) and (cuc'), we obtain $(\mathsf{admin}$ says $\mathsf{deletefile1}) \supset \perp$ says $\mathsf{deletefile1}$, and hence $\perp$ says $\mathsf{deletefile1}$. Finally, using (trust), we conclude deletefile1.

4.2 Translation from ICL$^{\mathcal{B}}$ to S4

The translation from ICL to S4 works virtually unchanged for ICL$^{\mathcal{B}}$. In the clause $\ulcorner \mathsf{A}$ says $s \urcorner = \Box(\mathsf{A} \vee \ulcorner s \urcorner)$, we interpret A as a formula in S4 in the most obvious way: each Boolean connective in A is mapped to the corresponding connective in S4, and each atomic principal in A is interpreted as an atomic formula in S4 (without any added boxes). For instance, the translation of

$$(\mathsf{Bob} \supset \mathsf{admin})\ \mathsf{says}\ \mathsf{deletefile1}$$

is

$$\Box((\mathsf{Bob} \supset \mathsf{admin}) \vee \Box\,\mathsf{deletefile1})$$

Again, we have soundness and completeness results:

Theorem 3 (Soundness and Completeness) *For every ICL$^{\mathcal{B}}$ formula s, $\vdash s$ in ICL$^{\mathcal{B}}$ if and only if $\vdash \ulcorner s \urcorner$ in S4.*

Proof. See Appendix D.

4.3 Decidability and Kripke Models for $\text{ICL}^{\mathcal{B}}$

Once more we obtain a decidability result:

Corollary 4 (Decidability) *There is a polynomial space procedure that decides whether a given $ICL^{\mathcal{B}}$ formula is provable or not.*

Furthermore, Kripke models for $\text{ICL}^{\mathcal{B}}$ may be obtained by generalizing those for ICL. The view map θ is defined only for atomic principals a. It is lifted to the function $\hat{\theta}$ that maps all principals to $\mathcal{P}(W)$ as follows:

$$
\begin{array}{rcl}
\hat{\theta}(a) & = & \theta(a) \\
\hat{\theta}(\mathsf{A} \wedge \mathsf{B}) & = & \hat{\theta}(\mathsf{A}) \cap \hat{\theta}(\mathsf{B}) \\
\hat{\theta}(\mathsf{A} \vee \mathsf{B}) & = & \hat{\theta}(\mathsf{A}) \cup \hat{\theta}(\mathsf{B}) \\
\hat{\theta}(\mathsf{A} \supset \mathsf{B}) & = & (W - \hat{\theta}(\mathsf{A})) \cup \hat{\theta}(\mathsf{B}) \\
\hat{\theta}(\top) & = & W \\
\hat{\theta}(\bot) & = & \emptyset
\end{array}
$$

The definition of satisfaction ($w \models s$) is modified to use $\hat{\theta}$ instead of θ:

$w \models \mathsf{A}\ \mathsf{says}\ s$ iff for all $w' \geq w$, either $w' \in \hat{\theta}(\mathsf{A})$ or $w' \models s$.

Again, these Kripke models are sound and complete in the sense of Corollary 2.

Thus, while the analysis of the translations requires special (and often difficult) arguments for each logic, the way in which decidability and semantics follow from translations is almost identical across logics. In the remainder of the paper, we turn to more unexpected consequences of the translations.

5 From $\text{ICL}^{\Rightarrow}$ to $\text{ICL}^{\mathcal{B}}$: Expressing "Speaks For" via Boolean Principals

We prove that $\mathsf{A} \Rightarrow \mathsf{B}$ can be encoded as $(\mathsf{A} \supset \mathsf{B})\ \mathsf{says}\ \bot$. More precisely, we analyze the following translation $\overline{(\cdot)}$ from $\text{ICL}^{\Rightarrow}$ to $\text{ICL}^{\mathcal{B}}$. It maps every connective except $\Rightarrow$ to itself.

$$
\begin{array}{rcl}
\overline{p} & = & p \\
\overline{s \wedge t} & = & \overline{s} \wedge \overline{t} \\
\overline{s \vee t} & = & \overline{s} \vee \overline{t} \\
\overline{s \supset t} & = & \overline{s} \supset \overline{t} \\
\overline{\top} & = & \top \\
\overline{\bot} & = & \bot \\
\overline{\mathsf{A}\ \mathsf{says}\ s} & = & \mathsf{A}\ \mathsf{says}\ \overline{s} \\
\overline{\mathsf{A} \Rightarrow \mathsf{B}} & = & (\mathsf{A} \supset \mathsf{B})\ \mathsf{says}\ \bot
\end{array}
$$

(Alternatively, we could translate an extension of $\text{ICL}^{\mathcal{B}}$ with $\Rightarrow$ to $\text{ICL}^{\mathcal{B}}$.) The encoding of $\Rightarrow$ is correct, in the following sense:

Theorem 4 *For every ICL$^{\Rightarrow}$ formula s, $\vdash s$ in ICL$^{\Rightarrow}$ if and only if $\vdash \overline{s}$ in ICL$^{\mathcal{B}}$.*

This theorem is easy to establish using the translations from ICL$^{\Rightarrow}$ and ICL$^{\mathcal{B}}$ to S4. First we show that for every formula s in ICL$^{\Rightarrow}$, $\ulcorner s \urcorner$ and $\ulcorner \overline{s} \urcorner$ are provably equivalent in S4. This argument is by a structural induction on s. The only interesting case is for s of the form $\mathsf{A} \Rightarrow \mathsf{B}$, where we observe that $\ulcorner \mathsf{A} \Rightarrow \mathsf{B} \urcorner = \Box(\mathsf{A} \supset \mathsf{B}) \equiv \Box((\mathsf{A} \supset \mathsf{B}) \vee \bot) = \ulcorner \overline{\mathsf{A} \Rightarrow \mathsf{B}} \urcorner$. It then follows from Theorems 2 and 3 that $\vdash s$ iff $\vdash \ulcorner s \urcorner$ iff $\vdash \ulcorner \overline{s} \urcorner$ iff $\vdash \overline{s}$.

6 On Second-Order Quantification

In this section we consider a logic with second-order quantification. In this logic, $\mathsf{A} \Rightarrow \mathsf{B}$ has a well-known, compelling definition, as an abbreviation for

$$\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X$$

Our main technical goal is to relate this definition to the quantifier-free axiomatizations of Sections 3–5. We prove that those axiomatizations are sound and complete with respect to the second-order definition. Thus, the full power and complexity of second-order quantification is not required for reasoning about $\Rightarrow$. A decidable fragment of the second-order logic suffices.

(This result was far from obvious to us: a priori, it seemed entirely possible that the axiomatizations were missing some subtle consequence of the second-order definition. Its proof was also surprising, as it includes a non-constructive detour through Kripke models, thus leveraging the work of Sections 3–5.)

6.1 The Logic

The second-order logic is the straightforward extension of ICL with universal quantification over propositions, with the rules of System F [12, 21].

This logic is not entirely new. It has previously been defined [2, Section 8] and used [18] under the name CDD (with only minor syntactic differences). Here we call it ICL$^{\forall}$ for the sake of uniformity.

The addition of second-order quantification provides great expressiveness, as illustrated by the definition of $\Rightarrow$ given above. On the other hand, it immediately leads to undecidability as well as to other difficulties. Nevertheless, this logic is an obvious and elegant extension of ICL.

6.2 Main Results

Though we do not discuss the theory of ICL$^{\forall}$ in detail, we have had to develop some of it in the course of our study of $\Rightarrow$. In this section we present only our main result on $\Rightarrow$ and mention other developments to the extent that they are relevant to this result.

There is an obvious embedding of ICL$^{\Rightarrow}$ into ICL$^{\forall}$:

$$\begin{array}{rcl} [\![p]\!] & = & p \\ [\![s \wedge t]\!] & = & [\![s]\!] \wedge [\![t]\!] \\ [\![s \vee t]\!] & = & [\![s]\!] \vee [\![t]\!] \\ [\![s \supset t]\!] & = & [\![s]\!] \supset [\![t]\!] \\ [\![\top]\!] & = & \top \\ [\![\bot]\!] & = & \bot \\ [\![\mathsf{A}\ \mathtt{says}\ s]\!] & = & \mathsf{A}\ \mathtt{says}\ [\![s]\!] \\ [\![\mathsf{A} \Rightarrow \mathsf{B}]\!] & = & \forall X.\ \mathsf{A}\ \mathtt{says}\ X \supset \mathsf{B}\ \mathtt{says}\ X \end{array}$$

This embedding is correct, in the following sense:

Theorem 5 *For every ICL$^{\Rightarrow}$ formula s, $\vdash s$ in ICL$^{\Rightarrow}$ if and only if $\vdash [\![s]\!]$ in ICL$^{\forall}$.*

Soundness (the implication from left to right) is easy to establish. It suffices to show that each axiom of ICL$^{\Rightarrow}$ can be simulated in ICL$^{\forall}$ after translation. A formal proof is described in Appendix E.

Completeness (the implication from right to left) is much harder. Complications arise because a proof of $[\![s]\!]$ may contain formulas which are not in the image of $[\![\cdot]\!]$. Even if we wish to restrict attention to a fragment in which the universal quantifier is restricted to formulas of the form $\forall X.\ \mathsf{A}\ \mathtt{says}\ X \supset \mathsf{B}\ \mathtt{says}\ X$, the proofs of theorems in this fragment may mention formulas that contain universal quantifiers in other positions. Although it is conceivable that a constructive proof-theoretic argument would be viable, this difficulty leads us to a non-constructive argument through acyclic Kripke models.

Our approach seems to be new, so we discuss it in some detail (Appendix E contains a full proof). It is as follows.

- First we define a translation from ICL$^{\forall}$ to second-order S4 (called S4$^{\forall}$), that is, classical S4 with a second-order universal quantifier. Let us call this translation $\ulcorner\cdot\urcorner$. This translation essentially mimics the translation of ICL to S4, and in addition maps $\forall X.\ s$ to $\Box\forall X.\ \ulcorner s\urcorner$.
  We show that this translation is sound, in the sense that $\vdash s$ in ICL$^{\forall}$ implies $\vdash \ulcorner s\urcorner$ in S4$^{\forall}$. It follows immediately that $\vdash [\![s]\!]$ in ICL$^{\forall}$ implies $\vdash \ulcorner[\![s]\!]\urcorner$ in S4$^{\forall}$.
  (We do not need to be concerned about the completeness of this translation for our purposes.)
- Next we may try to show that for every ICL$^{\Rightarrow}$ formula s, if $\vdash \ulcorner[\![s]\!]\urcorner$ in S4$^{\forall}$ then $\vdash \ulcorner s\urcorner$ in S4. If this were true, Theorem 2 would yield that $\vdash [\![s]\!]$ in ICL$^{\forall}$ implies $\vdash s$ in ICL$^{\Rightarrow}$ (because $\vdash [\![s]\!]$ in ICL$^{\forall}$ implies $\vdash \ulcorner[\![s]\!]\urcorner$ in S4$^{\forall}$, as noted above).
  Thus, it would suffice to establish that $\vdash \ulcorner[\![s]\!]\urcorner$ in S4$^{\forall}$ implies $\vdash \ulcorner s\urcorner$ in S4. We try to prove this by induction on s. Unfortunately, the proof does not go through. The argument fails for a formula of the form $\mathsf{A} \Rightarrow \mathsf{B}$, since

$$\ulcorner[\![\mathsf{A} \Rightarrow \mathsf{B}]\!]\urcorner = \Box\forall X.\ \Box\,(\Box\,(\mathsf{A} \vee \Box\, X) \supset \Box\,(\mathsf{B} \vee \Box\, X))$$

and

$$\ulcorner \mathsf{A} \Rightarrow \mathsf{B} \urcorner = \Box\,(\mathsf{A} \supset \mathsf{B})$$

In $S4^{\forall}$, the latter implies the former, but the former does not imply the latter.

- Two observations allow the proof to go through:
  1. On all acyclic models, $\ulcorner [\![\mathsf{A} \Rightarrow \mathsf{B}]\!] \urcorner$ implies $\ulcorner \mathsf{A} \Rightarrow \mathsf{B} \urcorner$.
     Therefore, we can establish that all acyclic models satisfy $\ulcorner [\![s]\!] \urcorner$ if and only if all acyclic models satisfy $\ulcorner s \urcorner$.
  2. Quantifier-free S4 is sound and complete with respect to acyclic models. (A model can be "unrolled", and the resulting acyclic model satisfies the same quantifier-free formulas as the original model.)
- Using these observations we can complete our proof as follows.
  - Suppose that $\vdash [\![s]\!]$ in $\mathrm{ICL}^{\forall}$.
  - By the soundness of the translation from $\mathrm{ICL}^{\forall}$ to $S4^{\forall}$, we obtain $\vdash \ulcorner [\![s]\!] \urcorner$ in $S4^{\forall}$.
  - Therefore every acyclic model of $S4^{\forall}$ satisfies $\ulcorner [\![s]\!] \urcorner$.
  - By (1), every acyclic model of $S4^{\forall}$ satisfies $\ulcorner s \urcorner$.
  - Since, for S4 formulas (without quantifiers), the models of $S4^{\forall}$ are the same as the models of S4, every acyclic model of S4 satisfies $\ulcorner s \urcorner$.
  - By (2), every model of S4 satisfies $\ulcorner s \urcorner$.
  - By the completeness of S4 for its models, it follows that $\vdash \ulcorner s \urcorner$ in S4.
  - By Theorem 2, we conclude that $\vdash s$ in $\mathrm{ICL}^{\Rightarrow}$.

7 Conclusion

Starting with a basic logic with a `says` operator, this paper describes simple translations of three logics of access control to S4. The translations lead to decidability results and semantics, and also to comparison of the logics. In particular, the translations enable us to study definitions and axiomatization of the "speaks for" relation.

Going further, one may attempt to carry out a similar programme for some of the diverse logics that appear in the literature. At present, there is no metric to compare these logics against each other, nor a method for integrating more than one logic into a single system. Translation to a standard logic such as S4 seems a promising approach for addressing both of these issues. Of course, first-order and second-order constructs may sometimes be necessary, and more substantial deviations from S4 may arise too—for instance, towards S5, or by the addition of special-purpose operators. Understanding those deviations may be instructive.

Going further, too, our results may be of practical value. They may serve as the basis for theorem provers for logics of access control, with the help of existing algorithms and provers for S4. More speculatively, finite models (of the kind that we obtain from our semantics) may also play a role in a new variant of proof-carrying authentication [6]. A model can serve as evidence that a particular formula is not valid, thus enabling the use of such negative information as an input to authorization decisions. These applications of our results are intriguing; they still require considerable design and experimentation.

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$$\frac{}{\Delta; \Gamma, p \Longrightarrow \Phi, p}(\texttt{init}) \qquad \frac{\Delta, s; \Gamma, s \Longrightarrow \Phi}{\Delta, s; \Gamma \Longrightarrow \Phi}(\texttt{copy})$$

$$\frac{\Delta; \Gamma, s \Longrightarrow \Phi, t}{\Delta; \Gamma \Longrightarrow \Phi, s \supset t}(\supset \texttt{R}) \qquad \frac{\Delta; \Gamma \Longrightarrow \Phi, s \quad \Delta; \Gamma, t \Longrightarrow \Phi}{\Delta; \Gamma, s \supset t \Longrightarrow \Phi}(\supset \texttt{L})$$

$$\frac{\Delta; \Gamma \Longrightarrow \Phi, s \quad \Delta; \Gamma \Longrightarrow \Phi, t}{\Delta; \Gamma \Longrightarrow \Phi, s \wedge t}(\wedge \texttt{R}) \qquad \frac{\Delta; \Gamma, s, t \Longrightarrow \Phi}{\Delta; \Gamma, s \wedge t \Longrightarrow \Phi}(\wedge \texttt{L})$$

$$\frac{\Delta; \Gamma \Longrightarrow \Phi, s, t}{\Delta; \Gamma \Longrightarrow \Phi, s \vee t}(\vee \texttt{R}) \qquad \frac{\Delta; \Gamma, s \Longrightarrow \Phi \quad \Delta; \Gamma, t \Longrightarrow \Phi}{\Delta; \Gamma, s \vee t \Longrightarrow \Phi}(\vee \texttt{L})$$

$$\frac{}{\Delta; \Gamma \Longrightarrow \Phi, \top}(\top\texttt{R}) \qquad \text{no } (\top\texttt{L}) \text{ rule}$$

$$\text{no } (\bot\texttt{R}) \text{ rule} \qquad \frac{}{\Delta; \Gamma, \bot \Longrightarrow \Phi}(\bot\texttt{L})$$

$$\frac{\Delta; \cdot \Longrightarrow s}{\Delta; \Gamma \Longrightarrow \Phi, \Box\, s}(\Box\texttt{R}) \qquad \frac{\Delta, s; \Gamma \Longrightarrow \Phi}{\Delta; \Gamma, \Box\, s \Longrightarrow \Phi}(\Box\texttt{L})$$

Fig. 2. Cut-free sequent calculus for S4

A Classical Modal Logic S4

This appendix summarizes modal logic S4 and describes a sequent calculus for it. We do not take negation as a primitive; instead it may be defined as $\neg A = (A \supset \bot)$. We omit the modality $\Diamond$ because it is not relevant to our discussion. It may be defined as $\Diamond s = \neg\Box\neg s$. Formulas in S4 are described by the following grammar:

$$s, t \ ::= \ p \mid s \supset t \mid s \wedge t \mid s \vee t \mid \top \mid \bot \mid \Box\, s$$

A.1 Sequent Calculus for S4

A sequent is written $\Delta; \Gamma \Longrightarrow \Phi$, where Δ, Γ and Φ are multisets of formulas. The sequent means that if we assume that all formulas in Δ are tautologies, and all formulas in Γ are true, then the disjunction of all formulas in Φ is true. The rules for this sequent calculus are shown in Figure 2. Cut is not a rule in the calculus. It is admissible, as Theorem 6 shows.

Lemma 1

1. *(Weakening) If $\Delta; \Gamma \Longrightarrow \Phi$, then each of the following holds by an equal or shorter derivation*
   (a) $\Delta, s; \Gamma \Longrightarrow \Phi$

   (b) $\Delta; \Gamma, s \Longrightarrow \Phi$
   (c) $\Delta; \Gamma \Longrightarrow \Phi, s$
2. *(Inversion) For each rule other than* ($\mathtt{init}$), ($\top\mathtt{R}$), ($\bot\mathtt{L}$) *and* ($\Box\mathtt{R}$), *if the conclusion of the rule holds, then the premises hold by an equal or shorter derivation.*

Proof. (Outline) In each case by induction on the height of the given derivation. For (2) a separate induction has to be performed for each rule.

Lemma 2 (Strengthening)

1. *If* $\Delta, s, s; \Gamma \Longrightarrow \Phi$*, then* $\Delta, s; \Gamma \Longrightarrow \Phi$.
2. *If* $\Delta; \Gamma, s, s \Longrightarrow \Phi$*, then* $\Delta; \Gamma, s \Longrightarrow \Phi$.
3. *If* $\Delta; \Gamma \Longrightarrow \Phi, s, s$*, then* $\Delta; \Gamma \Longrightarrow \Phi, s$.

Proof. By a simultaneous induction on the given derivations. The proof makes use of inversion from Lemma 1. Here's one representative case. Suppose we are proving (3) and the last rule is ($\wedge\,\mathtt{R}$). Let us assume that the formula s is principal. Thus we have the following situation:

$$\frac{\Delta; \Gamma \Longrightarrow \Phi, s \wedge t, s \qquad \Delta; \Gamma \Longrightarrow \Phi, s \wedge t, t}{\Delta; \Gamma \Longrightarrow \Phi, s \wedge t, s \wedge t}(\wedge\,\mathtt{R})$$

We want to show that $\Delta; \Gamma \Longrightarrow \Phi, s \wedge t$. Here's a proof:

1. (Inversion on first premise) $\Delta; \Gamma \Longrightarrow \Phi, s, s$
2. (i.h. on (1)) $\Delta; \Gamma \Longrightarrow \Phi, s$
3. (Inversion on second premise) $\Delta; \Gamma \Longrightarrow \Phi, t, t$
4. (i.h. on (3)) $\Delta; \Gamma \Longrightarrow \Phi, t$
5. (($\wedge\,\mathtt{R}$) on (2),(4)) $\Delta; \Gamma \Longrightarrow \Phi, s \wedge t$

Theorem 6 (Admissibility of Cut)

1. *If* $\Delta; \Gamma \Longrightarrow \Phi, s$ *and* $\Delta; \Gamma', s \Longrightarrow \Phi'$*, then* $\Delta; \Gamma, \Gamma' \Longrightarrow \Phi, \Phi'$.
2. *If* $\Delta; \cdot \Longrightarrow s$ *and* $\Delta, s; \Gamma \Longrightarrow \Phi$*, then* $\Delta; \Gamma \Longrightarrow \Phi$.

Proof. By simultaneous lexicographic induction, first on the size of the cut formula and then on the size of the derivations. We assume that a formula s in Δ is larger than the same formula in Γ.

Theorem 7 (Identity) $\Delta; \Gamma, s \Longrightarrow \Phi, s$ *for every formula* s.

Proof. By induction on the formula s.

A.2 Hilbert System and Kripke Semantics for S4

A Hilbert-style system may be defined for S4 by taking all standard axioms of classical logic (omitted here for brevity) and the four rules:

– From $\vdash s$ infer $\vdash \Box s$
– $\vdash \Box(s \supset t) \supset \Box s \supset \Box t$
– $\vdash \Box s \supset s$
– $\vdash \Box s \supset \Box\Box s$

A Kripke model of S4 is a 3-tuple $\mathcal{S} = \langle W, \leq, \rho \rangle$, where W is a set of worlds, $\leq$ is a reflexive and transitive binary relation on worlds, and ρ is map from atomic formulas to the $\mathcal{P}(W)$. Given $w \in W$, the relation "w satisfies s" (written $w \models s$) is defined by induction on s as follows:

- $w \models p$ iff $w \in \rho(p)$
- $w \models s \wedge t$ iff $w \models s$ and $w \models t$
- $w \models s \vee t$ iff $w \models s$ or $w \models t$
- $w \models s \supset t$ iff $w \not\models s$ or $w \models t$
- $w \models \top$ for every w
- $w \not\models \bot$ for any w
- $w \models \Box s$ iff for every $w' \geq w$, $w' \models s$.

We say that a model $\mathcal{S} = \langle W, \leq, \rho \rangle$ satisfies s (written $\mathcal{S} \models s$) if for every $w \in W$, $w \models s$. A formula s is called valid, written $\models s$, if every Kripke model satisfies it.

Theorem 8 (Equivalence) *The following are equivalent for a formula s in S4:*

1. $\cdot;\cdot \Longrightarrow s$ *in the sequent calculus*
2. $\models s$ *in Kripke models*
3. $\vdash s$ *in the Hilbert system*

Proof. (1) $\Rightarrow$ (2). We can show a more general result by induction on sequent calculus derivations. If $\Delta; \Gamma \Longrightarrow \Phi$, then for every model $\mathcal{S}$ and every world $w \in \mathcal{S}$, $w \models (((\Box \Delta) \wedge \Gamma) \supset (\vee \Phi))$.
(2) $\Rightarrow$ (3) is standard. See for instance [**?**].
(3) $\Rightarrow$ (1). This is easily established by showing that each rule of the Hilbert system can be derived in the sequent calculus.

B Details from Section 2

In this appendix we present a sequent calculus for ICL and prove the theorems of Section 2. A sequent has the form $\Gamma \Longrightarrow s$, where Γ is a multiset of formulas. Rules for the sequent calculus are described in Figure 3. It can be shown that this calculus is equivalent to the Hilbert-style axiomatization described in Section 2.

Lemma 3 (Structural Properties)

$$\frac{}{\Gamma, p \Longrightarrow p}(\texttt{init}) \qquad \frac{}{\Gamma \Longrightarrow \top}(\top\texttt{R}) \qquad \frac{}{\Gamma, \bot \Longrightarrow s}(\bot\texttt{L})$$

$$\frac{\Gamma, s \Longrightarrow t}{\Gamma \Longrightarrow s \supset t}(\supset \texttt{R}) \qquad \frac{\Gamma, s \supset t \Longrightarrow s \qquad \Gamma, t, s \supset t \Longrightarrow s'}{\Gamma, s \supset t \Longrightarrow s'}(\supset \texttt{L})$$

$$\frac{\Gamma \Longrightarrow s \qquad \Gamma \Longrightarrow t}{\Gamma \Longrightarrow s \wedge t}(\wedge \texttt{R}) \qquad \frac{\Gamma, s, t, s \wedge t \Longrightarrow s'}{\Gamma, s \wedge t \Longrightarrow s'}(\wedge \texttt{L})$$

$$\frac{\Gamma \Longrightarrow s}{\Gamma \Longrightarrow s \vee t}(\vee \texttt{R}_1) \qquad \frac{\Gamma \Longrightarrow t}{\Gamma \Longrightarrow s \vee t}(\vee \texttt{R}_2)$$

$$\frac{\Gamma, s, s \vee t \Longrightarrow s' \qquad \Gamma, t, s \vee t \Longrightarrow s'}{\Gamma, s \vee t \Longrightarrow s'}(\vee \texttt{L})$$

$$\frac{\Gamma \Longrightarrow s}{\Gamma \Longrightarrow \mathsf{A}\ \mathsf{says}\ s}(\mathsf{saysR}) \qquad \frac{\Gamma, s, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{A}\ \mathsf{says}\ s'}{\Gamma, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{A}\ \mathsf{says}\ s'}(\mathsf{saysL})$$

Fig. 3. Cut-free sequent calculus for ICL

1. *(Weakening) If $\Gamma \Longrightarrow s$, then $\Gamma, s' \Longrightarrow s$.*
2. *(Inversion) The rules* $(\wedge\ \texttt{R})$ *and* $(\supset\ \texttt{R})$ *are invertible.*
3. *(Strengthening) If $\Gamma, s, s \Longrightarrow s'$, then $\Gamma, s \Longrightarrow s'$.*

Proof. In each case by induction on the given derivation.

Theorem 9 (Admissibility of Cut)

If $\Gamma \Longrightarrow s$ and $\Gamma, s \Longrightarrow s'$ then $\Gamma \Longrightarrow s'$.

Proof. By a lexicographic induction, first on the size of the cut formula, and then on the size of the given derivations.

Theorem 10 (Identity) *For every formula s, $\Gamma, s \Longrightarrow s$.*

Proof. By induction on s.

B.1 Proofs from Section 2

Lemma 4 *For every ICL formula s, $\vdash \ulcorner s \urcorner \supset \Box \ulcorner s \urcorner$ in S4. (Or equivalently, $\cdot; \ulcorner s \urcorner \Longrightarrow \Box \ulcorner s \urcorner$.)*

Proof. We show that for every S4 Kripke frame $\mathcal{S}$, $\mathcal{S} \models \ulcorner s \urcorner \supset \Box \ulcorner s \urcorner$. Then it follows by definition that $\models \ulcorner s \urcorner \supset \Box \ulcorner s \urcorner$ and hence by Theorem 8 that $\vdash \ulcorner s \urcorner \supset \Box \ulcorner s \urcorner$. So let $\mathcal{S} = \langle W, \leq, \rho \rangle$ be an S4 Kripke frame. We use induction on s to show that for any world $w \in W$, $w \models \ulcorner s \urcorner \supset \Box \ulcorner s \urcorner$.

Case $s = p$**.** We must show that $w \models \Box p \supset \Box\Box p$. So assume that $w \models \Box p$. Now pick any $w' \geq w$. It suffices to show that $w' \models \Box p$. To show that, pick any $w'' \geq w'$. It is enough to show that $w'' \models p$. From transitivity of $\leq$ we get that $w'' \geq w$ and by assumption $w \models \Box p$, it follows that $w'' \models p$ as required.

Cases $s = \mathsf{A}\ \mathtt{says}\ s'$ **and** $s = t_1 \supset t_2$ are similar to the previous case because $\ulcorner s \urcorner$ is boxed in these cases.

Case $s = s_1 \wedge s_2$**.** We must show that $w \models (\ulcorner s_1 \urcorner \wedge \ulcorner s_2 \urcorner) \supset \Box(\ulcorner s_1 \urcorner \wedge \ulcorner s_2 \urcorner)$. Assume $w \models \ulcorner s_1 \urcorner \wedge \ulcorner s_2 \urcorner$ and pick any $w' \geq w$. It suffices to show that $w' \models \ulcorner s_i \urcorner$ for each $i = 1, 2$.

From the assumption $w \models \ulcorner s_1 \urcorner \wedge \ulcorner s_2 \urcorner$, it follows that $w \models \ulcorner s_i \urcorner$. Hence by i.h., $w \models \Box \ulcorner s_i \urcorner$. By definition, $w' \models \ulcorner s_i \urcorner$, as required.

Case $s = s_1 \vee s_2$ is similar to the previous case.

Cases $s = \top$ **and** $s = \bot$ are trivial.

Lemma 5 (Soundness) *If $\Gamma \Longrightarrow s$ in ICL, then $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \ulcorner s \urcorner$ in S4.*[4]

Proof. By induction on the derivation of $\Gamma \Longrightarrow s$. We analyze cases on the last rule in the derivation. We often use the cut theorem (Theorem 6). We write cut(a,b) to mean that we cut a formula in derivation b using the conclusion of a.

Case. $\dfrac{}{\Gamma, p \Longrightarrow p}(\mathtt{init})$

To show: $\cdot; \ulcorner \Gamma \urcorner, \Box p \Longrightarrow \Box p$. This follows from Theorem 7.

Case. $\dfrac{}{\Gamma \Longrightarrow \top}(\top\mathtt{R})$

To show: $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \top$. This follows by rule $(\top\mathtt{R})$.

Case. $\dfrac{}{\Gamma, \bot \Longrightarrow s}(\bot\mathtt{L})$

To show: $\cdot; \ulcorner \Gamma \urcorner, \bot \Longrightarrow \ulcorner s \urcorner$. This follows by rule $(\bot\mathtt{L})$.

Case. $\dfrac{\Gamma, s \Longrightarrow t}{\Gamma \Longrightarrow s \supset t}(\supset \mathtt{R})$

To show: $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box(\ulcorner s \urcorner \supset \ulcorner t \urcorner)$. We have the following proof:

1. (i.h.) $\cdot; \ulcorner \Gamma \urcorner, \ulcorner s \urcorner \Longrightarrow \ulcorner t \urcorner$.
2. (Weakening on 1) $\ulcorner \Gamma \urcorner; \ulcorner \Gamma \urcorner, \ulcorner s \urcorner \Longrightarrow \ulcorner t \urcorner$.
3. (($\mathtt{copy}$) on 2) $\ulcorner \Gamma \urcorner; \ulcorner s \urcorner \Longrightarrow \ulcorner t \urcorner$.
4. (($\supset \mathtt{R}$) on 3) $\ulcorner \Gamma \urcorner; \cdot \Longrightarrow \ulcorner s \urcorner \supset \ulcorner t \urcorner$.
5. (($\Box\mathtt{R}$) on 4) $\ulcorner \Gamma \urcorner; \cdot \Longrightarrow \Box(\ulcorner s \urcorner \supset \ulcorner t \urcorner)$.
6. (($\Box\mathtt{L}$) on 5) $\cdot; \Box \ulcorner \Gamma \urcorner \Longrightarrow \Box(\ulcorner s \urcorner \supset \ulcorner t \urcorner)$.
7. (Lemma 4) $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box \ulcorner \Gamma \urcorner$.
8. (Cut(7,6)) $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box(\ulcorner s \urcorner \supset \ulcorner t \urcorner)$.

Case. $\dfrac{\Gamma, s \supset t \Longrightarrow s \qquad \Gamma, t, s \supset t \Longrightarrow s'}{\Gamma, s \supset t \Longrightarrow s'}(\supset \mathtt{L})$

To show: $\cdot; \ulcorner \Gamma \urcorner, \Box(\ulcorner s \urcorner \supset \ulcorner t \urcorner) \Longrightarrow \ulcorner s' \urcorner$.

1. (i.h.) $\cdot; \ulcorner \Gamma \urcorner, \Box(\ulcorner s \urcorner \supset \ulcorner t \urcorner) \Longrightarrow \ulcorner s \urcorner$.

[4] $\ulcorner \Gamma \urcorner$ is the context obtained by translating each formula in Γ separately.

2. (i.h.) $\cdot; \ulcorner\Gamma\urcorner, \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner), \ulcorner t\urcorner \Longrightarrow \ulcorner s'\urcorner$.
3. (($\supset$ L) on 1,2) $\cdot; \ulcorner\Gamma\urcorner, \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner), \ulcorner s\urcorner \supset \ulcorner t\urcorner \Longrightarrow \ulcorner s'\urcorner$.
4. (Weakening on 3) $\ulcorner s\urcorner \supset \ulcorner t\urcorner; \ulcorner\Gamma\urcorner, \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner), \ulcorner s\urcorner \supset \ulcorner t\urcorner \Longrightarrow \ulcorner s'\urcorner$.
5. ((copy) on 4) $\ulcorner s\urcorner \supset \ulcorner t\urcorner; \ulcorner\Gamma\urcorner, \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner) \Longrightarrow \ulcorner s'\urcorner$.
6. (($\Box$L) on 5) $\cdot; \ulcorner\Gamma\urcorner, \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner), \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner) \Longrightarrow \ulcorner s'\urcorner$.
7. (Lemma 2 on 6) $\cdot; \ulcorner\Gamma\urcorner, \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner) \Longrightarrow \ulcorner s'\urcorner$.

Case. $\dfrac{\Gamma \Longrightarrow s \qquad \Gamma \Longrightarrow t}{\Gamma \Longrightarrow s \wedge t}(\wedge\ \mathtt{R})$

To show: $\cdot; \Gamma \Longrightarrow \ulcorner s\urcorner \wedge \ulcorner t\urcorner$.
1. (i.h.) $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \ulcorner s\urcorner$.
2. (i.h.) $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \ulcorner t\urcorner$.
3. (($\wedge$ R) on 1,2) $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \ulcorner s\urcorner \wedge \ulcorner t\urcorner$.

Case. $\dfrac{\Gamma, s, t, s \wedge t \Longrightarrow s'}{\Gamma, s \wedge t \Longrightarrow s'}(\wedge\ \mathtt{L})$

To show: $\cdot; \ulcorner\Gamma\urcorner, \ulcorner s\urcorner \wedge \ulcorner t\urcorner \Longrightarrow \ulcorner s'\urcorner$.
1. (i.h.) $\cdot; \ulcorner\Gamma\urcorner, \ulcorner s\urcorner, \ulcorner t\urcorner, \ulcorner s\urcorner \wedge \ulcorner t\urcorner \Longrightarrow \ulcorner s'\urcorner$.
2. (($\wedge$ L) on 1) $\cdot; \ulcorner\Gamma\urcorner, \ulcorner s\urcorner \wedge \ulcorner t\urcorner, \ulcorner s\urcorner \wedge \ulcorner t\urcorner \Longrightarrow \ulcorner s'\urcorner$.
3. (Lemma 2 on 2) $\cdot; \ulcorner\Gamma\urcorner, \ulcorner s\urcorner \wedge \ulcorner t\urcorner \Longrightarrow \ulcorner s'\urcorner$.

Case. $\dfrac{\Gamma \Longrightarrow s}{\Gamma \Longrightarrow s \vee t}(\vee\ \mathtt{R}_1)$

To show: $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \ulcorner s\urcorner \vee \ulcorner t\urcorner$.
1. (i.h.) $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \ulcorner s\urcorner$.
2. (Weakening on 1) $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \ulcorner s\urcorner, \ulcorner t\urcorner$.
3. (($\vee$ R) on 2) $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \ulcorner s\urcorner \vee \ulcorner t\urcorner$.

Case. $\dfrac{\Gamma \Longrightarrow t}{\Gamma \Longrightarrow s \vee t}(\vee\ \mathtt{R}_2)$

Similar to the previous case.

Case. $\dfrac{\Gamma, s, s \vee t \Longrightarrow s' \qquad \Gamma, t, s \vee t \Longrightarrow s'}{\Gamma, s \vee t \Longrightarrow s'}(\vee\ \mathtt{L})$

To show: $\cdot; \ulcorner\Gamma\urcorner, \ulcorner s\urcorner \vee \ulcorner t\urcorner \Longrightarrow \ulcorner s'\urcorner$
1. (i.h.) $\cdot; \ulcorner\Gamma\urcorner, \ulcorner s\urcorner, \ulcorner s\urcorner \vee \ulcorner t\urcorner \Longrightarrow \ulcorner s'\urcorner$.
2. (i.h.) $\cdot; \ulcorner\Gamma\urcorner, \ulcorner t\urcorner, \ulcorner s\urcorner \vee \ulcorner t\urcorner \Longrightarrow \ulcorner s'\urcorner$.
3. (($\vee$ L) on 1,2) $\cdot; \ulcorner\Gamma\urcorner, \ulcorner s\urcorner \vee \ulcorner t\urcorner, \ulcorner s\urcorner \vee \ulcorner t\urcorner \Longrightarrow \ulcorner s'\urcorner$.
4. (Lemma 2 on 3) $\cdot; \ulcorner\Gamma\urcorner, \ulcorner s\urcorner \vee \ulcorner t\urcorner \Longrightarrow \ulcorner s'\urcorner$.

Case. $\dfrac{\Gamma \Longrightarrow s}{\Gamma \Longrightarrow \mathsf{A\ says}\ s}(\mathtt{saysR})$

To show: $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \Box(\mathsf{A} \vee \ulcorner s\urcorner)$.
1. (i.h.) $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \ulcorner s\urcorner$.
2. (Lemma 4) $\cdot; \ulcorner s\urcorner \Longrightarrow \Box\ulcorner s\urcorner$.
3. (Cut(1,2)) $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \Box\ulcorner s\urcorner$.
4. (Provable in S4) $\cdot; \Box\ulcorner s\urcorner \Longrightarrow \Box(\mathsf{A} \vee \ulcorner s\urcorner)$.
5. (Cut(3,4)) $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \Box(\mathsf{A} \vee \ulcorner s\urcorner)$.

Case. $\dfrac{\Gamma, s, \mathsf{A\ says}\ s \Longrightarrow \mathsf{A\ says}\ s'}{\Gamma, \mathsf{A\ says}\ s \Longrightarrow \mathsf{A\ says}\ s'}(\mathtt{saysL})$

To show: $\cdot; \ulcorner\Gamma\urcorner, \Box(\mathsf{A} \vee \ulcorner s\urcorner) \Longrightarrow \Box(\mathsf{A} \vee \ulcorner s'\urcorner)$.

1. (i.h.) $\cdot; \ulcorner\Gamma\urcorner, \ulcorner s\urcorner, \Box(\mathsf{A} \vee \ulcorner s\urcorner) \Longrightarrow \Box(\mathsf{A} \vee \ulcorner s'\urcorner)$.
2. (Inversion for ($\Box$L) on 1) $\mathsf{A} \vee \ulcorner s\urcorner; \ulcorner\Gamma\urcorner, \ulcorner s\urcorner \Longrightarrow \Box(\mathsf{A} \vee \ulcorner s'\urcorner)$.
3. (Provable in S4) $\mathsf{A} \vee \ulcorner s\urcorner; \Box(\mathsf{A} \vee \ulcorner s'\urcorner) \Longrightarrow \mathsf{A} \vee \ulcorner s'\urcorner$.
4. (Cut(2,3)) $\mathsf{A} \vee \ulcorner s\urcorner; \ulcorner\Gamma\urcorner, \ulcorner s\urcorner \Longrightarrow \mathsf{A} \vee \ulcorner s'\urcorner$.
5. (Provable in S4) $\mathsf{A} \vee \ulcorner s\urcorner; \ulcorner\Gamma\urcorner, \mathsf{A} \Longrightarrow \mathsf{A} \vee \ulcorner s'\urcorner$.
6. (($\vee$ L) on 4,5) $\mathsf{A} \vee \ulcorner s\urcorner; \ulcorner\Gamma\urcorner, \mathsf{A} \vee \ulcorner s\urcorner \Longrightarrow \mathsf{A} \vee \ulcorner s'\urcorner$.
7. ((copy) on 6) $\mathsf{A} \vee \ulcorner s\urcorner; \ulcorner\Gamma\urcorner \Longrightarrow \mathsf{A} \vee \ulcorner s'\urcorner$.
8. (Weaken 7) $\ulcorner\Gamma\urcorner, \mathsf{A} \vee \ulcorner s\urcorner; \ulcorner\Gamma\urcorner \Longrightarrow \mathsf{A} \vee \ulcorner s'\urcorner$.
9. ((copy) on 8) $\ulcorner\Gamma\urcorner, \mathsf{A} \vee \ulcorner s\urcorner; \cdot \Longrightarrow \mathsf{A} \vee \ulcorner s'\urcorner$.
10. (($\Box$R) on 9) $\ulcorner\Gamma\urcorner, \mathsf{A} \vee \ulcorner s\urcorner; \cdot \Longrightarrow \Box(\mathsf{A} \vee \ulcorner s'\urcorner)$.
11. (($\Box$L) on 10) $\mathsf{A} \vee \ulcorner s\urcorner; \Box\ulcorner\Gamma\urcorner \Longrightarrow \Box(\mathsf{A} \vee \ulcorner s'\urcorner)$.
12. (Lemma 4) $\mathsf{A} \vee \ulcorner s\urcorner; \ulcorner\Gamma\urcorner \Longrightarrow \Box\ulcorner\Gamma\urcorner$.
13. (Cut(12,11)) $\mathsf{A} \vee \ulcorner s\urcorner; \ulcorner\Gamma\urcorner \Longrightarrow \Box(\mathsf{A} \vee \ulcorner s'\urcorner)$.
14. (($\Box$L) on 13) $\cdot; \ulcorner\Gamma\urcorner, \Box(\mathsf{A} \vee \ulcorner s\urcorner) \Longrightarrow \Box(\mathsf{A} \vee \ulcorner s'\urcorner)$.

In order to prove completeness, we define an inverse translation ($\llcorner\cdot\lrcorner$) from S4 to ICL. Since, not every sequent in S4 corresponds to an ICL sequent, this translation is only partial. We first define a notion of a *regular* S4 sequent. Intuitively, an S4 sequent is regular if it can occur in a proof of a sequent that is obtained by translating from ICL. Thus if we start from a translated ICL sequent and construct its proof backwards, we will only encounter regular sequents.

Definition 3 (Regular Sequents) An S4 sequent is called regular if it has one of the forms:

(i) $\Delta; \Gamma, \gamma \Longrightarrow \Phi$
(ii) $\Delta; \Gamma, \gamma \Longrightarrow \Phi, \mathsf{A}$ where $\mathsf{A} \notin \gamma$

and the following hold:

1. γ is a multiset of principals.
2. Δ contains assumptions of the form $\mathsf{A} \vee \ulcorner s\urcorner$, $\ulcorner s\urcorner \supset \ulcorner t\urcorner$ and p only.
3. Γ contains assumptions of the form $\mathsf{A} \vee \ulcorner s\urcorner$, $\ulcorner s\urcorner \supset \ulcorner t\urcorner$, p and $\ulcorner s\urcorner$ only.
4. Φ contains assumptions of the form $\ulcorner s\urcorner$ and p only.

Definition 4 ($\llcorner\cdot\lrcorner$) The inverse translation for formulas occurring in regular sequents is the following:

$$\begin{aligned} \llcorner\ulcorner s\urcorner\lrcorner &= s \\ \llcorner p\lrcorner &= p \\ \llcorner\ulcorner s\urcorner \supset \ulcorner t\urcorner\lrcorner &= s \supset t \\ \llcorner\mathsf{A} \vee \ulcorner s\urcorner\lrcorner &= \mathsf{A}\ \mathsf{says}\ s \end{aligned}$$

For multisets Δ, Γ, Φ the inverse translation is defined pointwise.

Lemma 6 (Completeness)

1. *If* $\Delta; \Gamma, \gamma \Longrightarrow \Phi$ *is regular, then* $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$.
2. *If* $\Delta; \Gamma, \gamma \Longrightarrow \Phi, \mathsf{A}$ *is regular, then* $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \mathsf{A}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.

Proof. We prove both statements simultaneously by induction on the given derivations. We analyze the last rule of the derivation by cases. We make heavy use of regularity to restrict the cases. We also use weakening implicitly.

Case. $\dfrac{}{\Delta; \Gamma, p, \gamma \Longrightarrow \Phi, p}(\texttt{init})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee p$. Clearly, this holds by rules $(\texttt{init})$ and $(\vee \, \texttt{R}_2)$.

Case. $\dfrac{}{\Delta; \Gamma, p, \gamma \Longrightarrow \Phi, p, \mathsf{A}}(\texttt{init})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p \Longrightarrow \mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee p)$.

1. $((\texttt{init}))$ $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p \Longrightarrow p$.
2. $((\vee \, \texttt{R}_2)$ on 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee p$.
3. $((\texttt{saysR})$ on 2) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p \Longrightarrow \mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee p)$.

Case. $\dfrac{\Delta, s; \Gamma, s, \gamma \Longrightarrow \Phi}{\Delta, s; \Gamma, \gamma \Longrightarrow \Phi}(\texttt{copy})$

To show: $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$.

1. (i.h.) $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner, \llcorner s\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$.
2. (Lemma 3 on 1) $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$.

Case. $\dfrac{\Delta, s; \Gamma, s, \gamma \Longrightarrow \Phi, \mathsf{A}}{\Delta, s; \Gamma, \gamma \Longrightarrow \Phi, \mathsf{A}}(\texttt{copy})$

To show: $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \mathsf{A}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.

1. (i.h.) $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner, \llcorner s\lrcorner \Longrightarrow \mathsf{A}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.
2. (Lemma 3 on 1) $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \mathsf{A}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.

Case. $(\supset \texttt{R})$ does not arise due to regularity.

Case. $\dfrac{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner s\urcorner \qquad \Delta; \Gamma, \ulcorner t\urcorner, \gamma \Longrightarrow \Phi}{\Delta; \Gamma, \ulcorner s\urcorner \supset \ulcorner t\urcorner, \gamma \Longrightarrow \Phi}(\supset \texttt{L})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t \Longrightarrow \vee \llcorner\Phi\lrcorner$.

1. (i.h.) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee s$.
2. (i.h.) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, t \Longrightarrow \vee \llcorner\Phi\lrcorner$.
3. (Theorem 10) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \Longrightarrow s$.
4. $((\supset \texttt{L})$ on 3,2) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t, s \Longrightarrow \vee \llcorner\Phi\lrcorner$.
5. (Theorem 10) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t, \vee \llcorner\Phi\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$.
6. $((\vee \, \texttt{L})$ on 5,4) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t, (\vee \llcorner\Phi\lrcorner) \vee s \Longrightarrow \vee \llcorner\Phi\lrcorner$.
7. (Cut(1,6)) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t \Longrightarrow \vee \llcorner\Phi\lrcorner$.

Case. $\dfrac{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner s\urcorner, \mathsf{A} \qquad \Delta; \Gamma, \ulcorner t\urcorner, \gamma \Longrightarrow \Phi, \mathsf{A}}{\Delta; \Gamma, \ulcorner s\urcorner \supset \ulcorner t\urcorner, \gamma \Longrightarrow \Phi, \mathsf{A}}(\supset \texttt{L})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t \Longrightarrow \mathsf{A}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.

1. (i.h.) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee s)$.
2. (Theorem 10) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t, \vee \llcorner\Phi\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$.
3. $((\texttt{saysR})$ on 2) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t, \vee \llcorner\Phi\lrcorner \Longrightarrow \mathsf{A}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.
4. (i.h.) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, t \Longrightarrow \mathsf{A}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.
5. (Theorem 10) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \Longrightarrow s$.
6. $((\supset \texttt{L})$ on 5,4) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t, s \Longrightarrow \mathsf{A}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.
7. $((\vee \, \texttt{L})$ on 6,3) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t, (\vee \llcorner\Phi\lrcorner) \vee s \Longrightarrow \mathsf{A}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.
8. $((\texttt{saysL})$ on 7) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t, \mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee s) \Longrightarrow \mathsf{A}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.

9. (Cut(1,8)) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t \Longrightarrow \mathsf{A}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.

Case. $$\frac{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner s \urcorner \qquad \Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner t \urcorner}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner s \urcorner \wedge \ulcorner t \urcorner}(\wedge\ \mathtt{R})$$

To show: $\llcorner\Delta\lrcorner, \llcorner\Delta\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee (s \wedge t)$.

1. (i.h.) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee s$.
2. (i.h.) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee t$.
3. (Provable in ICL) $(\vee \llcorner\Phi\lrcorner) \vee s, (\vee \llcorner\Phi\lrcorner) \vee t \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee (s \wedge t)$.
4. (Cut(1,3)) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, (\vee \llcorner\Phi\lrcorner) \vee t \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee (s \wedge t)$.
5. (Cut(2,4)) $\llcorner\Delta\lrcorner, \llcorner\Delta\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee (s \wedge t)$.

Case. $$\frac{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner s \urcorner, \mathsf{A} \qquad \Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner t \urcorner, \mathsf{A}}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner s \urcorner \wedge \ulcorner t \urcorner, \mathsf{A}}(\wedge\ \mathtt{R})$$

To show: $\llcorner\Delta\lrcorner, \llcorner\Delta\lrcorner \Longrightarrow \mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee (s \wedge t))$.

1. (i.h.) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee s)$.
2. (i.h.) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee t)$.
3. (Provable in ICL) $\mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee s), \mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee t) \Longrightarrow \mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee (s \wedge t))$.
4. (Cut(1,3)) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee t) \Longrightarrow \mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee (s \wedge t))$.
5. (Cut(2,4)) $\llcorner\Delta\lrcorner, \llcorner\Delta\lrcorner \Longrightarrow \mathsf{A}\ \mathsf{says}\ ((\vee \llcorner\Phi\lrcorner) \vee (s \wedge t))$.

Case. $(\wedge\ \mathtt{L})$ is straightforward. (The principal formula can only be $\ulcorner s \urcorner \wedge \ulcorner t \urcorner$.)

Case. $(\vee\ \mathtt{R})$ is straightforward. (The principal formula can only be $\ulcorner s \urcorner \vee \ulcorner t \urcorner$.)

Case. $$\frac{\Delta; \Gamma, \ulcorner s \urcorner, \gamma \Longrightarrow \Phi \qquad \Delta; \Gamma, \ulcorner t \urcorner, \gamma \Longrightarrow \Phi}{\Delta; \Gamma, \ulcorner s \urcorner \vee \ulcorner t \urcorner, \gamma \Longrightarrow \Phi}(\vee\ \mathtt{L})$$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \vee t \Longrightarrow \vee \llcorner\Phi\lrcorner$.

1. (i.h) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \Longrightarrow \vee \llcorner\Phi\lrcorner$.
2. (i.h.) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, t \Longrightarrow \vee \llcorner\Phi\lrcorner$.
3. ($(\vee\ \mathtt{L})$ on 1,2) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \vee t \Longrightarrow \vee \llcorner\Phi\lrcorner$.

Case. $$\frac{\Delta; \Gamma, \ulcorner s \urcorner, \gamma \Longrightarrow \Phi, \mathsf{A} \qquad \Delta; \Gamma, \ulcorner t \urcorner, \gamma \Longrightarrow \Phi, \mathsf{A}}{\Delta; \Gamma, \ulcorner s \urcorner \vee \ulcorner t \urcorner, \gamma \Longrightarrow \Phi, \mathsf{A}}(\vee\ \mathtt{L})$$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \vee t \Longrightarrow \vee\ \mathsf{A}\ \mathsf{says}\ (\llcorner\Phi\lrcorner)$.

1. (i.h) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \Longrightarrow \vee\ \mathsf{A}\ \mathsf{says}\ (\llcorner\Phi\lrcorner)$.
2. (i.h.) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, t \Longrightarrow \vee\ \mathsf{A}\ \mathsf{says}\ (\llcorner\Phi\lrcorner)$.
3. ($(\vee\ \mathtt{L})$ on 1,2) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \vee t \Longrightarrow \mathsf{A}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.

Case. $$\frac{\Delta; \Gamma, \gamma, \mathsf{A} \Longrightarrow \Phi \qquad \Delta; \Gamma, \ulcorner s \urcorner, \gamma \Longrightarrow \Phi}{\Delta; \Gamma, \mathsf{A} \vee \ulcorner s \urcorner, \gamma \Longrightarrow \Phi}(\vee\ \mathtt{L})$$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \vee \llcorner\Phi\lrcorner$.

1. (i.h. on 1st premise) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$.
2. (Weakening on 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \vee \llcorner\Phi\lrcorner$.

Case. $$\frac{\Delta; \Gamma, \gamma, \mathsf{A} \Longrightarrow \Phi, \mathsf{B} \qquad \Delta; \Gamma, \ulcorner s \urcorner, \gamma \Longrightarrow \Phi, \mathsf{B}}{\Delta; \Gamma, \mathsf{A} \vee \ulcorner s \urcorner, \gamma \Longrightarrow \Phi, \mathsf{B}}(\vee\ \mathtt{L}) \qquad [\mathsf{A} \neq \mathsf{B}]$$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{B}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.

In this case the first premise is regular because $B \notin A, \gamma$ if $B \notin \gamma$.

1. (i.h. on 1st premise) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \mathsf{B}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner)$.

2. (Weakening on 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \mathsf{A}\ \texttt{says}\ s \Longrightarrow \mathsf{B}\ \texttt{says}\ (\vee \llcorner\Phi\lrcorner)$.

Case. $$\frac{\Delta; \Gamma, \gamma, \mathsf{A} \Longrightarrow \Phi, \mathsf{A} \qquad \Delta; \Gamma, \ulcorner s\urcorner, \gamma \Longrightarrow \Phi, \mathsf{A}}{\Delta; \Gamma, \mathsf{A} \vee \ulcorner s\urcorner, \gamma \Longrightarrow \Phi, \mathsf{A}}(\vee\, \mathtt{L})$$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \mathsf{A}\ \texttt{says}\ s \Longrightarrow \mathsf{A}\ \texttt{says}\ (\vee \llcorner\Phi\lrcorner)$.

1. (i.h. on 2nd premise) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \Longrightarrow \mathsf{A}\ \texttt{says}\ (\vee \llcorner\Phi\lrcorner)$.
2. ((saysL) on 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \mathsf{A}\ \texttt{says}\ s \Longrightarrow \mathsf{A}\ \texttt{says}\ (\vee \llcorner\Phi\lrcorner)$.

Case. $$\frac{}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \top}(\top\mathtt{R})$$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee \top$. This follows by rules ($\top\mathtt{R}$) and ($\vee\, \mathtt{R}_2$).

Case. $$\frac{}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \top, \mathsf{A}}(\top\mathtt{R})$$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \mathsf{A}\ \texttt{says}\ ((\vee \llcorner\Phi\lrcorner) \vee \top)$. This follows by rules ($\top\mathtt{R}$), ($\vee\, \mathtt{R}_2$) and (saysR).

Case. $$\frac{}{\Delta; \Gamma, \bot, \gamma \Longrightarrow \Phi}(\bot\mathtt{L})$$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \bot \Longrightarrow \vee \llcorner\Phi\lrcorner$. This is immediate from rule ($\bot\mathtt{L}$).

Case. $$\frac{}{\Delta; \Gamma, \bot, \gamma \Longrightarrow \Phi, \mathsf{A}}(\bot\mathtt{L})$$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \bot \Longrightarrow \mathsf{A}\ \texttt{says}\ (\vee \llcorner\Phi\lrcorner)$. This is immediate from rule ($\bot\mathtt{L}$).

Case. $$\frac{\Delta; \cdot \Longrightarrow s}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \Box\, s}(\Box\mathtt{R})$$

By regularity, s can only have the forms p, $\ulcorner s_1\urcorner \supset \ulcorner s_2\urcorner$ and $\mathsf{A} \vee \ulcorner t\urcorner$.

Subcase. $s = p$.

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee p$.

By i.h. we get $\llcorner\Delta\lrcorner \Longrightarrow p$. Hence the result follows by weakening and ($\vee\, \mathtt{R}_2$).

Subcase. $s = \ulcorner s_1\urcorner \supset \ulcorner s_2\urcorner$.

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee (s_1 \supset s_2)$.

(a) (Inversion on premise) $\Delta; \ulcorner s_1\urcorner \Longrightarrow \ulcorner s_2\urcorner$.
(b) (i.h. on a) $\llcorner\Delta\lrcorner, s_1 \Longrightarrow s_2$.
(c) (($\supset\mathtt{R}$) on b) $\llcorner\Delta\lrcorner \Longrightarrow s_1 \supset s_2$.
(d) (($\vee\, \mathtt{R}_2$) on c) $\llcorner\Delta\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee (s_1 \supset s_2)$.
(e) (Weakening on d) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee (s_1 \supset s_2)$.

Subcase. $s = \mathsf{A} \vee \ulcorner t\urcorner$.

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee \mathsf{A}\ \texttt{says}\ s$.

(a) (Inversion on premise) $\Delta; \cdot \Longrightarrow \ulcorner t\urcorner, \mathsf{A}$.
(b) (i.h. on a) $\llcorner\Delta\lrcorner \Longrightarrow \mathsf{A}\ \texttt{says}\ s$.
(c) (($\vee\, \mathtt{R}_2$) on b) $\llcorner\Delta\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee \mathsf{A}\ \texttt{says}\ s$.
(d) (Weakening on c) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee \mathsf{A}\ \texttt{says}\ s$.

Case. $$\frac{\Delta; \cdot \Longrightarrow s}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \Box\, s, \mathsf{B}}(\Box\mathtt{R})$$

This case is similar to the previous case. We analyze the three possible forms of s. However, at the end of each possible subcase, we must also use the additional rule (saysR) to add a $\mathsf{B}\ \texttt{says}$ on the right.

Case. $\dfrac{\Delta, s; \Gamma, \gamma \Longrightarrow \Phi}{\Delta; \Gamma, \Box\, s, \gamma \Longrightarrow \Phi}(\Box\mathtt{L})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \llcorner\Box\, s\lrcorner \Longrightarrow^{\vee} \llcorner\Phi\lrcorner$.

By regularity, s must have the form p, $\ulcorner s_1 \urcorner \supset \ulcorner s_2 \urcorner$ or $\mathsf{A} \vee \ulcorner t \urcorner$. In each case observe that $\llcorner s\lrcorner = \llcorner\Box\, s\lrcorner$.

By i.h. we get $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow^{\vee} \llcorner\Phi\lrcorner$.

Therefore, $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \llcorner\Box\, s\lrcorner \Longrightarrow^{\vee} \llcorner\Phi\lrcorner$.

Case. $\dfrac{\Delta, s; \Gamma, \gamma \Longrightarrow \Phi, \mathsf{A}}{\Delta; \Gamma, \Box\, s, \gamma \Longrightarrow \Phi, \mathsf{A}}(\Box\mathtt{L})$

Similar to the previous case.

Proof (Proof of Theorem 1).

1. Suppose $\vdash s$ in ICL. Then $\cdot \Longrightarrow s$. By Lemma 5, $\cdot; \cdot \Longrightarrow \ulcorner s \urcorner$ in S4. Hence by Theorem 8, $\vdash \ulcorner s \urcorner$.
2. Suppose $\vdash \ulcorner s \urcorner$ in S4. By Theorem 8, $\cdot; \cdot \Longrightarrow \ulcorner s \urcorner$. Hence by Lemma 6, $\cdot \Longrightarrow \llcorner\ulcorner s \urcorner\lrcorner$ in ICL. By definition, $\llcorner\ulcorner s \urcorner\lrcorner = s$. Therefore, $\cdot \Longrightarrow s$, i.e., $\vdash s$.

Proof (Proof of Corollary 2). We prove this theorem by constructing from every ICL Kripke model of s, an equivalent S4 Kripke model of $\ulcorner s \urcorner$ and vice-versa.

For an ICL Kripke model $\mathcal{K} = \langle W, \leq, \rho, \theta \rangle$, define an S4 Kripke model $\phi(\mathcal{K}) = \langle W, \leq, \rho' \rangle$, where $\rho'(p) = \rho(p)$ on atomic formulas and $\rho'(A) = \theta(A)$ for principals. It is easy to show by induction on s that $\mathcal{K} \models s$ if and only if $\phi(\mathcal{K}) \models \ulcorner s \urcorner$. Now suppose that $\vdash s$. Then by Theorem 1, $\vdash \ulcorner s \urcorner$. By soundness of Kripke models with respect to S4 (Theorem 8), it follows that $\phi(\mathcal{K}) \models \ulcorner s \urcorner$ for every ICL model $\mathcal{K}$. Hence, $\mathcal{K} \models s$ and therefore, $\models s$.

For an S4 Kripke model $\mathcal{S} = \langle W, \leq, \rho' \rangle$, define an ICL Kripke model $\psi(\mathcal{S}) = \langle W, \leq, \rho, \theta \rangle$, by taking $\rho(p) = \{w \in W | \forall w' \geq w, w' \in \rho'(p)\}$ and $\theta(\mathsf{A}) = \rho'(\mathsf{A})$. It is again easy to show by induction on ICL formulas s, that $\mathcal{S} \models \ulcorner s \urcorner$ if and only if $\psi(\mathcal{S}) \models s$. Now suppose $\models s$. Then for any frame $\mathcal{S}$, $\psi(\mathcal{S}) \models s$. Therefore, $\mathcal{S} \models \ulcorner s \urcorner$. By completeness of Kripke models for S4 (Theorem 8), we get $\vdash \ulcorner s \urcorner$. Hence by Theorem 1, $\vdash s$.

C Details from Section 3

We first describe a sequent calculus for the logic $\text{ICL}^{\Rightarrow}$. This extends the sequent calculus for ICL (Figure 3) with additional rules for $\Rightarrow$. The cut rule is no longer admissible and must be added explicitly. Further, the init rule must be generalized to account for formulas of the form $\mathsf{A} \Rightarrow \mathsf{B}$. This sequent calculus is shown in Figure 4. It is easy to show that this calculus corresponds exactly to the Hilbert style axiomatization.

Lemma 7 *For every $\text{ICL}^{\Rightarrow}$ formula s, $\vdash \ulcorner s \urcorner \supset \Box \ulcorner s \urcorner$ in S4. (Or equivalently, $\cdot; \ulcorner s \urcorner \Longrightarrow \Box \ulcorner s \urcorner$.)*

(ALL RULES OF FIGURE 3 except (init) ARE INCLUDED)

$$\frac{}{\Gamma \Longrightarrow \mathsf{A} \Rightarrow \mathsf{A}}(\texttt{refl}) \qquad \frac{}{\Gamma \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{B}) \supset (\mathsf{B} \Rightarrow \mathsf{C}) \supset (\mathsf{A} \Rightarrow \mathsf{C})}(\texttt{trans})$$

$$\frac{}{\Gamma \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{B}) \supset (\mathsf{A}\ \mathsf{says}\ s) \supset (\mathsf{B}\ \mathsf{says}\ s)}(\texttt{delegate})$$

$$\frac{}{\Gamma \Longrightarrow (\mathsf{B}\ \mathsf{says}\ (\mathsf{A} \Rightarrow \mathsf{B})) \supset (\mathsf{A} \Rightarrow \mathsf{B})}(\texttt{handoff}) \qquad \frac{}{\Gamma, s \Longrightarrow s}(\texttt{init})$$

$$\frac{\Gamma \Longrightarrow s \qquad \Gamma, s \Longrightarrow s'}{\Gamma \Longrightarrow s'}(\texttt{cut})$$

Fig. 4. Sequent calculus for ICL$^{\Rightarrow}$

Proof. As in the proof of Lemma 4, we induct on s to show that for every S4 Kripke frame $\mathcal{S}$, $\mathcal{S} \models \ulcorner s \urcorner \supset \Box \ulcorner s \urcorner$. By Theorem 8, it follows that $\vdash \ulcorner s \urcorner \supset \Box \ulcorner s \urcorner$. All cases of the induction are the same as that of Lemma 4. There is just one new case: $s = \mathsf{A} \Rightarrow \mathsf{B}$. In this case, $\ulcorner s \urcorner = \Box (\mathsf{A} \supset \mathsf{B})$. So we must show for any world w in a Kripke model $\mathcal{S}$ that $w \models \Box (\mathsf{A} \supset \mathsf{B}) \supset \Box \Box (\mathsf{A} \supset \mathsf{B})$. This proof is exactly like the case $s = p$ from Lemma 4.

Lemma 8 (Soundness) *If $\Gamma \Longrightarrow s$ in ICL$^{\Rightarrow}$, then $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \ulcorner s \urcorner$ in S4.*

Proof. Exactly like Lemma 5. We show here the additional cases in ICL$^{\Rightarrow}$.

Case. $\frac{}{\Gamma \Longrightarrow \mathsf{A} \Rightarrow \mathsf{A}}(\texttt{refl})$

To show: $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box (\mathsf{A} \supset \mathsf{A})$.

1. ((init)) $\cdot; \mathsf{A} \Longrightarrow \mathsf{A}$.
2. (($\supset$ R) on 1) $\cdot; \cdot \Longrightarrow \mathsf{A} \supset \mathsf{A}$.
3. (($\Box$R) on 2) $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box (\mathsf{A} \supset \mathsf{A})$.

Case. $\frac{}{\Gamma \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{B}) \supset (\mathsf{B} \Rightarrow \mathsf{C}) \supset (\mathsf{A} \Rightarrow \mathsf{C})}(\texttt{trans})$

To show: $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box ((\Box (\mathsf{A} \supset \mathsf{B})) \supset \Box ((\Box (\mathsf{B} \supset \mathsf{C})) \supset \Box (\mathsf{A} \supset \mathsf{C})))$.

1. (Provable in S4) $\cdot; \mathsf{A} \supset \mathsf{B}, \mathsf{B} \supset \mathsf{C} \Longrightarrow \mathsf{A} \supset \mathsf{C}$.
2. (Weakening, (copy) on 1) $\mathsf{A} \supset \mathsf{B}, \mathsf{B} \supset \mathsf{C}; \cdot \Longrightarrow \mathsf{A} \supset \mathsf{C}$.
3. (($\Box$R) on 2) $\mathsf{A} \supset \mathsf{B}, \mathsf{B} \supset \mathsf{C}; \cdot \Longrightarrow \Box (\mathsf{A} \supset \mathsf{C})$.
4. (($\Box$L) on 3) $\mathsf{A} \supset \mathsf{B}; \Box (\mathsf{B} \supset \mathsf{C}) \Longrightarrow \Box (\mathsf{A} \supset \mathsf{C})$.
5. (($\supset$ R) on 4) $\mathsf{A} \supset \mathsf{B}; \cdot \Longrightarrow (\Box (\mathsf{B} \supset \mathsf{C})) \supset \Box (\mathsf{A} \supset \mathsf{C})$.
6. (($\Box$R) on 5) $\mathsf{A} \supset \mathsf{B}; \cdot \Longrightarrow \Box ((\Box (\mathsf{B} \supset \mathsf{C})) \supset \Box (\mathsf{A} \supset \mathsf{C}))$.
7. (($\Box$L) on 6) $\cdot; \Box (\mathsf{A} \supset \mathsf{B}) \Longrightarrow \Box ((\Box (\mathsf{B} \supset \mathsf{C})) \supset \Box (\mathsf{A} \supset \mathsf{C}))$.
8. (($\supset$ R) on 7) $\cdot; \cdot \Longrightarrow (\Box (\mathsf{A} \supset \mathsf{B})) \supset \Box ((\Box (\mathsf{B} \supset \mathsf{C})) \supset \Box (\mathsf{A} \supset \mathsf{C}))$.
9. (($\Box$R) on 8) $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box ((\Box (\mathsf{A} \supset \mathsf{B})) \supset \Box ((\Box (\mathsf{B} \supset \mathsf{C})) \supset \Box (\mathsf{A} \supset \mathsf{C})))$.

Case. $\frac{}{\Gamma \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{B}) \supset (\mathsf{A}\ \mathsf{says}\ s) \supset (\mathsf{B}\ \mathsf{says}\ s)}(\texttt{delegate})$

To show: $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box ((\Box (\mathsf{A} \supset \mathsf{B})) \supset \Box ((\Box (\mathsf{A} \vee \ulcorner s \urcorner)) \supset \Box (\mathsf{B} \vee \ulcorner s \urcorner)))$.

1. (Provable in S4) $\cdot; \mathsf{A} \supset \mathsf{B}, \mathsf{A} \vee \ulcorner s \urcorner \Longrightarrow \mathsf{B} \vee \ulcorner s \urcorner$.
   We repeat steps (2)-(9) from the previous case by replacing $(\mathsf{B} \supset \mathsf{C})$ with $(\mathsf{A} \vee \ulcorner s \urcorner)$ and $(\mathsf{A} \supset \mathsf{C})$ with $(\mathsf{B} \vee \ulcorner s \urcorner)$. We finally get:

9 $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box((\Box(\mathsf{A} \supset \mathsf{B})) \supset \Box((\Box(\mathsf{A} \vee \ulcorner s \urcorner)) \supset \Box(\mathsf{B} \vee \ulcorner s \urcorner)))$.

Case. $\dfrac{}{\Gamma \Longrightarrow (\mathsf{B}\ \mathtt{says}\ (\mathsf{A} \Rightarrow \mathsf{B})) \supset (\mathsf{A} \Rightarrow \mathsf{B})}(\mathtt{handoff})$

To show: $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box((\Box(\mathsf{B} \vee \Box(\mathsf{A} \supset \mathsf{B}))) \supset \Box(\mathsf{A} \supset \mathsf{B}))$.

1. (Provable in S4) $\cdot; \mathsf{B} \Longrightarrow \mathsf{A} \supset \mathsf{B}$.
2. (Provable in S4) $\cdot; \Box(\mathsf{A} \supset \mathsf{B}) \Longrightarrow \mathsf{A} \supset \mathsf{B}$.
3. (($\vee$ L) on 1,2) $\cdot; \mathsf{B} \vee \Box(\mathsf{A} \supset \mathsf{B}) \Longrightarrow \mathsf{A} \supset \mathsf{B}$.
4. (Weakening, (copy) on 3) $\mathsf{B} \vee \Box(\mathsf{A} \supset \mathsf{B}); \cdot \Longrightarrow \mathsf{A} \supset \mathsf{B}$.
5. (($\Box$R) on 4) $\mathsf{B} \vee \Box(\mathsf{A} \supset \mathsf{B}); \cdot \Longrightarrow \Box(\mathsf{A} \supset \mathsf{B})$.
6. (($\Box$L) on 5) $\cdot; \Box(\mathsf{B} \vee \Box(\mathsf{A} \supset \mathsf{B})) \Longrightarrow \Box(\mathsf{A} \supset \mathsf{B})$.
7. (($\supset$ R) on 6) $\cdot; \cdot \Longrightarrow (\Box(\mathsf{B} \vee \Box(\mathsf{A} \supset \mathsf{B}))) \supset \Box(\mathsf{A} \supset \mathsf{B})$.
8. (($\Box$R) on 7) $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box((\Box(\mathsf{B} \vee \Box(\mathsf{A} \supset \mathsf{B}))) \supset \Box(\mathsf{A} \supset \mathsf{B}))$.

Case. $\dfrac{}{\Gamma, s \Longrightarrow s}(\mathtt{init})$

To show: $\cdot; \ulcorner \Gamma \urcorner, \ulcorner s \urcorner \Longrightarrow \ulcorner s \urcorner$.

Follows immediately from Theorem 7.

Case. $\dfrac{\Gamma \Longrightarrow s \qquad \Gamma, s \Longrightarrow s'}{\Gamma \Longrightarrow s'}(\mathtt{cut})$

To show: $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \ulcorner s' \urcorner$.

1. (i.h.) $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \ulcorner s \urcorner$.
2. (i.h.) $\cdot; \ulcorner \Gamma \urcorner, \ulcorner s \urcorner \Longrightarrow \ulcorner s' \urcorner$.
3. (Cut(1,2) in S4) $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \ulcorner s' \urcorner$.

Next we prove completeness of the translation from $\mathrm{ICL}^{\Rightarrow}$ to S4. As before, we syntactically characterize regular sequents, i.e., those sequents that may potentially occur in a derivation whose conclusion is obtained by translation. In this case there are two kinds of regular sequents: α-regular sequents and β-regular sequents. A translated sequent is always an α-regular sequent. A β-regular sequent also carries a label with it. This label is the name of principal. It is written over the sequent arrow, as in $\Gamma \overset{\mathsf{A}}{\Longrightarrow} s$. These labels are *inferred* from a given derivation. Hence they are merely a proof tool, not a modification to the proof system of S4. The exact manner in which labels are inferred is described after the definition of regular sequents.

Definition 5 (Regular Sequents) An S4 sequent is called α**-regular** if it has the form:

$$\Delta; \Gamma, \gamma \Longrightarrow \Phi$$

and the following hold:

1. γ is a multiset of principals.
2. Δ contains assumptions of the form $\mathsf{A} \vee \ulcorner s \urcorner$, $\ulcorner s \urcorner \supset \ulcorner t \urcorner$, p and $\mathsf{A} \supset \mathsf{B}$ only.
3. Γ contains assumptions of the form $\mathsf{A} \vee \ulcorner s \urcorner$, $\ulcorner s \urcorner \supset \ulcorner t \urcorner$, p, $\mathsf{A} \supset \mathsf{B}$ and $\ulcorner s \urcorner$ only.

4. Φ contains assumptions of the form $\ulcorner s \urcorner$ and p only.

An S4 sequent is called **β-regular** if it has the form:

$$\Delta; \Gamma, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi$$

and the following hold:

1. C is an inferred label (described later).
2. γ and ϕ are multisets of principals.
3. Δ contains assumptions of the form $\mathsf{A} \vee \ulcorner s \urcorner$, $\ulcorner s \urcorner \supset \ulcorner t \urcorner$, p and $\mathsf{A} \supset \mathsf{B}$ only.
4. Γ contains assumptions of the form $\mathsf{A} \vee \ulcorner s \urcorner$, $\ulcorner s \urcorner \supset \ulcorner t \urcorner$, p, $\mathsf{A} \supset \mathsf{B}$ and $\ulcorner s \urcorner$ only.
5. Φ contains assumptions of the form $\ulcorner s \urcorner$ and p only.

Inferring labels on β-regular sequents. Given a derivation of a translated sequent, we infer labels on its sequents starting from the *conclusion* and proceeding to the leaves, according to the following rules.

1. The final conclusion of the entire derivation has no label since it is an α-regular sequent.
2. If the conclusion of a rule other than $\Box R$ has some label, then the premises have the same label. If the conclusion has no label, nor do the premises.
3. For a rule $\Box R$, the label on the premise is determined by the principal formula. The following table lists the possible principal formulas and the corresponding labels on the premises. If "No label" is listed, it means that the premise is an α-regular sequent.

Principal Formula	Label
$\Box\, p$	No label
$\Box\, (\mathsf{A} \supset \mathsf{B})$	B
$\Box\, (\mathsf{A} \vee \ulcorner s \urcorner)$	A
$\Box\, (\ulcorner s \urcorner \supset \ulcorner t \urcorner)$	No label

Now we define an inverse translation from formulas and principals occurring in regular sequents to formulas of $\text{ICL}^{\Rightarrow}$.

Definition 6 ($\llcorner \cdot \lrcorner$) The inverse translation for formulas occurring in regular sequents is the following:

$$\begin{aligned}
\llcorner \ulcorner s \urcorner \lrcorner &= s \\
\llcorner p \lrcorner &= p \\
\llcorner \ulcorner s \urcorner \supset \ulcorner t \urcorner \lrcorner &= s \supset t \\
\llcorner \mathsf{A} \vee \ulcorner s \urcorner \lrcorner &= \mathsf{A}\ \mathsf{says}\ s \\
\llcorner \mathsf{A} \supset \mathsf{B} \lrcorner &= \mathsf{A} \Rightarrow \mathsf{B}
\end{aligned}$$

For principals, the inverse translation is defined relative to another principal as follows:

$$\llcorner \mathsf{A} \lrcorner_{\mathsf{C}} = \mathsf{A} \Rightarrow \mathsf{C}$$

For multisets $\Delta, \Gamma, \Phi, \phi, \gamma$ the inverse translation is defined pointwise.

Lemma 9 (Completeness)

1. *If* $\Delta; \Gamma, \gamma \Longrightarrow \Phi$ *is* α*-regular and provable in S4, then* $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$ *is provable in* $ICL^{\Rightarrow}$.
2. *If* $\Delta; \Gamma, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi$ *is* β*-regular and provable in S4, then* $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$ *is provable in* $ICL^{\Rightarrow}$.

Proof. The proof is by simultaneous induction on the height of the given S4 derivation. We analyze cases of the last rule in the derivation.

Case. $\dfrac{}{\Delta; \Gamma, p, \gamma \Longrightarrow \Phi, p}(\mathtt{init})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p \Longrightarrow \vee \llcorner\Phi\lrcorner \vee p$.

1. (Rule $(\mathtt{init})$) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p \Longrightarrow p$.
2. (Rule $(\vee \mathtt{R}_2)$ on 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p \Longrightarrow \vee \llcorner\Phi\lrcorner \vee p$.

Case. $\dfrac{}{\Delta; \Gamma, p, \gamma \Longrightarrow^{\mathsf{C}} \Phi, p, \phi}(\mathtt{init})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner \vee p \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.

1. (Rule $(\mathtt{init})$) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow p$.
2. (Rule $(\vee \mathtt{R}_2)$, $(\vee \mathtt{R}_1)$ on 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \vee \llcorner\Phi\lrcorner \vee p \vee \llcorner\gamma\lrcorner_{\mathsf{C}}$.
3. (Rule $(\mathtt{says}\ \mathtt{R})$ on 2) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner \vee p \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.

Case. $\dfrac{}{\Delta; \Gamma, \gamma, \mathsf{A} \Longrightarrow^{\mathsf{C}} \Phi, \phi, \mathsf{A}}(\mathtt{init})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \llcorner\phi\lrcorner_{\mathsf{C}}, (\mathsf{A} \Rightarrow \mathsf{C}) \Longrightarrow \mathsf{C}\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}} \vee (\mathsf{A} \Rightarrow \mathsf{C}))$.

1. ($(\mathtt{init})$) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \llcorner\phi\lrcorner_{\mathsf{C}}, (\mathsf{A} \Rightarrow \mathsf{C}) \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{C})$.
2. ($(\vee \mathtt{R}_1)$ on 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \llcorner\phi\lrcorner_{\mathsf{C}}, (\mathsf{A} \Rightarrow \mathsf{C}) \Longrightarrow \vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}} \vee (\mathsf{A} \Rightarrow \mathsf{C})$.
3. ($(\mathtt{says}\ \mathtt{R})$ on 2) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \llcorner\phi\lrcorner_{\mathsf{C}}, (\mathsf{A} \Rightarrow \mathsf{C}) \Longrightarrow \mathsf{C}\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}} \vee (\mathsf{A} \Rightarrow \mathsf{C}))$.

Case. $\dfrac{\Delta, s; \Gamma, s, \gamma \Longrightarrow \Phi}{\Delta, s; \Gamma, \gamma \Longrightarrow \Phi}(\mathtt{copy})$

To show: $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$.

1. (i.h.) $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner, \llcorner s\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$.
2. (Strengthening on 1) $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$.

Case. $\dfrac{\Delta, s; \Gamma, s, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi}{\Delta, s; \Gamma, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi}(\mathtt{copy})$

To show: $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.

1. (i.h.) $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner, \llcorner s\lrcorner, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.
2. (Strengthening on 1) $\llcorner\Delta\lrcorner, \llcorner s\lrcorner, \llcorner\Gamma\lrcorner, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.

Case. $\dfrac{\Delta; \Gamma, s \Longrightarrow \Phi, t}{\Delta; \Gamma \Longrightarrow \Phi, s \supset t}(\supset \mathtt{R})$

Does not arise since $s \supset t$ cannot be in Φ in any regular sequent.

Case. $\dfrac{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner s\urcorner \qquad \Delta; \Gamma, \ulcorner t\urcorner, \gamma \Longrightarrow \Phi}{\Delta; \Gamma, \ulcorner s\urcorner \supset \ulcorner t\urcorner, \gamma \Longrightarrow \Phi}(\supset \mathtt{L})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t \Longrightarrow \vee \llcorner\Phi\lrcorner$.

1. ($(\mathtt{init})$) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t, \vee \llcorner\Phi\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$.
2. ($(\mathtt{init})$) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \supset t, s \Longrightarrow s$.

3. (i.h. 2) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, t \Longrightarrow \vee \llcorner \Phi \lrcorner$.
4. (($\supset$ L) on 2,3) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, s \supset t, s \Longrightarrow \vee \llcorner \Phi \lrcorner$.
5. (($\vee$ L) on 1,4) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, s \supset t, \vee \llcorner \Phi \lrcorner \vee s \Longrightarrow \vee \llcorner \Phi \lrcorner$.
6. (i.h. 1) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow \vee \llcorner \Phi \lrcorner \vee s$.
7. (Cut (6,5)) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, s \supset t \Longrightarrow \vee \llcorner \Phi \lrcorner$.

Case. $$\frac{\Delta; \Gamma, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \ulcorner s \urcorner, \phi \qquad \Delta; \Gamma, \ulcorner t \urcorner, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi}{\Delta; \Gamma, \ulcorner s \urcorner \supset \ulcorner t \urcorner, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi}(\supset \mathtt{L})$$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, s \supset t, \llcorner \phi \lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
Let $\Psi = \llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \llcorner \phi \lrcorner_{\mathsf{C}}$.

1. ((init)) $\Psi, s \supset t, s \Longrightarrow s$.
2. (i.h. 2) $\Psi, t \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
3. (($\supset$ L) on 1,2) $\Psi, s, s \supset t \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
4. ((init)) $\Psi, (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}}) \Longrightarrow (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
5. ((says R) on 4) $\Psi, (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}}) \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
6. (($\vee$ L) on 3,5) $\Psi, (\vee \llcorner \Phi \lrcorner \vee s \vee \llcorner \gamma \lrcorner_{\mathsf{C}}), s \supset t \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
7. ((says L) on 6) $\Psi, \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee s \vee \llcorner \gamma \lrcorner_{\mathsf{C}}), s \supset t \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
8. (i.h. 1) $\Psi \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee s \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
9. (Cut(8,7)) $\Psi, s \supset t \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.

Case. $$\frac{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \mathsf{A} \qquad \Delta; \Gamma, \gamma, \mathsf{B} \Longrightarrow \Phi}{\Delta; \Gamma, \mathsf{A} \supset \mathsf{B}, \gamma \Longrightarrow \Phi}(\supset \mathtt{L})$$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, (\mathsf{A} \Rightarrow \mathsf{B}) \Longrightarrow \vee \llcorner \Phi \lrcorner$.
Note that the second premise is regular (the first one is not).

1. (i.h. 2) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow \vee \llcorner \Phi \lrcorner$.
2. (Weakening on 1) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, (\mathsf{A} \Rightarrow \mathsf{B}) \Longrightarrow \vee \llcorner \Phi \lrcorner$.

Case. $$\frac{\Delta; \Gamma, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi, \mathsf{A} \qquad \Delta; \Gamma, \gamma, \mathsf{B} \Longrightarrow^{\mathsf{C}} \Phi, \phi}{\Delta; \Gamma, \mathsf{A} \supset \mathsf{B}, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi}(\supset \mathtt{L})$$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, (\mathsf{A} \Rightarrow \mathsf{B}), \llcorner \phi \lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
Let $\Psi = \llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \llcorner \phi \lrcorner_{\mathsf{C}}$. Then we have to show that:

$\Psi, (\mathsf{A} \Rightarrow \mathsf{B}) \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$

1. (Provable in $\text{ICL}^{\Rightarrow}$) $\Psi, \mathsf{A} \Rightarrow \mathsf{B}, \mathsf{B} \Rightarrow \mathsf{C}, (\mathsf{A} \Rightarrow \mathsf{B}) \supset (\mathsf{B} \Rightarrow \mathsf{C}) \supset (\mathsf{A} \Rightarrow \mathsf{C}) \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{C})$.
2. ((trans)) $\Psi \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{B}) \supset (\mathsf{B} \Rightarrow \mathsf{C}) \supset (\mathsf{A} \Rightarrow \mathsf{C})$.
3. (Cut (2,1)) $\Psi, (\mathsf{A} \Rightarrow \mathsf{B}), (\mathsf{B} \Rightarrow \mathsf{C}) \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{C})$.
4. (i.h. 1) $\Psi, (\mathsf{A} \Rightarrow \mathsf{C}) \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
5. (Cut (4,3)) $\Psi, (\mathsf{A} \Rightarrow \mathsf{B}), (\mathsf{B} \Rightarrow \mathsf{C}) \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
6. ((init)) $\Psi, (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}}) \Longrightarrow (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
7. ((saysR) on 6) $\Psi, (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}}) \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
8. (($\vee$ L) on 7,5) $\Psi, (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}} \vee (\mathsf{B} \Rightarrow \mathsf{C})), (\mathsf{A} \Rightarrow \mathsf{B}) \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
9. ((says L) on 8) $\Psi, \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}} \vee (\mathsf{B} \Rightarrow \mathsf{C})), (\mathsf{A} \Rightarrow \mathsf{B}) \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
10. (i.h. 2) $\Psi \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}} \vee (\mathsf{B} \Rightarrow \mathsf{C}))$.
11. (Cut (10,9)) $\Psi, (\mathsf{A} \Rightarrow \mathsf{B}) \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.

Case. $\dfrac{\Delta;\Gamma,\gamma \Longrightarrow \Phi,\ulcorner s\urcorner \qquad \Delta;\Gamma,\gamma \Longrightarrow \Phi,\ulcorner t\urcorner}{\Delta;\Gamma,\gamma \Longrightarrow \Phi,\ulcorner s\urcorner \wedge \ulcorner t\urcorner}(\wedge\ \mathtt{R})$

To show: $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow \vee\llcorner\Phi\lrcorner \vee (s\wedge t)$.

1. (Provable in ICL$^{\Rightarrow}$) $\vee\llcorner\Phi\lrcorner \vee s, \vee\llcorner\Phi\lrcorner \vee t \Longrightarrow \vee\llcorner\Phi\lrcorner \vee (s\wedge t)$.
2. (i.h. 1) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow \vee\llcorner\Phi\lrcorner \vee s$.
3. (i.h. 2) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow \vee\llcorner\Phi\lrcorner \vee t$.
4. (Cut (2,1)) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, \vee\llcorner\Phi\lrcorner \vee t \Longrightarrow \vee\llcorner\Phi\lrcorner \vee (s\wedge t)$.
5. (Cut (3,4)) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow \vee\llcorner\Phi\lrcorner \vee (s\wedge t)$.

Case. $\dfrac{\Delta;\Gamma,\gamma \Longrightarrow^{\mathsf{C}} \Phi,\ulcorner s\urcorner,\phi \qquad \Delta;\Gamma,\gamma \Longrightarrow^{\mathsf{C}} \Phi,\ulcorner t\urcorner,\phi}{\Delta;\Gamma,\gamma \Longrightarrow^{\mathsf{C}} \Phi,\ulcorner s\urcorner \wedge \ulcorner t\urcorner,\phi}(\wedge\ \mathtt{R})$

To show: $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner,\llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee\llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}} \vee (s\wedge t))$.

Let $\Psi = \llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner,\llcorner\phi\lrcorner_{\mathsf{C}}$, and $\psi = \vee\llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}}$. Then we have to show that: $\Psi \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\psi \vee (s\wedge t))$.

1. (Provable in ICL$^{\Rightarrow}$) $\psi\vee s, \psi\vee t \Longrightarrow \psi\vee(s\wedge t)$.
2. ((says R) on 1) $\psi\vee s, \psi\vee t \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\psi\vee(s\wedge t))$.
3. ((says L) on 2, two times) $\mathsf{C}\ \mathsf{says}\ (\psi\vee s), \mathsf{C}\ \mathsf{says}\ (\psi\vee t) \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\psi\vee(s\wedge t))$.
4. (i.h. 1) $\Psi \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\psi\vee s)$.
5. (i.h. 2) $\Psi \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\psi\vee t)$.
6. (Cut(5, Cut(4,3)) $\Psi \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\psi\vee(s\wedge t))$.

Case. $\dfrac{\Delta;\Gamma,\ulcorner s\urcorner,\ulcorner t\urcorner,\gamma \Longrightarrow \Phi}{\Delta;\Gamma,\ulcorner s\urcorner \wedge \ulcorner t\urcorner,\gamma \Longrightarrow \Phi}(\wedge\ \mathtt{L})$

To show: $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s\wedge t \Longrightarrow \vee\llcorner\Phi\lrcorner$.

1. (i.h.) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s, t \Longrightarrow \vee\llcorner\Phi\lrcorner$.
2. (Weakening on 1) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s, t, s\wedge t \Longrightarrow \vee\llcorner\Phi\lrcorner$.
3. (($\wedge$ L) on 2) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s\wedge t \Longrightarrow \vee\llcorner\Phi\lrcorner$.

Case. $\dfrac{\Delta;\Gamma,\ulcorner s\urcorner,\ulcorner t\urcorner,\gamma \Longrightarrow^{\mathsf{C}} \Phi,\phi}{\Delta;\Gamma,\ulcorner s\urcorner \wedge \ulcorner t\urcorner,\gamma \Longrightarrow^{\mathsf{C}} \Phi,\phi}(\wedge\ \mathtt{L})$

To show: $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s\wedge t, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee\llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.

1. (i.h.) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s, t, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee\llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.
2. (Weakening on 1) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s, t, s\wedge t, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee\llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.
3. (($\wedge$ L) on 2) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s\wedge t, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee\llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.

Case. $\dfrac{\Delta;\Gamma,\gamma \Longrightarrow \Phi,\ulcorner s\urcorner,\ulcorner t\urcorner}{\Delta;\Gamma,\gamma \Longrightarrow \Phi,\ulcorner s\urcorner \vee \ulcorner t\urcorner}(\vee\ \mathtt{R})$

To show: $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow \vee\llcorner\Phi\lrcorner \vee (s\vee t)$.

This follows immediately from the i.h.

Case. $\dfrac{\Delta;\Gamma,\gamma \Longrightarrow^{\mathsf{C}} \Phi,\ulcorner s\urcorner,\ulcorner t\urcorner,\phi}{\Delta;\Gamma,\gamma \Longrightarrow^{\mathsf{C}} \Phi,\ulcorner s\urcorner \vee \ulcorner t\urcorner,\phi}(\vee\ \mathtt{R})$

To show: $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner,\llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee\llcorner\Phi\lrcorner \vee (s\vee t) \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.

This follows immediately from the i.h.

Case. $\dfrac{\Delta;\Gamma,\ulcorner s\urcorner,\gamma \Longrightarrow \Phi \qquad \Delta;\Gamma,\ulcorner t\urcorner,\gamma \Longrightarrow \Phi}{\Delta;\Gamma,\ulcorner s\urcorner \vee \ulcorner t\urcorner,\gamma \Longrightarrow \Phi}(\vee\ \mathtt{L})$

To show: $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s\vee t \Longrightarrow \vee\llcorner\Phi\lrcorner$.

1. (i.h. 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \Longrightarrow \vee \llcorner\Phi\lrcorner$.
2. (Weakening on 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s, s \vee t \Longrightarrow \vee \llcorner\Phi\lrcorner$.
3. (i.h. 2) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, t \Longrightarrow \vee \llcorner\Phi\lrcorner$.
4. (Weakening on 3) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, t, s \vee t \Longrightarrow \vee \llcorner\Phi\lrcorner$.
5. (($\vee$ $\mathtt{L}$) on 2,4) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \vee t \Longrightarrow \vee \llcorner\Phi\lrcorner$.

Case. $\dfrac{\Delta; \Gamma, \ulcorner s \urcorner, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi \qquad \Delta; \Gamma, \ulcorner t \urcorner, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi}{\Delta; \Gamma, \ulcorner s \urcorner \vee \ulcorner t \urcorner, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi}(\vee\ \mathtt{L})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \vee t, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.

1. (i.h. 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.
2. (Weakening on 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s, s \vee t, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.
3. (i.h. 2) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, t, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.
4. (Weakening on 3) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, t, s \vee t, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.
5. (($\vee$ $\mathtt{L}$) on 2,4) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, s \vee t, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.

Case. $\dfrac{\Delta; \Gamma, \gamma, \mathsf{A} \Longrightarrow \Phi \qquad \Delta; \Gamma, \ulcorner s \urcorner, \gamma \Longrightarrow \Phi}{\Delta; \Gamma, \mathsf{A} \vee \ulcorner s \urcorner, \gamma \Longrightarrow \Phi}(\vee\ \mathtt{L})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \vee \llcorner\Phi\lrcorner$.

1. (i.h. 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner$.
2. (Weakening on 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \vee \llcorner\Phi\lrcorner$.

Case. $\dfrac{\Delta; \Gamma, \gamma, \mathsf{A} \Longrightarrow^{\mathsf{C}} \Phi, \phi \qquad \Delta; \Gamma, \ulcorner s \urcorner, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi}{\Delta; \Gamma, \mathsf{A} \vee \ulcorner s \urcorner, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi}(\vee\ \mathtt{L})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \mathsf{A}\ \mathsf{says}\ s, \llcorner\phi\lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}})$.

Let $\Psi = \llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, \llcorner\phi\lrcorner_{\mathsf{C}}$, and $\psi = \vee \llcorner\Phi\lrcorner \vee \llcorner\gamma\lrcorner_{\mathsf{C}}$. Then we have to show that: $\Psi, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{C}\ \mathsf{says}\ \psi$.

1. (($\mathtt{init}$)) $\Psi, \psi, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \psi$.
2. ((says $\mathtt{R}$) on 1) $\Psi, \psi, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{C}\ \mathsf{says}\ \psi$.
3. (i.h. 2) $\Psi, s \Longrightarrow \mathsf{C}\ \mathsf{says}\ \psi$.
4. ((says $\mathtt{L}$) on 3) $\Psi, \mathsf{C}\ \mathsf{says}\ s \Longrightarrow \mathsf{C}\ \mathsf{says}\ \psi$.
5. (Weakening on 4) $\Psi, \mathsf{A} \Rightarrow \mathsf{C}, \mathsf{A}\ \mathsf{says}\ s, \mathsf{C}\ \mathsf{says}\ s \Longrightarrow \mathsf{C}\ \mathsf{says}\ \psi$.
6. (($\mathtt{init}$)) $\Psi, \mathsf{A} \Rightarrow \mathsf{C}, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{A}\ \mathsf{says}\ s$.
7. (($\supset$ $\mathtt{L}$) on 6,5) $\Psi, \mathsf{A} \Rightarrow \mathsf{C}, \mathsf{A}\ \mathsf{says}\ s, (\mathsf{A}\ \mathsf{says}\ s) \supset (\mathsf{C}\ \mathsf{says}\ s) \Longrightarrow \mathsf{C}\ \mathsf{says}\ \psi$.
8. (($\mathtt{init}$)) $\Psi, \mathsf{A} \Rightarrow \mathsf{C}, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{A} \Rightarrow \mathsf{C}$.
9. (($\supset$ $\mathtt{L}$) on 8,7) $\Psi, \mathsf{A} \Rightarrow \mathsf{C}, \mathsf{A}\ \mathsf{says}\ s, (\mathsf{A} \Rightarrow \mathsf{C}) \supset (\mathsf{A}\ \mathsf{says}\ s) \supset (\mathsf{C}\ \mathsf{says}\ s) \Longrightarrow \mathsf{C}\ \mathsf{says}\ \psi$.
10. (($\mathtt{delegate}$)) $\Psi, \mathsf{A} \Rightarrow \mathsf{C}, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{C}) \supset (\mathsf{A}\ \mathsf{says}\ s) \supset (\mathsf{C}\ \mathsf{says}\ s)$.
11. (Cut (10,9)) $\Psi, \mathsf{A} \Rightarrow \mathsf{C}, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{C}\ \mathsf{says}\ \psi$.
12. (($\vee$ $\mathtt{L}$) on 2,11) $\Psi, \psi \vee (\mathsf{A} \Rightarrow \mathsf{C}), \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{C}\ \mathsf{says}\ \psi$.
13. ((says $\mathtt{L}$) on 12) $\Psi, \mathsf{C}\ \mathsf{says}\ (\psi \vee (\mathsf{A} \Rightarrow \mathsf{C})), \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{C}\ \mathsf{says}\ \psi$.
14. (i.h. 1) $\Psi \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\psi \vee (\mathsf{A} \Rightarrow \mathsf{C}))$.
15. (Cut (14,13)) $\Psi, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{C}\ \mathsf{says}\ \psi$.

Case. $\dfrac{}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \top}(\top\mathtt{R})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \vee \llcorner\Phi\lrcorner \vee \top$.

1. (($\top\mathtt{R}$)) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow \top$.

2. ((∨ $\mathtt{R}_1$) on 1) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow \vee \llcorner \Phi \lrcorner \vee \top$.

Case. $\dfrac{}{\Delta; \Gamma, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \top, \phi}(\top\mathtt{R})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \llcorner \phi \lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \top \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.

1. (($\top\mathtt{R}$)) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \llcorner \phi \lrcorner_{\mathsf{C}} \Longrightarrow \top$.
2. ((∨ $\mathtt{R}_2$) on 1) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \llcorner \phi \lrcorner_{\mathsf{C}} \Longrightarrow \top \vee \llcorner \gamma \lrcorner_{\mathsf{C}}$.
3. ((∨ $\mathtt{R}_1$) on 2) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \llcorner \phi \lrcorner_{\mathsf{C}} \Longrightarrow \vee \llcorner \Phi \lrcorner \vee \top \vee \llcorner \gamma \lrcorner_{\mathsf{C}}$.
4. (($\mathsf{says}\ \mathtt{R}$) on 3) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \llcorner \phi \lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \top \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.

Case. $\dfrac{}{\Delta; \Gamma, \bot, \gamma \Longrightarrow \Phi}(\bot\mathtt{L})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \bot \Longrightarrow \vee \llcorner \Phi \lrcorner$.
Follows immediately by rule ($\bot\mathtt{L}$).

Case. $\dfrac{}{\Delta; \Gamma, \bot, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi}(\bot\mathtt{L})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \bot, \llcorner \phi \lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
Follows immediately by rule ($\bot\mathtt{L}$).

Case. $\dfrac{\Delta, s; \Gamma, \gamma \Longrightarrow \Phi}{\Delta; \Gamma, \Box\, s, \gamma \Longrightarrow \Phi}(\Box\mathtt{L})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \llcorner \Box\, s \lrcorner \Longrightarrow \vee \llcorner \Phi \lrcorner$.
By regularity conditions s can have only one of the following forms: $\ulcorner s_1 \urcorner \supset \ulcorner s_2 \urcorner$, $\mathsf{A} \vee \ulcorner s \urcorner$, p or $\mathsf{A} \supset \mathsf{B}$. In each case, $\llcorner s \lrcorner = \llcorner \Box\, s \lrcorner$. Thus we have:

(i.h.) $\llcorner \Delta \lrcorner, \llcorner s \lrcorner, \llcorner \Gamma \lrcorner, \Longrightarrow \vee \llcorner \Phi \lrcorner$ as required.

Case. $\dfrac{\Delta, s; \Gamma, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi}{\Delta; \Gamma, \Box\, s, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \phi}(\Box\mathtt{L})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \llcorner \Box\, s \lrcorner, \llcorner \phi \lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
By regularity conditions s can have only one of the following forms: $\ulcorner s_1 \urcorner \supset \ulcorner s_2 \urcorner$, $\mathsf{A} \vee \ulcorner s \urcorner$, p or $\mathsf{A} \supset \mathsf{B}$. In each case, $\llcorner s \lrcorner = \llcorner \Box\, s \lrcorner$. Thus we have:

(i.h.) $\llcorner \Delta \lrcorner, \llcorner s \lrcorner, \llcorner \Gamma \lrcorner, \llcorner \phi \lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$ as required.

Case. $\dfrac{\Delta; \cdot \Longrightarrow^{?} s}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \Box\, s}(\Box\mathtt{R})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow \vee \llcorner \Phi \lrcorner \vee \llcorner \Box\, s \lrcorner$.
We show a stronger condition
(1) $\llcorner \Delta \lrcorner \Longrightarrow \llcorner \Box\, s \lrcorner$.
Then the required statement follows by weakening and rule (∨ $\mathtt{R}_1$).
By regularity, $s = p$, $\ulcorner s_1 \urcorner \supset \ulcorner s_2 \urcorner$, $\mathsf{A} \vee \ulcorner s \urcorner$ or $\mathsf{A} \supset \mathsf{B}$. We analyze each of these cases, proving (1) for each of these.

Subcase. $s = p$: $\dfrac{\Delta; \cdot \Longrightarrow p}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \Box\, p}(\Box\mathtt{R})$

To show: $\llcorner \Delta \lrcorner \Longrightarrow p$.
Follows by i.h.

Subcase. $s = \ulcorner s_1 \urcorner \supset \ulcorner s_2 \urcorner$: $\dfrac{\Delta; \cdot \Longrightarrow \ulcorner s_1 \urcorner \supset \ulcorner s_2 \urcorner}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \Box\, (\ulcorner s_1 \urcorner \supset \ulcorner s_2 \urcorner)}(\Box\mathtt{R})$

To show: $\llcorner \Delta \lrcorner \Longrightarrow s_1 \supset s_2$.

1. (Inversion on premise) $\Delta; \ulcorner s_1 \urcorner \Longrightarrow \ulcorner s_2 \urcorner$.
2. (i.h. on 1) $\llcorner \Delta \lrcorner, s_1 \Longrightarrow s_2$.

3. $((\supset \mathtt{R})$ on 2) $\llcorner \Delta \lrcorner \Longrightarrow s_1 \supset s_2$.

Subcase. $s = \mathsf{A} \vee \ulcorner s \urcorner$: $\dfrac{\Delta; \cdot \Longrightarrow^{\mathsf{A}} \mathsf{A} \vee \ulcorner s \urcorner}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \Box\,(\mathsf{A} \vee \ulcorner s \urcorner)}(\Box\mathtt{R})$

To show: $\llcorner \Delta \lrcorner \Longrightarrow \mathsf{A}\ \mathtt{says}\ s$.

1. (Inversion on premise) $\Delta; \cdot \Longrightarrow^{\mathsf{A}} \mathsf{A}, \ulcorner s \urcorner$.
2. (i.h. on 1) $\llcorner \Delta \lrcorner, (\mathsf{A} \Rightarrow \mathsf{A}) \Longrightarrow \mathsf{A}\ \mathtt{says}\ s$.
3. ((refl)) $\llcorner \Delta \lrcorner \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{A})$.
4. (Cut (3,2)) $\llcorner \Delta \lrcorner \Longrightarrow \mathsf{A}\ \mathtt{says}\ s$.

Subcase. $s = \mathsf{A} \supset \mathsf{B}$: $\dfrac{\Delta; \cdot \Longrightarrow^{\mathsf{B}} \mathsf{A} \supset \mathsf{B}}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \Box\,(\mathsf{A} \supset \mathsf{B})}(\Box\mathtt{R})$

To show: $\llcorner \Delta \lrcorner \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{B})$.

1. (Inversion on premise) $\Delta; \mathsf{A} \Longrightarrow^{\mathsf{B}} \mathsf{B}$.
2. (i.h. on 1) $\llcorner \Delta \lrcorner, (\mathsf{B} \Rightarrow \mathsf{B}) \Longrightarrow \mathsf{B}\ \mathtt{says}\ (\mathsf{A} \Rightarrow \mathsf{B})$.
3. ((refl)) $\llcorner \Delta \lrcorner \Longrightarrow (\mathsf{B} \Rightarrow \mathsf{B})$.
4. (Cut (3,2)) $\llcorner \Delta \lrcorner \Longrightarrow \mathsf{B}\ \mathtt{says}\ (\mathsf{A} \Rightarrow \mathsf{B})$.
5. ((init)) $\llcorner \Delta \lrcorner, (\mathsf{A} \Rightarrow \mathsf{B}) \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{B})$.
6. $((\supset \mathtt{L})$ on 4,5) $\llcorner \Delta \lrcorner, (\mathsf{B}\ \mathtt{says}\ (\mathsf{A} \Rightarrow \mathsf{B})) \supset (\mathsf{A} \Rightarrow \mathsf{B}) \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{B})$.
7. ((handoff)) $\llcorner \Delta \lrcorner \Longrightarrow (\mathsf{B}\ \mathtt{says}\ (\mathsf{A} \Rightarrow \mathsf{B})) \supset (\mathsf{A} \Rightarrow \mathsf{B})$.
8. (Cut (7,6)) $\llcorner \Delta \lrcorner \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{B})$.

Case. $\dfrac{\Delta; \cdot \Longrightarrow^{?} s}{\Delta; \Gamma, \gamma \Longrightarrow^{\mathsf{C}} \Phi, \Box\, s, \phi}(\Box\mathtt{R})$

To show: To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \llcorner \phi \lrcorner_{\mathsf{C}} \Longrightarrow \mathsf{C}\ \mathtt{says}\ (\vee \llcorner \Phi \lrcorner \vee \llcorner \Box\, s \lrcorner \vee \llcorner \gamma \lrcorner_{\mathsf{C}})$.
We show that the following stronger statement holds:
(1) $\llcorner \Delta \lrcorner \Longrightarrow \llcorner \Box\, s \lrcorner$.
From this the required statement follows by weakening, $(\vee\, \mathtt{R}_1)$, $(\vee\, \mathtt{R}_2)$ and ($\mathtt{says}\ \mathtt{R}$). That (1) holds follows exactly as in the previous case by analyzing the forms of s.

Proof (Proof of Theorem 2).

1. Suppose $\vdash s$ in $\mathrm{ICL}^{\Rightarrow}$. Then $\cdot \Longrightarrow s$. By Lemma 8, $\cdot; \cdot \Longrightarrow \ulcorner s \urcorner$ in S4. Hence by Theorem 8, $\vdash \ulcorner s \urcorner$.
2. Suppose $\vdash \ulcorner s \urcorner$ in S4. By Theorem 8, $\cdot; \cdot \Longrightarrow \ulcorner s \urcorner$. Hence by Lemma 9.1, $\cdot \Longrightarrow \llcorner \ulcorner s \urcorner \lrcorner$ in $\mathrm{ICL}^{\Rightarrow}$. By definition, $\llcorner \ulcorner s \urcorner \lrcorner = s$. Therefore, $\cdot \Longrightarrow s$, i.e., $\vdash s$.

D Details from Section 4

We first describe a sequent calculus for $\mathrm{ICL}^{\mathcal{B}}$. This calculus extends the one for ICL with some additional initial sequents corresponding to the new axioms in $\mathrm{ICL}^{\mathcal{B}}$. One salient point is that cut is no longer admissible, hence we must add it as an explicit rule. We use the same notation $\Gamma \Longrightarrow s$ to denote sequents in $\mathrm{ICL}^{\mathcal{B}}$. Figure 5 summarizes the calculus. It is easy to show that this calculus corresponds exactly to the Hilbert style axiomatization.

Lemma 10 *If* $\mathsf{A} \equiv \mathsf{B}$*, then* $\cdot; \cdot \Longrightarrow \mathsf{A} \supset \mathsf{B}$ *in S4.*

(ALL RULES OF FIGURE 3 ARE INCLUDED)

$$\frac{}{\Gamma \Longrightarrow (\bot \ \mathsf{says}\ s) \supset s}(\texttt{trust}) \qquad \frac{\mathsf{A} \equiv \top}{\Gamma \Longrightarrow \mathsf{A}\ \mathsf{says}\ \bot}(\texttt{untrust})$$

$$\frac{}{\Gamma \Longrightarrow (\mathsf{A} \supset \mathsf{B})\ \mathsf{says}\ s \supset \mathsf{A}\ \mathsf{says}\ s \supset \mathsf{B}\ \mathsf{says}\ s}(\texttt{cuc}')$$

$$\frac{\Gamma \Longrightarrow s \qquad \Gamma, s \Longrightarrow s'}{\Gamma \Longrightarrow s'}(\texttt{cut})$$

Fig. 5. Sequent calculus for $\mathrm{ICL}^{\mathcal{B}}$

Proof. This is immediate because S4 is a conservative extension of classical logic.

In order to prove Theorem 3, we first establish analogues of Lemmas 4, 5 and 6 for $\mathrm{ICL}^{\mathcal{B}}$. It is easy to see that the proofs of Lemmas 4 and 5 do not analyze principals at all, and hence work almost unchanged for $\mathrm{ICL}^{\mathcal{B}}$as well. In Lemma 5, we must additionally consider cases for each new rule in $\mathrm{ICL}^{\mathcal{B}}$.

Lemma 11 *For every $\mathrm{ICL}^{\mathcal{B}}$formula s, $\vdash \ulcorner s \urcorner \supset \Box \ulcorner s \urcorner$ in S4. (Or equivalently, $\cdot; \ulcorner s \urcorner \Longrightarrow \Box \ulcorner s \urcorner$.)*

Proof. Exactly like Lemma 4.

Lemma 12 (Soundness) *If $\Gamma \Longrightarrow s$ in $\mathrm{ICL}^{\mathcal{B}}$, then $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \ulcorner s \urcorner$ in S4.*

Proof. Exactly like Lemma 5. We show here the additional cases in $\mathrm{ICL}^{\mathcal{B}}$.

Case. $\dfrac{}{\Gamma \Longrightarrow (\bot \ \mathsf{says}\ s) \supset s}(\texttt{trust})$

To show: $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box\, ((\Box\, (\bot \vee \ulcorner s \urcorner)) \supset \ulcorner s \urcorner)$

1. (Provable in S4) $\cdot; \cdot \Longrightarrow \Box\, ((\Box\, (\bot \vee \ulcorner s \urcorner)) \supset \ulcorner s \urcorner)$.
2. (Weakening on 1) $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box\, ((\Box\, (\bot \vee \ulcorner s \urcorner)) \supset \ulcorner s \urcorner)$.

Case. $\dfrac{\mathsf{A} \equiv \top}{\Gamma \Longrightarrow \mathsf{A}\ \mathsf{says}\ \bot}(\texttt{untrust})$

To show: $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box\, (\mathsf{A} \vee \bot)$

1. (Lemma 10) $\cdot; \cdot \Longrightarrow \top \supset \mathsf{A}$.
2. (Inversion on 1) $\cdot; \top \Longrightarrow \mathsf{A}$.
3. (($\top$R)) $\cdot; \cdot \Longrightarrow \top$.
4. (Cut(3,2)) $\cdot; \cdot \Longrightarrow \mathsf{A}$.
5. (Weakening on 4) $\cdot; \cdot \Longrightarrow \mathsf{A}, \ulcorner s \urcorner$.
6. (($\vee$ R) on 5) $\cdot; \cdot \Longrightarrow \mathsf{A} \vee \ulcorner s \urcorner$.
7. (($\Box$R) on 6) $\cdot; \cdot \Longrightarrow \Box\, (\mathsf{A} \vee \ulcorner s \urcorner)$.
8. (Weakening on 7) $\cdot; \ulcorner \Gamma \urcorner \Longrightarrow \Box\, (\mathsf{A} \vee \ulcorner s \urcorner)$.

Case. $\dfrac{}{\Gamma \Longrightarrow (\mathsf{A} \supset \mathsf{B})\ \mathsf{says}\ s \supset \mathsf{A}\ \mathsf{says}\ s \supset \mathsf{B}\ \mathsf{says}\ s}(\mathtt{cuc}')$

To show: $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \Box((\Box((\mathsf{A} \supset \mathsf{B}) \vee \ulcorner s\urcorner)) \supset \Box((\Box(\mathsf{A} \vee \ulcorner s\urcorner)) \supset \Box(\mathsf{B} \vee \ulcorner s\urcorner)))$

1. (Provable in S4) $\cdot; \mathsf{A} \vee \ulcorner s\urcorner, (\mathsf{A} \supset \mathsf{B}) \vee \ulcorner s\urcorner \Longrightarrow \mathsf{B} \vee \ulcorner s\urcorner$.
2. (($\mathtt{copy}$) on 1) $\mathsf{A} \vee \ulcorner s\urcorner, (\mathsf{A} \supset \mathsf{B}) \vee \ulcorner s\urcorner; \cdot \Longrightarrow \mathsf{B} \vee \ulcorner s\urcorner$.
3. (($\Box\mathtt{R}$) on 2) $\mathsf{A} \vee \ulcorner s\urcorner, (\mathsf{A} \supset \mathsf{B}) \vee \ulcorner s\urcorner; \cdot \Longrightarrow \Box(\mathsf{B} \vee \ulcorner s\urcorner)$.
4. (($\Box\mathtt{L}$) on 3) $(\mathsf{A} \supset \mathsf{B}) \vee \ulcorner s\urcorner; \Box(\mathsf{A} \vee \ulcorner s\urcorner) \Longrightarrow \Box(\mathsf{B} \vee \ulcorner s\urcorner)$.
5. (($\supset\mathtt{R}$) on 4) $(\mathsf{A} \supset \mathsf{B}) \vee \ulcorner s\urcorner; \cdot \Longrightarrow (\Box(\mathsf{A} \vee \ulcorner s\urcorner)) \supset \Box(\mathsf{B} \vee \ulcorner s\urcorner)$.
6. (($\Box\mathtt{R}$) on 5) $(\mathsf{A} \supset \mathsf{B}) \vee \ulcorner s\urcorner; \cdot \Longrightarrow \Box((\Box(\mathsf{A} \vee \ulcorner s\urcorner)) \supset \Box(\mathsf{B} \vee \ulcorner s\urcorner))$.
7. (($\Box\mathtt{L}$) on 6) $\cdot; \Box((\mathsf{A} \supset \mathsf{B}) \vee \ulcorner s\urcorner) \Longrightarrow \Box((\Box(\mathsf{A} \vee \ulcorner s\urcorner)) \supset \Box(\mathsf{B} \vee \ulcorner s\urcorner))$.
8. (($\supset\mathtt{R}$) on 7) $\cdot; \cdot \Longrightarrow (\Box((\mathsf{A} \supset \mathsf{B}) \vee \ulcorner s\urcorner)) \supset \Box((\Box(\mathsf{A} \vee \ulcorner s\urcorner)) \supset \Box(\mathsf{B} \vee \ulcorner s\urcorner))$.
9. (($\Box\mathtt{R}$) on 8) $\cdot; \cdot \Longrightarrow \Box((\Box((\mathsf{A} \supset \mathsf{B}) \vee \ulcorner s\urcorner)) \supset \Box((\Box(\mathsf{A} \vee \ulcorner s\urcorner)) \supset \Box(\mathsf{B} \vee \ulcorner s\urcorner)))$.
10. (Weakening on 9) $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \Box((\Box((\mathsf{A} \supset \mathsf{B}) \vee \ulcorner s\urcorner)) \supset \Box((\Box(\mathsf{A} \vee \ulcorner s\urcorner)) \supset \Box(\mathsf{B} \vee \ulcorner s\urcorner)))$.

Case. $\dfrac{\Gamma \Longrightarrow s \qquad \Gamma, s \Longrightarrow s'}{\Gamma \Longrightarrow s'}(\mathtt{cut})$

To show: $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \ulcorner s'\urcorner$.

1. (i.h.) $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \ulcorner s\urcorner$.
2. (i.h.) $\cdot; \ulcorner\Gamma\urcorner, \ulcorner s\urcorner \Longrightarrow \ulcorner s'\urcorner$.
3. (Cut(1,2) in S4) $\cdot; \ulcorner\Gamma\urcorner \Longrightarrow \ulcorner s'\urcorner$.

Next we prove completeness for the translation. This proof has a similar structure to the proof of Lemma 6, but differs greatly in the details. We start by defining regular sequents in S4, and an inverse translation for them.

Definition 7 (Regular Sequents) An S4 sequent is called regular if it has the form:

$$\Delta; \Gamma, \gamma \Longrightarrow \Phi, \phi$$

and the following hold:

1. γ and ϕ are multisets of principals.
2. Δ contains assumptions of the form $\mathsf{A} \vee \ulcorner s\urcorner$, $\ulcorner s\urcorner \supset \ulcorner t\urcorner$ and p only.
3. Γ contains assumptions of the form $\mathsf{A} \vee \ulcorner s\urcorner$, $\ulcorner s\urcorner \supset \ulcorner t\urcorner$, p and $\ulcorner s\urcorner$ only.
4. Φ contains assumptions of the form $\ulcorner s\urcorner$ and p only.

Definition 8 ($\llcorner\cdot\lrcorner$) The inverse translation for formulas occurring in regular sequents is the following:

$$\begin{array}{rcl} \llcorner\ulcorner s\urcorner\lrcorner & = & s \\ \llcorner p\lrcorner & = & p \\ \llcorner\ulcorner s\urcorner \supset \ulcorner t\urcorner\lrcorner & = & s \supset t \\ \llcorner\mathsf{A} \vee \ulcorner s\urcorner\lrcorner & = & \mathsf{A}\ \mathsf{says}\ s \end{array}$$

For multisets Δ, Γ, Φ the inverse translation is defined pointwise.

Before showing completeness, we prove some basic lemmas about $\text{ICL}^{\mathcal{B}}$. These simplify our proof.

Lemma 13 (Basic Theorems in $\text{ICL}^{\mathcal{B}}$)

1. *If* $\mathsf{A} \equiv \top$*, then* $\Gamma \Longrightarrow \mathsf{A}\ \mathtt{says}\ s$.
2. *If* $\mathsf{A} \supset \mathsf{B}$ *classically and* $\Gamma \Longrightarrow \mathsf{A}\ \mathtt{says}\ s$*, then* $\Gamma \Longrightarrow \mathsf{B}\ \mathtt{says}\ s$.
3. *If* $\Gamma \Longrightarrow \mathsf{A}\ \mathtt{says}\ s$ *and* $\Gamma \Longrightarrow \mathsf{B}\ \mathtt{says}\ s$*, then* $\Gamma \Longrightarrow (\mathsf{A} \wedge \mathsf{B})\ \mathtt{says}\ s$.

Proof.

1. If $\mathsf{A} \equiv \top$, $\Gamma \Longrightarrow \mathsf{A}\ \mathtt{says}\ \bot$ by rule (untrust). Also it is easy to prove that $\mathsf{A}\ \mathtt{says}\ \bot \Longrightarrow \mathsf{A}\ \mathtt{says}\ s$. Thus by (cut), $\Gamma \Longrightarrow \mathsf{A}\ \mathtt{says}\ s$.
2. If $\mathsf{A} \supset \mathsf{B}$, then in classical logic, $(\mathsf{A} \supset \mathsf{B}) \equiv \top$. Thus $\Gamma \Longrightarrow (\mathsf{A} \supset \mathsf{B})\ \mathtt{says}\ s$ by rule (untrust). Now it is easy to show that $(\mathsf{A} \supset \mathsf{B})\ \mathtt{says}\ s, \mathsf{A}\ \mathtt{says}\ s \Longrightarrow \mathsf{B}\ \mathtt{says}\ s$ (through rule (cuc′)). Hence by (cut), $\Gamma, \mathsf{A}\ \mathtt{says}\ s \Longrightarrow \mathsf{B}\ \mathtt{says}\ s$. Using the given condition $\Gamma \Longrightarrow \mathsf{A}\ \mathtt{says}\ s$ and (cut), we get $\Gamma \Longrightarrow \mathsf{B}\ \mathtt{says}\ s$.
3. We consider the following proof:
   (a) (Classical theorem) $\mathsf{A} \supset (\mathsf{B} \supset (\mathsf{A} \wedge \mathsf{B})) \equiv \top$.
   (b) ((1) above) $\Gamma \Longrightarrow (\mathsf{A} \supset (\mathsf{B} \supset (\mathsf{A} \wedge \mathsf{B})))\ \mathtt{says}\ s$.
   (c) ((cuc′) and b) $\Gamma \Longrightarrow \mathsf{A}\ \mathtt{says}\ s \supset \mathsf{B}\ \mathtt{says}\ s \supset (\mathsf{A} \wedge \mathsf{B})\ \mathtt{says}\ s$.
   (d) (Given assumptions and c) $\Gamma \Longrightarrow (\mathsf{A} \wedge \mathsf{B})\ \mathtt{says}\ s$.

Lemma 14 (Completeness) *If* $\Delta; \Gamma, \gamma \Longrightarrow \Phi, \phi$ *is regular and provable in S4, then* $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner)$ *is provable in* $ICL^{\mathcal{B}}$.
[Note: If $\phi = \cdot$*, we take* $\vee\, \phi = \bot$*, and if* $\gamma = \cdot$*, we take* $\wedge\, \gamma = \top$*.]*

Proof. By induction on the size of the given S4 derivation. We analyze the last rule of the derivation by cases. Several cases follow the the pattern of corresponding cases in Lemma 6. In such situations, we write the name of the rule, the form of the principal formula and indicate that the proof is identical by writing s.t.p.l (similar to previous lemma).

Case. $\dfrac{}{\Delta; \Gamma, p, \gamma \Longrightarrow \Phi, p, \phi}(\mathtt{init})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner) \vee p$

1. ((init)) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p \Longrightarrow p$.
2. (($\vee\, \mathtt{R}_2$)) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p \Longrightarrow (\vee \llcorner\Phi\lrcorner) \vee p$.
3. ((saysR)) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner, p \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner) \vee p$.

Case. $\dfrac{}{\Delta; \Gamma, \gamma, a \Longrightarrow \Phi, \phi, a}(\mathtt{init})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow ((\wedge\, \gamma \wedge \mathsf{a}) \supset (\vee\, \phi \vee \mathsf{a}))\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner)$

1. (Classical theorem) $(\wedge\, \gamma \wedge a) \supset (\vee\, \phi \vee a) \equiv \top$.
2. (Lemma 13 on 1) $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow ((\wedge\, \gamma \wedge \mathsf{a}) \supset (\vee\, \phi \vee \mathsf{a}))\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner)$.

Case. (copy). s.t.p.l.

Case. $\dfrac{\Delta; \Gamma, \gamma, \mathsf{A} \Longrightarrow \Phi, \phi, \mathsf{B}}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \phi, \mathsf{A} \supset \mathsf{B}}(\supset \mathtt{R})$

To show: $\llcorner\Delta\lrcorner, \llcorner\Gamma\lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi \vee (\mathsf{A} \supset \mathsf{B})))\ \mathtt{says}\ (\vee \llcorner\Phi\lrcorner)$

1. (Classical theorem) $(\wedge\, \gamma \wedge \mathsf{A}) \supset (\vee\, \phi \vee \mathsf{B}) \equiv (\wedge\, \gamma) \supset (\vee\, \phi \vee (\mathsf{A} \supset \mathsf{B}))$.
2. (i.h.) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma \wedge \mathsf{A}) \supset (\vee\, \phi \vee \mathsf{B}))\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$.
3. (Lemma 13 on 1,2) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi \vee (\mathsf{A} \supset \mathsf{B})))\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$.

Case. $\dfrac{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner s \urcorner, \phi \qquad \Delta; \Gamma, \ulcorner t \urcorner, \gamma \Longrightarrow \Phi, \phi}{\Delta; \Gamma, \ulcorner s \urcorner \supset \ulcorner t \urcorner, \gamma \Longrightarrow \Phi, \phi} (\supset \mathtt{L})$

s.t.p.l.

Case. $\dfrac{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \phi, \mathsf{A} \qquad \Delta; \Gamma, \gamma, \mathsf{B} \Longrightarrow \Phi, \phi}{\Delta; \Gamma, \gamma, \mathsf{A} \supset \mathsf{B} \Longrightarrow \Phi, \phi} (\supset \mathtt{L})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma \wedge (\mathsf{A} \supset \mathsf{B})) \supset (\vee\, \phi))\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$

Define $\mathsf{D} = (\wedge\, \gamma) \supset (\vee\, \phi \vee \mathsf{A})$ and $\mathsf{E} = (\wedge\, \gamma \wedge \mathsf{B}) \supset (\vee\, \gamma)$.

1. (i.h.) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow \mathsf{D}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$.
2. (i.h.) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow \mathsf{E}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$.
3. (Lemma 13) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow (\mathsf{D} \wedge \mathsf{E})\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$.
4. (Classical theorem) $(\mathsf{D} \wedge \mathsf{E}) \supset ((\wedge\, \gamma \wedge (\mathsf{A} \supset \mathsf{B})) \supset (\vee\, \phi))$.
5. (Lemma 13) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma \wedge (\mathsf{A} \supset \mathsf{B})) \supset (\vee\, \phi))\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$.

Case. $\dfrac{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner s \urcorner, \phi \qquad \Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner t \urcorner, \phi}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner s \urcorner \wedge \ulcorner t \urcorner} (\wedge \mathtt{R})$

s.t.p.l.

Case. $\dfrac{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \phi, \mathsf{A} \qquad \Delta; \Gamma, \gamma \Longrightarrow \Phi, \phi, \mathsf{B}}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \phi, \mathsf{A} \wedge \mathsf{B}} (\wedge \mathtt{R})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi \vee (\mathsf{A} \wedge \mathsf{B})))\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$

Let $\mathsf{C} = (\wedge\, \gamma) \supset (\vee\, \phi \vee \mathsf{A})$ and $\mathsf{D} = (\wedge\, \gamma) \supset (\vee\, \phi \vee \mathsf{B})$.

1. (i.h.) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$.
2. (i.h.) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow \mathsf{D}\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$.
3. (Lemma 13) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow (\mathsf{C} \wedge \mathsf{D})\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$.
4. (Classical theorem) $\mathsf{C} \wedge \mathsf{D} \supset ((\wedge\, \gamma) \supset (\vee\, \phi \vee (\mathsf{A} \wedge \mathsf{B})))$
5. (Lemma 13) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi \vee (\mathsf{A} \wedge \mathsf{B})))\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$.

Case. $\dfrac{\Delta; \Gamma, \ulcorner s \urcorner, \ulcorner t \urcorner, \gamma \Longrightarrow \Phi, \phi}{\Delta; \Gamma, \ulcorner s \urcorner \wedge \ulcorner t \urcorner, \gamma \Longrightarrow \Phi, \phi} (\wedge \mathtt{L})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, s \wedge t \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$

1. (i.h.) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, s, t \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$.
2. ($(\wedge \mathtt{L})$ on 1) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, s \wedge t \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$.

Case. $\dfrac{\Delta; \Gamma, \gamma, \mathsf{A}, \mathsf{B} \Longrightarrow \Phi, \phi}{\Delta; \Gamma, \gamma, \mathsf{A} \wedge \mathsf{B} \Longrightarrow \Phi, \phi} (\wedge \mathtt{L})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma \wedge (\mathsf{A} \wedge \mathsf{B})) \supset (\vee\, \phi))\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$

Follows immediately by i.h.

Case. $\dfrac{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner s \urcorner, \ulcorner t \urcorner, \phi}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \ulcorner s \urcorner \vee \ulcorner t \urcorner, \phi} (\vee \mathtt{R})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \mathsf{says}\ ((\vee \llcorner \Phi \lrcorner) \vee s \vee t)$

Follows immediately by i.h.

Case. $\dfrac{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \phi, \mathsf{A}, \mathsf{B}}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \phi, \mathsf{A} \vee \mathsf{B}} (\vee \mathtt{R})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi \vee (\mathsf{A} \vee \mathsf{B})))\ \mathsf{says}\ (\vee \llcorner \Phi \lrcorner)$

Follows immediately by i.h.

Case. $\dfrac{\Delta;\Gamma,\ulcorner s\urcorner,\gamma \Longrightarrow \Phi,\phi \qquad \Delta;\Gamma,\ulcorner t\urcorner,\gamma \Longrightarrow \Phi,\phi}{\Delta;\Gamma,\ulcorner s\urcorner \vee \ulcorner t\urcorner,\gamma \Longrightarrow \Phi,\phi}(\vee\,\mathsf{L})$

To show: $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s \vee t \Longrightarrow ((\wedge\,\gamma) \supset (\vee\,\phi))\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner)$

1. (i.h.) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s \Longrightarrow ((\wedge\,\gamma) \supset (\vee\,\phi))\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner)$.
2. (i.h.) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, t \Longrightarrow ((\wedge\,\gamma) \supset (\vee\,\phi))\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner)$.
3. (($\vee\,\mathsf{L}$) on 1,2) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s \vee t \Longrightarrow ((\wedge\,\gamma) \supset (\vee\,\phi))\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner)$.

Case. $\dfrac{\Delta;\Gamma,\gamma,\mathsf{A} \Longrightarrow \Phi,\phi \qquad \Delta;\Gamma,\gamma,\mathsf{B} \Longrightarrow \Phi,\phi}{\Delta;\Gamma,\gamma,\mathsf{A}\vee\mathsf{B} \Longrightarrow \Phi,\phi}(\vee\,\mathsf{L})$

To show: $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow ((\wedge\,\gamma \wedge (\mathsf{A}\vee\mathsf{B})) \supset (\vee\,\phi))\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner)$

Let $\mathsf{C} = (\wedge\,\gamma \wedge \mathsf{A}) \supset (\vee\,\phi)$ and $\mathsf{D} = (\wedge\,\gamma \wedge \mathsf{B}) \supset (\vee\,\phi)$.

1. (i.h.) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow \mathsf{C}\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner)$.
2. (i.h.) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow \mathsf{D}\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner)$.
3. (Lemma 13) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow (\mathsf{C}\wedge\mathsf{D})\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner)$.
4. (Classical theorem) $\mathsf{C}\wedge\mathsf{D} \supset ((\wedge\,\gamma \wedge (\mathsf{A}\vee\mathsf{B})) \supset (\vee\,\phi))$
5. (Lemma 13) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow ((\wedge\,\gamma \wedge (\mathsf{A}\vee\mathsf{B})) \supset (\vee\,\phi))\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner)$.

Case. $\dfrac{\Delta;\Gamma,\gamma,\mathsf{A} \Longrightarrow \Phi,\phi \qquad \Delta;\Gamma,\ulcorner s\urcorner,\gamma \Longrightarrow \Phi,\phi}{\Delta;\Gamma,\mathsf{A}\vee\ulcorner s\urcorner,\gamma \Longrightarrow \Phi,\phi}(\vee\,\mathsf{L})$

To show: $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner,\mathsf{A}\ \mathsf{says}\ s \Longrightarrow ((\wedge\,\gamma) \supset (\vee\,\phi))\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner)$

To make the proof easier to read, we introduce some notation. Let $t = \vee\,\llcorner\Phi\lrcorner$ and $\mathsf{B} = (\wedge\,\gamma) \supset (\vee\,\phi)$. We have to show that $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner,\mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{B}\ \mathsf{says}\ t$.

1. (i.h.) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow ((\wedge\,\gamma \wedge \mathsf{A}) \supset (\vee\,\phi))\ \mathsf{says}\ t$.
2. (Classical theorem) $((\wedge\,\gamma \wedge \mathsf{A}) \supset (\vee\,\phi)) \equiv \mathsf{A} \supset \mathsf{B}$.
3. (Lemma 13 on 1,2) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow (\mathsf{A}\supset\mathsf{B})\ \mathsf{says}\ t$.
4. (Provable in ICL$^{\mathcal{B}}$) $(\mathsf{A}\supset\mathsf{B})\ \mathsf{says}\ t \Longrightarrow (\mathsf{A}\supset\mathsf{B})\ \mathsf{says}\ \mathsf{B}\ \mathsf{says}\ t$.
5. (Cut(3,4)) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow (\mathsf{A}\supset\mathsf{B})\ \mathsf{says}\ \mathsf{B}\ \mathsf{says}\ t$.
6. (i.h.) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s \Longrightarrow \mathsf{B}\ \mathsf{says}\ t$.
7. ((saysR) on 6) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner, s \Longrightarrow \mathsf{A}\ \mathsf{says}\ \mathsf{B}\ \mathsf{says}\ t$.
8. ((saysL) on 7) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner,\mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{A}\ \mathsf{says}\ \mathsf{B}\ \mathsf{says}\ t$.
9. (Weakening on 5) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner,\mathsf{A}\ \mathsf{says}\ s \Longrightarrow (\mathsf{A}\supset\mathsf{B})\ \mathsf{says}\ \mathsf{B}\ \mathsf{says}\ t$.
10. (Lemma 13 on 8,9) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner,\mathsf{A}\ \mathsf{says}\ s \Longrightarrow (\mathsf{A}\wedge(\mathsf{A}\supset\mathsf{B}))\ \mathsf{says}\ \mathsf{B}\ \mathsf{says}\ t$.
11. (Classical theorem) $(\mathsf{A}\wedge(\mathsf{A}\supset\mathsf{B})) \supset \mathsf{B}$.
12. (Lemma 13 on 10,11) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner,\mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{B}\ \mathsf{says}\ \mathsf{B}\ \mathsf{says}\ t$.
13. (Provable in ICL$^{\mathcal{B}}$) $\mathsf{B}\ \mathsf{says}\ \mathsf{B}\ \mathsf{says}\ t \Longrightarrow \mathsf{B}\ \mathsf{says}\ t$.
14. (Cut(12,13)) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner,\mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{B}\ \mathsf{says}\ t$.

Case. $\dfrac{}{\Delta;\Gamma,\gamma \Longrightarrow (\Phi,\top),\phi}(\top\mathtt{R})$

To show: $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow ((\wedge\,\gamma) \supset (\vee\,\phi))\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner \vee \top)$

1. (($\top\mathtt{R}$))$\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow \top$.
2. (($\vee\,\mathtt{R}_2$) on 1) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow (\vee\,\llcorner\Phi\lrcorner \vee \top)$.
3. ((saysR) on 2) $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow ((\wedge\,\gamma) \supset (\vee\,\phi))\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner \vee \top)$.

Case. $\dfrac{}{\Delta;\Gamma,\gamma \Longrightarrow \Phi,(\phi,\top)}(\top\mathtt{R})$

To show: $\llcorner\Delta\lrcorner,\llcorner\Gamma\lrcorner \Longrightarrow ((\wedge\,\gamma) \supset (\vee\,\phi \vee \top))\ \mathsf{says}\ (\vee\,\llcorner\Phi\lrcorner)$

1. (Classical theorem) $(\wedge\, \gamma) \supset (\vee\, \phi \vee \top) \equiv \top$.
2. (Lemma 13 on 1) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi \vee \top))\ \texttt{says}\ (\vee \llcorner \Phi \lrcorner)$.

Case. $\dfrac{}{\Delta; (\Gamma, \bot), \gamma \Longrightarrow \Phi, \phi}(\bot\texttt{L})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner, \bot \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \texttt{says}\ (\vee \llcorner \Phi \lrcorner)$

This follows immediately by rule $(\bot\texttt{L})$.

Case. $\dfrac{}{\Delta; \Gamma, (\gamma, \bot) \Longrightarrow \Phi, \phi}(\bot\texttt{L})$

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma \wedge \bot) \supset (\vee\, \phi))\ \texttt{says}\ (\vee \llcorner \Phi \lrcorner)$

1. (Classical theorem) $((\wedge\, \gamma \wedge \bot) \supset (\vee\, \phi)) \equiv \top$.
2. (Lemma 13 on 1) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma \wedge \bot) \supset (\vee\, \phi))\ \texttt{says}\ (\vee \llcorner \Phi \lrcorner)$.

Case. $\dfrac{\Delta; \cdot \Longrightarrow s}{\Delta; \Gamma, \gamma \Longrightarrow \Phi, \Box\, s, \phi}(\Box\texttt{R})$

By regularity, s can only have the forms p, $\ulcorner s_1 \urcorner \supset \ulcorner s_2 \urcorner$ and $\mathsf{A} \vee \ulcorner t \urcorner$.

Subcase. $s = p$.

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \texttt{says}\ (\vee \llcorner \Phi \lrcorner \vee p)$.

(a) (i.h.) $\llcorner \Delta \lrcorner \Longrightarrow (\top \supset \bot)\ \texttt{says}\ p$.
(b) (Classical theorem) $(\top \supset \bot) \equiv \bot$.
(c) (Lemma 13 on a,b) $\llcorner \Delta \lrcorner \Longrightarrow \bot\ \texttt{says}\ p$.
(d) ((trust) with c) $\llcorner \Delta \lrcorner \Longrightarrow p$.
(e) (($\vee\, \texttt{R}_2$) on d) $\llcorner \Delta \lrcorner \Longrightarrow (\vee \llcorner \Phi \lrcorner \vee p)$.
(f) ((saysR) on e) $\llcorner \Delta \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \texttt{says}\ (\vee \llcorner \Phi \lrcorner \vee p)$.
(g) (Weakening on f) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \texttt{says}\ (\vee \llcorner \Phi \lrcorner \vee p)$.

Subcase. $s = \ulcorner s_1 \urcorner \supset \ulcorner s_2 \urcorner$.

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \texttt{says}\ ((\vee \llcorner \Phi \lrcorner) \vee (s_1 \supset s_2))$.

(a) (Inversion on premise) $\Delta; \ulcorner s_1 \urcorner \Longrightarrow \ulcorner s_2 \urcorner$.
(b) (i.h. on a) $\llcorner \Delta \lrcorner, s_1 \Longrightarrow (\top \supset \bot)\ \texttt{says}\ s_2$.
(c) (Classical theorem) $(\top \supset \bot) \equiv \bot$.
(d) (Lemma 13 on b,c) $\llcorner \Delta \lrcorner, s_1 \Longrightarrow \bot\ \texttt{says}\ s_2$.
(e) ((trust) with d) $\llcorner \Delta \lrcorner, s_1 \Longrightarrow s_2$.
(f) (($\supset \texttt{R}$) on e) $\llcorner \Delta \lrcorner \Longrightarrow s_1 \supset s_2$.
(g) (($\vee\, \texttt{R}_2$) on f) $\llcorner \Delta \lrcorner \Longrightarrow (\vee \llcorner \Phi \lrcorner) \vee (s_1 \supset s_2)$.
(h) (Weakening on g) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow (\vee \llcorner \Phi \lrcorner) \vee (s_1 \supset s_2)$.
(i) ((saysR) on h) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \texttt{says}\ ((\vee \llcorner \Phi \lrcorner) \vee (s_1 \supset s_2))$.

Subcase. $s = \mathsf{A} \vee \ulcorner t \urcorner$.

To show: $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \texttt{says}\ ((\vee \llcorner \Phi \lrcorner) \vee \mathsf{A}\ \texttt{says}\ s)$.

(a) (Inversion on premise) $\Delta; \cdot \Longrightarrow \ulcorner t \urcorner, \mathsf{A}$.
(b) (i.h. on a) $\llcorner \Delta \lrcorner \Longrightarrow (\top \supset \mathsf{A})\ \texttt{says}\ s$.
(c) (Classical theorem) $(\top \supset \mathsf{A}) \equiv \mathsf{A}$.
(d) (Lemma 13 on b,c) $\llcorner \Delta \lrcorner \Longrightarrow \mathsf{A}\ \texttt{says}\ s$.
(e) (($\vee\, \texttt{R}_2$) on d) $\llcorner \Delta \lrcorner \Longrightarrow (\vee \llcorner \Phi \lrcorner) \vee \mathsf{A}\ \texttt{says}\ s$.
(f) (Weakening on e) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow (\vee \llcorner \Phi \lrcorner) \vee \mathsf{A}\ \texttt{says}\ s$.
(g) ((saysR) on f) $\llcorner \Delta \lrcorner, \llcorner \Gamma \lrcorner \Longrightarrow ((\wedge\, \gamma) \supset (\vee\, \phi))\ \texttt{says}\ ((\vee \llcorner \Phi \lrcorner) \vee \mathsf{A}\ \texttt{says}\ s)$.

Case. $(\Box\texttt{L})$. s.t.p.l.

Proof (Proof of Theorem 3).

1. Suppose $\vdash s$ in ICL$^{\mathcal{B}}$. Then $\cdot \Longrightarrow s$. By Lemma 12, $\cdot;\cdot \Longrightarrow \ulcorner s \urcorner$ in S4. Hence by Theorem 8, $\vdash \ulcorner s \urcorner$.
2. Suppose $\vdash \ulcorner s \urcorner$ in S4. By Theorem 8, $\cdot;\cdot \Longrightarrow \ulcorner s \urcorner$. Hence by Lemma 14, $\cdot \Longrightarrow (\top \supset \bot)$ $\texttt{says}$ $\llcorner \ulcorner s \urcorner \lrcorner$ in ICL$^{\mathcal{B}}$. By definition, $\llcorner \ulcorner s \urcorner \lrcorner = s$. Therefore, $\cdot \Longrightarrow (\top \supset \bot)$ $\texttt{says}$ s. Classically it is the case that $\top \supset \bot \equiv \bot$. Hence by Lemma 13, $\cdot \Longrightarrow \bot$ $\texttt{says}$ s. Using rule ($\texttt{trust}$), we derive that $\cdot \Longrightarrow s$, i.e., $\vdash s$.

E Details from Section 6

In this appendix, we prove Theorem 5 from Section 6. A proof system for ICL$^{\Rightarrow}$ was presented in appendix C. A proof system for ICL$^{\forall}$ can be obtained by adding the following rules for the universal quantifier to the proof system in figure 3 (Appendix B).

$$\frac{\Gamma \Longrightarrow s}{\Gamma \Longrightarrow \forall X.\ s}(\forall \mathtt{R})(X \text{ fresh}) \qquad \frac{\Gamma, \forall X.\ s, s[t/X] \Longrightarrow s'}{\Gamma, \forall X.\ s \Longrightarrow s'}(\forall \mathtt{L})$$

$$\frac{\Gamma \Longrightarrow s \qquad \Gamma, s \Longrightarrow s'}{\Gamma \Longrightarrow s'}(\mathtt{cut})$$

Even though we expect that cut-elimination holds for ICL$^{\forall}$, we do not prove it since we do not require it. The identity principle, however can be proved for this logic as shown below:

Theorem 11 (Identity) *For every formula s in ICL$^{\forall}$, $\Gamma, s \Longrightarrow s$.*

Proof. By induction on s.

Now we prove the soundness part of Theorem 5.

Lemma 15 (Soundness) *For every ICL$^{\Rightarrow}$ formula s, if $\vdash s$ (i.e., $\cdot \Longrightarrow s$), then $\vdash [\![s]\!]$ (i.e., $\cdot \Longrightarrow [\![s]\!]$) in ICL$^{\forall}$.*

Proof. We prove a more general result: if $\Gamma \Longrightarrow s$ in ICL$^{\Rightarrow}$, then $[\![\Gamma]\!] \Longrightarrow [\![s]\!]$ in ICL$^{\forall}$. We induct induct on the proof of $\Gamma \Longrightarrow s$ and analyze cases on the last rule in the proof. Most cases are straightforward (since the rules from Figure 3 are common to the two logics). We show cases of the rules that are not common (Figure 4).

Case. $\dfrac{}{\Gamma \Longrightarrow \mathsf{A} \Rightarrow \mathsf{A}}(\mathtt{refl})$

To show: $[\![\Gamma]\!] \Longrightarrow \forall X.\ \mathsf{A}\ \texttt{says}\ X \supset \mathsf{A}\ \texttt{says}\ X$.

1. (Provable in ICL$^{\forall}$) $\mathsf{A}\ \texttt{says}\ X \Longrightarrow \mathsf{A}\ \texttt{says}\ X$.

2. $((\supset \texttt{R})$ on 1) $\cdot \Longrightarrow \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{A}\ \mathsf{says}\ X$.
3. $((\forall\texttt{R})$ on 2) $\cdot \Longrightarrow \forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{A}\ \mathsf{says}\ X$.
4. (Weakening on 3) $[\![\Gamma]\!] \Longrightarrow \forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{A}\ \mathsf{says}\ X$.

Case. $$\frac{}{\Gamma \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{B}) \supset (\mathsf{B} \Rightarrow \mathsf{C}) \supset (\mathsf{A} \Rightarrow \mathsf{C})}(\texttt{trans})$$

To show: $[\![\Gamma]\!] \Longrightarrow (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X) \supset (\forall X.\ \mathsf{B}\ \mathsf{says}\ X \supset \mathsf{C}\ \mathsf{says}\ X) \supset (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{C}\ \mathsf{says}\ X)$.

1. (Provable in $\mathrm{ICL}^{\forall}$) $\mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X, \mathsf{B}\ \mathsf{says}\ X \supset \mathsf{C}\ \mathsf{says}\ X \Longrightarrow \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{C}\ \mathsf{says}\ X$.
2. $((\forall\texttt{L})$ on 1) $\forall X.\ (\mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X), \forall X.\ (\mathsf{B}\ \mathsf{says}\ X \supset \mathsf{C}\ \mathsf{says}\ X) \Longrightarrow \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{C}\ \mathsf{says}\ X$.
3. $((\forall\texttt{R})$ on 2) $\forall X.\ (\mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X), \forall X.\ (\mathsf{B}\ \mathsf{says}\ X \supset \mathsf{C}\ \mathsf{says}\ X) \Longrightarrow \forall X.\ (\mathsf{A}\ \mathsf{says}\ X \supset \mathsf{C}\ \mathsf{says}\ X)$.
4. $((\supset \texttt{R})$ on 3) $\cdot \Longrightarrow (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X) \supset (\forall X.\ \mathsf{B}\ \mathsf{says}\ X \supset \mathsf{C}\ \mathsf{says}\ X) \supset (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{C}\ \mathsf{says}\ X)$.
5. (Weakening on 4) $[\![\Gamma]\!] \Longrightarrow (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X) \supset (\forall X.\ \mathsf{B}\ \mathsf{says}\ X \supset \mathsf{C}\ \mathsf{says}\ X) \supset (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{C}\ \mathsf{says}\ X)$.

Case. $$\frac{}{\Gamma \Longrightarrow (\mathsf{A} \Rightarrow \mathsf{B}) \supset (\mathsf{A}\ \mathsf{says}\ s) \supset (\mathsf{B}\ \mathsf{says}\ s)}(\texttt{delegate})$$

To show: $[\![\Gamma]\!] \Longrightarrow (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X) \supset \mathsf{A}\ \mathsf{says}\ [\![s]\!] \supset \mathsf{B}\ \mathsf{says}\ [\![s]\!]$.

1. (Provable in $\mathrm{ICL}^{\forall}$) $\mathsf{A}\ \mathsf{says}\ [\![s]\!] \supset \mathsf{B}\ \mathsf{says}\ [\![s]\!] \Longrightarrow \mathsf{A}\ \mathsf{says}\ [\![s]\!] \supset \mathsf{B}\ \mathsf{says}\ [\![s]\!]$.
2. $((\forall\texttt{L})$ on 1) $(\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X) \Longrightarrow \mathsf{A}\ \mathsf{says}\ [\![s]\!] \supset \mathsf{B}\ \mathsf{says}\ [\![s]\!]$.
3. $((\supset \texttt{R})$ on 2) $\cdot \Longrightarrow (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X) \supset \mathsf{A}\ \mathsf{says}\ [\![s]\!] \supset \mathsf{B}\ \mathsf{says}\ [\![s]\!]$.
4. (Weakening on 3) $[\![\Gamma]\!] \Longrightarrow (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X) \supset \mathsf{A}\ \mathsf{says}\ [\![s]\!] \supset \mathsf{B}\ \mathsf{says}\ [\![s]\!]$.

Case. $$\frac{}{\Gamma \Longrightarrow (\mathsf{B}\ \mathsf{says}\ (\mathsf{A} \Rightarrow \mathsf{B})) \supset (\mathsf{A} \Rightarrow \mathsf{B})}(\texttt{handoff})$$

To show: $[\![\Gamma]\!] \Longrightarrow \mathsf{B}\ \mathsf{says}\ (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X) \supset (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X)$.

1. (Provable in $\mathrm{ICL}^{\forall}$) $\mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X, \mathsf{A}\ \mathsf{says}\ X \Longrightarrow \mathsf{B}\ \mathsf{says}\ X$.
2. $((\forall\texttt{L})$ on 1) $\forall X.\ (\mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X), \mathsf{A}\ \mathsf{says}\ X \Longrightarrow \mathsf{B}\ \mathsf{says}\ X$.
3. $((\mathsf{says}\ \texttt{L})$ on 2) $\mathsf{B}\ \mathsf{says}\ (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X), \mathsf{A}\ \mathsf{says}\ X \Longrightarrow \mathsf{B}\ \mathsf{says}\ X$.
4. $((\supset \texttt{R})$ on 3) $\mathsf{B}\ \mathsf{says}\ (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X) \Longrightarrow \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X$.
5. $((\forall\texttt{R})$ on 4) $\mathsf{B}\ \mathsf{says}\ (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X) \Longrightarrow \forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X$.
6. $((\supset \texttt{R})$ on 5) $\cdot \Longrightarrow \mathsf{B}\ \mathsf{says}\ (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X) \supset (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X)$.
7. (Weakening on 6) $[\![\Gamma]\!] \Longrightarrow \mathsf{B}\ \mathsf{says}\ (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X) \supset (\forall X.\ \mathsf{A}\ \mathsf{says}\ X \supset \mathsf{B}\ \mathsf{says}\ X)$.

Case. $$\frac{}{\Gamma, s \Longrightarrow s}(\texttt{init})$$

To show: $[\![\Gamma]\!], [\![s]\!] \Longrightarrow [\![s]\!]$.
Follows immediately by Theorem 11.

Case. $\dfrac{\Gamma \Longrightarrow s \qquad \Gamma, s \Longrightarrow s'}{\Gamma \Longrightarrow s'}(\texttt{cut})$

To show: $[\![\Gamma]\!] \Longrightarrow [\![s']\!]$.

1. (i.h. 1) $[\![\Gamma]\!] \Longrightarrow [\![s]\!]$.
2. (i.h. 2) $[\![\Gamma]\!], [\![s]\!] \Longrightarrow [\![s']\!]$.
3. ((cut) on 1,2) $[\![\Gamma]\!] \Longrightarrow [\![s']\!]$.

E.1 S4$^\forall$: Second-order S4

Proving the completeness part of Theorem 5 requires more technical machinery. We first define a second-order extension of S4 called S4$^\forall$. Formulas in S4$^\forall$ are the same as those in S4 with the exception that they may also contain propositional variables (written X, Y, etc.) and universal quantifiers over formulas, written $\forall X.\ s$. A sequent calculus is obtained by adding the following rules to those of figure 2 (Appendix A):

$$\frac{\Delta; \Gamma \Longrightarrow s}{\Delta; \Gamma \Longrightarrow \forall X.\ s}(\forall\texttt{R})(X \text{ fresh}) \qquad \frac{\Delta; \Gamma, \forall X.\ s, s[t/X] \Longrightarrow s'}{\Delta; \Gamma, \forall X.\ s \Longrightarrow s'}(\forall\texttt{L})$$

Kripke Models. A Kripke model of S4$^\forall$ is a model of S4, where in addition, ρ maps propositional variables to $\mathcal{P}(W)$. Satisfaction for $\forall X.\ s$ at a world w is defined as follows:

- $w \models \forall X.\ s$ if for every $S \subseteq W$, $w \models s$ with map $\rho[X \mapsto S]$ in place of ρ.

For S4$^\forall$ Kripke models (as opposed to S4 Kripke models), the mapping ρ changes with the formula. Thus we often write it in the satisfaction relation with the world, as in $w, \rho \models s$. As usual, satisfaction with respect to models $\models s$ is defined by lifting this relation to all worlds and all models.

Lemma 16 *Let $\mathcal{S} = \langle W, \leq, \rho\rangle$ be an S4$^\forall$ model, and $s(X)$ and t be S4$^\forall$ formulas. Let $U = \{w \in W | w, \rho \models t\}$. Then $w, \rho[X \mapsto U] \models s(X)$ if and only if $w, \rho \models s[t/X]$.*

Proof. We induct on s to prove the lemma, showing some representative cases.

Case. $s = X$. Suppose $w, \rho[X \mapsto U] \models X$. By definition, $w \in U$. Therefore, by definition of U, $w, \rho \models t = s[t/X]$.
Conversely, suppose that $w, \rho \models s[t/X] = t$. By definition, $w \in U$. Therefore $w, \rho[X \mapsto U] \models X$.
Case. $s = Y \neq X$. Suppose $w, \rho[X \mapsto U] \models Y$. By definition, $w \in \rho(Y)$. Hence, $w, \rho \models Y = s[t/X]$.
Conversely, suppose $w, \rho \models s[t/X] = Y$. Then, $w \in \rho(Y)$. Thus $w \in \rho[X \mapsto U]$. Therefore, $w, \rho[X \mapsto U] \models Y$.
Case. $s = p$ is similar to the previous case.

Case. $s = \Box s'$. Suppose $w, \rho[X \mapsto U] \models \Box s'$. We want to show that $w, \rho \models \Box (s'[t/X])$. Choose any $w' \geq w$. It suffices to show that $w', \rho \models s'[t/X]$. From the given assumption, $w', \rho[X \mapsto U] \models s'$. Thus by i.h., $w', \rho \models s'[t/X]$, as required.

Conversely, suppose that $w, \rho \models \Box (s'[t/X])$. We want to show that $w, \rho[X \mapsto U] \models \Box s'$. Choose any $w' \geq w$. It suffices to show that $w', \rho[X \mapsto U] \models s'$. By given assumption, $w', \rho \models s'[t/X]$. Hence by i.h., $w', \rho[X \mapsto U] \models s'$ as required.

Case. $s = \forall Y.s'$. Suppose $w, \rho[X \mapsto U] \models \forall Y.s'$. We want to show that $w, \rho \models \forall Y.(s'[t/X])$. Choose any $V \subseteq W$. If suffices to show that $w, \rho[Y \mapsto V] \models s'[t/X]$. By given condition, $w, \rho[X \mapsto U] \models \forall Y.s'$. Hence, $w, \rho[X \mapsto U][Y \mapsto V] \models s'$, i.e., $w, \rho[Y \mapsto V][X \mapsto U] \models s'$. By i.h., $w, \rho[Y \mapsto V] \models s'[t/X]$, as required.

Conversely, suppose $w, \rho \models \forall Y.(s'[t/X])$. We want to show that $w, \rho[X \mapsto U] \models \forall Y.s'$. Choose any $V \subseteq W$. It suffices to show that $w, \rho[X \mapsto U][Y \mapsto V] \models s'$, i.e., $w, \rho[Y \mapsto V][X \mapsto U] \models s'$. By the given condition, $w, \rho[Y \mapsto V] \models s'[t/X]$. Thus by the i.h., $w, \rho[Y \mapsto V][X \mapsto U] \models s'$, as required.

Case. $s = s_1 \supset s_2$. Suppose $w, \rho[X \mapsto U] \models s_1 \supset s_2$. We want to show that $w, \rho \models (s_1[t/X]) \supset (s_2[t/X])$. Assume that $w, \rho \models s_1[t/X]$. It suffices to show that $w, \rho \models s_2[t/X]$. By i.h., $w, \rho[X \mapsto U] \models s_1$. Combining with the assumed condition, $w, \rho[X \mapsto U] \models s_2$. By i.h. again, we get that $w, \rho \models s_2[t/X]$ as required.

Conversely, suppose that $w, \rho \models (s_1[t/X]) \supset (s_2[t/X])$. We want to show that $w, \rho[X \mapsto U] \models s_1 \supset s_2$. Assume that $w, \rho[X \mapsto U] \models s_1$. It suffices to show that $w, \rho[X \mapsto U] \models s_2$. By i.h., $w, \rho \models s_1[t/X]$. Combining with our initial assumption, $w, \rho \models s_2[t/X]$. Hence by i.h., $w, \rho[X \mapsto U] \models s_2$ as required.

The remaining cases are straightforward.

Lemma 17 *In any Kripke model of $S4^{\forall}$, $w, \rho \models (\forall X.s) \supset s[t/X]$.*

Proof. Suppose $w, \rho \models \forall X.s$. We want to show that $w, \rho \models s[t/X]$. Let $U = \{w \in W | w, \rho \models t\}$. By definition, $w, \rho[X \mapsto U] \models s$. Hence by Lemma 16, $w, \rho \models s[t/X]$, as required.

E.2 Quantifier-free Formulas in $S4^{\forall}$

Quantifier-free formulas in $S4^{\forall}$ are those that do not contain any occurrence of the universal quantifier, or of propositional variables. By the way we have defined $S4^{\forall}$, these formulas are also in the syntax of S4. The following elementary lemma shows that on such formulas, satisfiability in Kripke models of $S4^{\forall}$ and S4 coincides.

Lemma 18 *The following are equivalent for any quantifier free $S4^{\forall}$ formula s.*

1. *$\models s$ in $S4^{\forall}$ Kripke structures.*
2. *$\models s$ in S4 Kripke structures.*

Proof. For an S4$^\forall$ Kripke structure $\mathcal{S} = \langle W, \leq, \rho \rangle$, define S4 Kripke structure $\overline{\mathcal{S}} = \langle W, \leq, \overline{\rho} \rangle$, where $\overline{\rho}$ is the restriction of ρ to atomic formulas only (ignoring the map ρ on propositional variables. Now using a straightforward induction on quantifier free s, one can establish that $\mathcal{S} \models s$ if and only if $\overline{\mathcal{S}} \models s$. From this the result follows immediately.

Definition 9 (Acyclic Kripke Structure) A Kripke structure for S4$^\forall$ is called acyclic if the accessibility relation $\leq$ has no cycles, i.e., it is a partial order (rather than a pre-order).

Definition 10 (Structure Unrolling) Let $\mathcal{S} = \langle W, \leq, \rho \rangle$ be an S4$^\forall$ Kripke structure. We define an *acyclic* S4$^\forall$ Kripke structure $\text{un}(\mathcal{S}) = \langle \text{un}(W), \text{un}(\leq), \text{un}(\rho) \rangle$ as follows.

- $\text{un}(W) = \{w_0 \ldots w_n \mid w_i \in W, n \geq 0, w_i \leq w_{i+1}\}$. Thus elements of $\text{un}(W)$ are accessible sequences of worlds in W.
- $\text{un}(\leq) = \{(\overline{w}, \overline{w'}) \mid \overline{w} \text{ is a prefix of } \overline{w'}\}$.
- $\text{un}(\rho)(X) = \{\overline{w} \mid \mathit{last}(\overline{w}) \in \rho(X)\}$.

Clearly, $\text{un}(\leq)$ is reflexive and transitive (because the prefix relation is). Further it is acyclic since prefixes always get longer under the relation.

Lemma 19 *Let s be a quantifier-free S4$^\forall$ formula. Let $\mathcal{S} = \langle W, \leq, \rho \rangle$ be a Kripke structure. Then the following hold.*

1. *If $w, \rho \models s$ then $(\overline{w}, w), un(\rho) \models s$ for every $\overline{w}$ such that $(\overline{w}, w) \in un(W)$.*
2. *If $(\overline{w}, w) \in un(W)$ and $(\overline{w}, w), un(\rho) \models s$, then $w, \rho \models s$.*

Proof. We prove both statements by a simultaneous induction on s. We case analyze the structure of s.

Case. $s = p$.
1. Suppose $w, \rho \models p$. Then by definition, $w \in \rho(p)$. By Definition 10, $(\overline{w}, w) \in \text{un}(\rho)(p)$. Thus $(\overline{w}, w), \text{un}(\rho) \models p$.
2. Suppose $(\overline{w}, w), \text{un}(\rho) \models p$. Thus, $(\overline{w}, w) \in \text{un}(\rho)(p)$. By definition, $w \in \rho(p)$. Hence $w, \rho \models p$.

Case. $s = X$ does not arise since s is assumed quantifier free.

Case. $s = \top$.
1. By definition of satisfaction in Kripke models, it is always the case that $(\overline{w}, w), \text{un}(\rho) \models \top$.
2. It is always the case that $w, \rho \models \top$.

Case. $s = \bot$.
1. Statement is vacuously true since it is never the case that $w, \rho \models s$.
2. Statement is vacuously true since it is never the case that $(\overline{w}, w), \text{un}(\rho) \models \rho$.

Case. $s = s_1 \wedge s_2$.
1. Suppose $w, \rho \models s_1 \wedge s_2$. By definition of satisfaction, $w, \rho \models s_1$ and $w, \rho \models s_2$. Thus by i.h.(1) we have $(\overline{w}, w), \text{un}(\rho) \models s_1$ and $(\overline{w}, w), \text{un}(\rho) \models s_2$. Hence $(\overline{w}, w), \text{un}(\rho) \models s_1 \wedge s_2$.

2. Similar to argument in (1) except that we use i.h.(2).

Case. $s = s_1 \vee s_2$. Similar to the previous case.

Case. $s = s_1 \supset s_2$.

1. Suppose $w, \rho \models s_1 \supset s_2$. We want to show that $(\overline{w}, w), \mathrm{un}(\rho) \models s_1 \supset s_2$. So assume that $(\overline{w}, w), \mathrm{un}(\rho) \models s_1$. By i.h.(2), $w, \rho \models s_1$. Hence from the assumption $w, \rho \models s_2$. By i.h.(1), $(\overline{w}, w), \mathrm{un}(\rho) \models s_2$, as required.
2. Similar to (1).

Case. $s = \forall X.s'$ does not arise since s is assumed quantifier free.

Case. $s = \Box t$.

1. Suppose $w, \rho \models \Box t$. We want to show that $(\overline{w}, w), \mathrm{un}(\rho) \models \Box t$. So pick any world $\overline{w'}$ such that $((\overline{w}, w), \overline{w'}) \in \mathrm{un}(\leq)$. It suffices to show that $\overline{w'}, \mathrm{un}(\rho) \models t$.
   By definition of $\mathrm{un}(\leq)$ it follows that $\overline{w'} = \overline{w}, w, w_1 \dots w_n$ where $n \geq 0$, $w \leq w_1$ and $w_i \leq w_{i+1}$. By transitivity of $\leq$, $w \leq w_n$. Thus from the given assumption it follows that $w_n, \rho \models t$. Hence by i.h.(1), $\overline{w'}, \mathrm{un}(\rho) \models t$.
2. Suppose $(\overline{w}, w), \mathrm{un}(\rho) \models \Box t$. We want to show that $w, \rho \models \Box t$. Pick any $w' \geq w$. It suffices to show that $w', \rho \models t$.
   From the assumption $w' \geq w$, it follows that $((\overline{w}, w), (\overline{w}, w, w')) \in \mathrm{un}(\leq)$. Thus from our assumption, it follows that $(\overline{w}, w, w'), \mathrm{un}(\rho) \models t$. By i.h.(2), $w', \rho \models t$.

Lemma 20 (Completeness of Acyclic Structures) *If s is a quantifier free $S4^{\forall}$ formula and $\mathcal{S}$ is a Kripke structure, then $\mathcal{S} \models s$ if and only if $un(\mathcal{S}) \models s$.*

Proof. Suppose $\mathcal{S} = \langle W, \leq, \rho \rangle \models s$. Let $\overline{w} \in \mathrm{un}(W)$. By assumption $last(\overline{w}), \rho \models s$. Thus by Lemma 19, $\overline{w}, \mathrm{un}(\rho) \models s$. Hence $\mathrm{un}(\mathcal{S}) \models s$.

Conversely, suppose $\mathrm{un}(\mathcal{S}) \models s$. Let $w \in W$. By assumption, $w, \mathrm{un}(\rho) \models s$. Therefore, by Lemma 19, $w, \rho \models s$. Hence $\mathcal{S} \models s$.

E.3 From $\mathrm{ICL}^{\forall}$ to $\mathrm{S4}^{\forall}$

We define a translation from $\mathrm{ICL}^{\forall}$ to $\mathrm{S4}^{\forall}$. With a slight abuse of notation, we refer to this translation as $\ulcorner \cdot \urcorner$. This should not cause any confusion, since the source of the translation will always be clear.

$$\begin{array}{rcl}
\ulcorner p \urcorner & = & \Box p \\
\ulcorner s \wedge t \urcorner & = & \ulcorner s \urcorner \wedge \ulcorner t \urcorner \\
\ulcorner s \vee t \urcorner & = & \ulcorner s \urcorner \vee \ulcorner t \urcorner \\
\ulcorner s \supset t \urcorner & = & \Box (\ulcorner s \urcorner \supset \ulcorner t \urcorner) \\
\ulcorner \bot \urcorner & = & \bot \\
\ulcorner \top \urcorner & = & \top \\
\ulcorner \mathsf{A} \text{ says } s \urcorner & = & \Box (\mathsf{A} \vee \ulcorner s \urcorner) \\
\ulcorner \forall X.\, s \urcorner & = & \Box \forall X.\, \ulcorner s \urcorner \\
\ulcorner X \urcorner & = & \Box X
\end{array}$$

Lemma 21 *Let s be a ICL$^\forall$ formula, and $\mathcal{S}$ be an S4$^\forall$ Kripke model. Then $\mathcal{S} \models \ulcorner s \urcorner \supset \Box \ulcorner s \urcorner$.*

Proof. We induct on s to prove that for any model $\mathcal{S}$, and any $w \in \mathcal{S}$, $w \models \ulcorner s \urcorner \supset \Box \ulcorner s \urcorner$.

Case. $s = p$. Then we have to show that $w \models \Box p \supset \Box \Box p$. So assume that $w \models \Box p$. It suffices to show that for any $w' \geq w$, $w' \models \Box p$. It suffices to show that for any $w'' \geq w'$, $w'' \models p$. But by transitivity of $\geq$, $w'' \geq w$. Hence from the condition $w \models \Box p$, it follows that $w'' \models p$.

Case. $s = s_1 \supset s_2$, $s = \mathsf{A}\ \mathtt{says}\ s'$, $s = X$ and $s = \forall X.\ t$ follow the same structure as the previous case because $\ulcorner s \urcorner$ has a $\Box$ in front.

Case. $s = \bot$. We must show that $w \models \bot \supset \Box \bot$. This is immediate because $w \not\models \bot$.

Case. $s = \top$. We must show that $w \models \top \supset \Box \top$. Assume that $w \models \top$. It suffices to show that for any $w' \geq w$, $w' \models \top$. This is immediate because every $w' \models \top$ by definition.

Case. $s = s_1 \wedge s_2$. To show: $w \models (\ulcorner s_1 \urcorner \wedge \ulcorner s_2 \urcorner) \supset \Box (\ulcorner s_1 \urcorner \wedge \ulcorner s_2 \urcorner)$. Assume that $w \models \ulcorner s_1 \urcorner \wedge \ulcorner s_2 \urcorner$. Therefore $w \models \ulcorner s_1 \urcorner$ and $w \models \ulcorner s_2 \urcorner$. By induction, $w \models \ulcorner s_1 \urcorner \supset \Box \ulcorner s_1 \urcorner$. Therefore, $w \models \Box \ulcorner s_1 \urcorner$. Hence for any $w' \geq w$, $w' \models \ulcorner s_1 \urcorner$. Similarly, $w' \models \ulcorner s_2 \urcorner$.

Hence it follows that $w' \models \ulcorner s_1 \urcorner \wedge \ulcorner s_2 \urcorner$. Since w' is arbitrary, $w \models \Box (\ulcorner s_1 \urcorner \wedge \ulcorner s_2 \urcorner)$. This is what we wanted to show.

Case. $s = s_1 \vee s_2$. To show: $w \models (\ulcorner s_1 \urcorner \vee \ulcorner s_2 \urcorner) \supset \Box (\ulcorner s_1 \urcorner \vee \ulcorner s_2 \urcorner)$. Assume that $w \models \ulcorner s_1 \urcorner \vee \ulcorner s_2 \urcorner$. Thus $w \models \ulcorner s_1 \urcorner$ or $w \models \ulcorner s_2 \urcorner$. Let us take the case $w \models \ulcorner s_1 \urcorner$ (the other case is symmetric).

By i.h., $w \models \ulcorner s_1 \urcorner \supset \Box \ulcorner s_1 \urcorner$. Therefore, $w \models \Box \ulcorner s_1 \urcorner$. Hence for any $w' \geq w$, $w' \models \ulcorner s_1 \urcorner$. By definition, $w' \models \ulcorner s_1 \urcorner \vee \ulcorner s_2 \urcorner$. Since w' is arbitrary, $w \models \Box (\ulcorner s_1 \urcorner \vee \ulcorner s_2 \urcorner)$, which is what we needed to show.

Lemma 22 (Substitution) *In any Kripke model $\mathcal{S}$ of S4$^\forall$, and for any ICL$^\forall$ formulas s, t, it is the case that $w, \rho \models \ulcorner s \urcorner [\ulcorner t \urcorner / X]$ if and only if $w, \rho \models \ulcorner s[t/X] \urcorner$.*

Proof. We prove this lemma by induction on s. The cases are straightforward. The only interesting case is where $s = X$. Then we have $\ulcorner s \urcorner [\ulcorner t \urcorner / X] = \Box X [\ulcorner t \urcorner / X] = \Box \ulcorner t \urcorner$. And $\ulcorner s[t/X] \urcorner = \ulcorner X[t/X] \urcorner = \ulcorner t \urcorner$. Thus we have to show two things:

1. $w, \rho \models \Box \ulcorner t \urcorner$ implies $w, \rho \models \ulcorner t \urcorner$.
2. $w, \rho \models \ulcorner t \urcorner$ implies $w, \rho \models \Box \ulcorner t \urcorner$.

(1) is trivial. For (2), it follows by Lemma 21 that $w, \rho \models \ulcorner t \urcorner \supset \Box \ulcorner t \urcorner$. By definition of satisfaction, this is the same as (2).

Lemma 23 (Soundness of $\ulcorner \cdot \urcorner$) *Suppose $\Gamma \Longrightarrow s$ is provable in ICL$^\forall$. Then for any Kripke model $\mathcal{S}$ of S4$^\forall$, it is the case that $\mathcal{S} \models (\ulcorner \Gamma \urcorner \supset \ulcorner s \urcorner)$.*
[By $\ulcorner \Gamma \urcorner$ we mean translation of conjunction of all formulas in Γ. If Γ is empty, we take this translation to be $\top$.]

Proof. We prove by induction on the given proof of $\Gamma \Longrightarrow s$ that for any world w and any mapping ρ, it is the case that $w, \rho \models \ulcorner\Gamma\urcorner \supset \ulcorner s\urcorner$. We analyze cases of the last rule, and show some representative cases here.

Case. $\dfrac{}{\Gamma, p \Longrightarrow p}(\texttt{init})$

To show: $w, \rho \models (\ulcorner\Gamma\urcorner \wedge \Box p) \supset \Box p$.

1. [Assume] $w, \rho \models \ulcorner\Gamma\urcorner \wedge \Box p$.
2. (From 1) $w, \rho \models \Box p$.
3. $(1 \Rightarrow 2)$ $w, \rho \models (\ulcorner\Gamma\urcorner \wedge \Box p)$ implies $w, \rho \models \Box p$.
4. (Definition of $\models$ and 3) $w, \rho \models (\ulcorner\Gamma\urcorner \wedge \Box p) \supset \Box p$.

Case. $\dfrac{\Gamma, s \supset t \Longrightarrow s \qquad \Gamma, t, s \supset t \Longrightarrow s'}{\Gamma, s \supset t \Longrightarrow s'}(\supset \texttt{L})$

To show: $w, \rho \models (\ulcorner\Gamma\urcorner \wedge \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner)) \supset \ulcorner s'\urcorner$.

1. [Assume] $w, \rho \models \ulcorner\Gamma\urcorner \wedge \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner)$.
2. (From 1) $w, \rho \models \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner)$.
3. (From 2) $w, \rho \models \ulcorner s\urcorner \supset \ulcorner t\urcorner$.
4. (i.h. 1) $w, \rho \models (\ulcorner\Gamma\urcorner \wedge \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner)) \supset \ulcorner s\urcorner$.
5. (From 4,1) $w, \rho \models \ulcorner s\urcorner$.
6. (From 5,3) $w, \rho \models \ulcorner t\urcorner$.
7. (From 1,6) $w, \rho \models \ulcorner\Gamma\urcorner \wedge \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner) \wedge \ulcorner t\urcorner$.
8. (i.h. 2) $w, \rho \models (\ulcorner\Gamma\urcorner \wedge \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner) \wedge \ulcorner t\urcorner) \supset \ulcorner s'\urcorner$.
9. (From 8,7) $w, \rho \models \ulcorner s'\urcorner$.
10. $(1 \Rightarrow 9)$ $w, \rho \models \ulcorner\Gamma\urcorner \wedge \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner)$ implies $w, \rho \models \ulcorner s'\urcorner$.
11. (Definition of $\models$ and 10) $w, \rho \models (\ulcorner\Gamma\urcorner \wedge \Box(\ulcorner s\urcorner \supset \ulcorner t\urcorner)) \supset \ulcorner s'\urcorner$.

Case. $\dfrac{\Gamma \Longrightarrow s}{\Gamma \Longrightarrow \mathsf{A}\ \mathsf{says}\ s}(\texttt{saysR})$

To show: $w, \rho \models \ulcorner\Gamma\urcorner \supset \Box(\mathsf{A} \vee \ulcorner s\urcorner)$.

1. [Assume] $w, \rho \models \ulcorner\Gamma\urcorner$.
2. [Assume] $w' \geq w$.
3. (i.h. at w' instead of w) $w', \rho \models \ulcorner\Gamma\urcorner \supset \ulcorner s\urcorner$.
4. (Lemma 21) $w, \rho \models \ulcorner\Gamma\urcorner \supset \Box\ulcorner\Gamma\urcorner$.
5. (From 4,1) $w, \rho \models \Box\ulcorner\Gamma\urcorner$.
6. (From 5,2) $w', \rho \models \ulcorner\Gamma\urcorner$.
7. (From 3,6) $w', \rho \models \ulcorner s\urcorner$.
8. (From 7) $w', \rho \models \mathsf{A} \vee \ulcorner s\urcorner$.
9. $(2 \Rightarrow 8)$ $w' \geq w$ implies $w', \rho \models \mathsf{A} \vee \ulcorner s\urcorner$.
10. (From 9) $w', \rho \models \Box(\mathsf{A} \vee \ulcorner s\urcorner)$.
11. $(1 \Rightarrow 10)$ $w, \rho \models \ulcorner\Gamma\urcorner$ implies $w', \rho \models \Box(\mathsf{A} \vee \ulcorner s\urcorner)$.
12. (From 11) $w, \rho \models \ulcorner\Gamma\urcorner \supset \Box(\mathsf{A} \vee \ulcorner s\urcorner)$.

Case. $\dfrac{\Gamma, s, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{A}\ \mathsf{says}\ s'}{\Gamma, \mathsf{A}\ \mathsf{says}\ s \Longrightarrow \mathsf{A}\ \mathsf{says}\ s'}(\texttt{saysL})$

To show: $w, \rho \models (\ulcorner\Gamma\urcorner \wedge \Box(\mathsf{A} \vee \ulcorner s\urcorner)) \supset \Box(\mathsf{A} \vee \ulcorner s'\urcorner)$.

1. [Assume] $w, \rho \models \ulcorner\Gamma\urcorner \wedge \Box(\mathsf{A} \vee \ulcorner s\urcorner)$.
2. [Assume] $w' \geq w$.

3. (From 1) $w, \rho \models \ulcorner \Gamma \urcorner$.
4. (From 1) $w, \rho \models \Box(\mathsf{A} \vee \ulcorner s \urcorner)$.
5. (From 4,2) $w', \rho \models \mathsf{A} \vee \ulcorner s \urcorner$.
6. (From 5) $w', \rho \models \mathsf{A}$ or $w', \rho \models \ulcorner s \urcorner$.
   A **Case.** $w', \rho \models \mathsf{A}$.
      A.1 (From A) $w', \rho \models \mathsf{A} \vee \ulcorner s' \urcorner$.
   B **Case.** $w', \rho \models \ulcorner s \urcorner$.
      B.1 (Lemma 21) $w, \rho \models \ulcorner \Gamma \urcorner \supset \Box \ulcorner \Gamma \urcorner$.
      B.2 (From B.1,3) $w, \rho \models \Box \ulcorner \Gamma \urcorner$.
      B.3 (From B.2,2) $w', \rho \models \ulcorner \Gamma \urcorner$.
      B.4 [Assume] $w'' \geq w'$.
      B.5 (From B.4,2) $w'' \geq w$.
      B.6 (From 3,B.5) $w'', \rho \models \mathsf{A} \vee \ulcorner s \urcorner$.
      B.7 (B.4 $\Rightarrow$ B.6) $w'' \geq w'$ implies $w'', \rho \models \mathsf{A} \vee \ulcorner s \urcorner$.
      B.8 (From B.7) $w', \rho \models \Box(\mathsf{A} \vee \ulcorner s \urcorner)$.
      B.9 (From B,B.3,B.8) $w', \rho \models \ulcorner \Gamma \urcorner \wedge \Box(\mathsf{A} \vee \ulcorner s \urcorner) \wedge \ulcorner s \urcorner$.
      B.10 (i.h. at w' instead of w) $w', \rho \models (\ulcorner \Gamma \urcorner \wedge \Box(\mathsf{A} \vee \ulcorner s \urcorner) \wedge \ulcorner s \urcorner) \supset \Box(\mathsf{A} \vee \ulcorner s' \urcorner)$.
      B.11 (From B.10,B.9) $w', \rho \models \Box(\mathsf{A} \vee \ulcorner s' \urcorner)$.
      B.12 (From B.11) $w', \rho \models \mathsf{A} \vee \ulcorner s' \urcorner$.
7. (From A.1,B.11) $w', \rho \models \mathsf{A} \vee \ulcorner s' \urcorner$.
8. (2 $\Rightarrow$ 7) $w' \geq w$ implies $w', \rho \models \mathsf{A} \vee \ulcorner s' \urcorner$.
9. (From 8) $w, \rho \models \Box(\mathsf{A} \vee \ulcorner s' \urcorner)$.
10. (1 $\Rightarrow$ 9) $w, \rho \models \ulcorner \Gamma \urcorner \wedge \Box(\mathsf{A} \vee \ulcorner s \urcorner)$ implies $w, \rho \models \Box(\mathsf{A} \vee \ulcorner s' \urcorner)$.
11. (From 10) $w, \rho \models (\ulcorner \Gamma \urcorner \wedge \Box(\mathsf{A} \vee \ulcorner s \urcorner)) \supset \Box(\mathsf{A} \vee \ulcorner s' \urcorner)$.

Case. $\dfrac{\Gamma \Longrightarrow s}{\Gamma \Longrightarrow \forall X.\ s}(\forall\mathtt{R})(X \text{ fresh})$

To show: $w, \rho \models \ulcorner \Gamma \urcorner \supset \Box \forall X.\ \ulcorner s \urcorner$.

1. [Assume] $w, \rho \models \ulcorner \Gamma \urcorner$.
2. [Assume] $w' \geq w$.
3. [Assume] $U \subseteq W$.
4. (i.h. at w' and $\rho[X \mapsto U]$) $w', \rho[X \mapsto U] \models \ulcorner \Gamma \urcorner \supset \ulcorner s \urcorner$.
5. (Lemma 21) $w, \rho \models \ulcorner \Gamma \urcorner \supset \Box \ulcorner \Gamma \urcorner$.
6. (From 5,1) $w, \rho \models \Box \ulcorner \Gamma \urcorner$.
7. (From 6,2) $w', \rho \models \ulcorner \Gamma \urcorner$.
8. ($X \notin \ulcorner \Gamma \urcorner$ and 7) $w', \rho[X \mapsto U] \models \ulcorner \Gamma \urcorner$.
9. (From 4,7) $w', \rho[X \mapsto U] \models \ulcorner s \urcorner$.
10. (3 $\Rightarrow$ 8) $U \subseteq W$ implies $w', \rho[X \mapsto U] \models \ulcorner s \urcorner$.
11. (From 9) $w', \rho \models \forall X.\ \ulcorner s \urcorner$.
12. (2 $\Rightarrow$ 10) $w' \geq w$ implies $w', \rho \models \forall X.\ \ulcorner s \urcorner$.
13. (From 11) $w, \rho \models \Box \forall X.\ \ulcorner s \urcorner$.
14. (1 $\Rightarrow$ 12) $w, \rho \models \ulcorner \Gamma \urcorner$ implies $w, \rho \models \Box \forall X.\ \ulcorner s \urcorner$.
15. (From 13) $w, \rho \models \ulcorner \Gamma \urcorner \supset \Box \forall X.\ \ulcorner s \urcorner$.

Case. $\dfrac{\Gamma, \forall X.\ s, s[t/X] \Longrightarrow s'}{\Gamma, \forall X.\ s \Longrightarrow s'}(\forall\mathtt{L})$

To show: $w, \rho \models (\ulcorner \Gamma \urcorner \wedge \Box \forall X.\ \ulcorner s \urcorner) \supset \ulcorner s' \urcorner$.

1. [Assume] $w, \rho \models \ulcorner \Gamma \urcorner \wedge \Box \forall X. \ulcorner s \urcorner$.
2. (From 1) $w, \rho \models \Box \forall X. \ulcorner s \urcorner$.
3. (From 2) $w, \rho \models \forall X. \ulcorner s \urcorner$.
4. (Lemma 17) $w, \rho \models (\forall X. \ulcorner s \urcorner) \supset \ulcorner s \urcorner [\ulcorner t \urcorner / X]$.
5. (From 4,3) $w, \rho \models \ulcorner s \urcorner [\ulcorner t \urcorner / X]$.
6. (Lemma 22 on 5) $w, \rho \models \ulcorner s[t/X] \urcorner$.
7. (From 1,6) $w, \rho \models \ulcorner \Gamma \urcorner \wedge \Box \forall X. \ulcorner s \urcorner \wedge \ulcorner s[t/X] \urcorner$.
8. (i.h.) $w, \rho \models (\ulcorner \Gamma \urcorner \wedge \Box \forall X. \ulcorner s \urcorner \wedge \ulcorner s[t/X] \urcorner) \supset \ulcorner s' \urcorner$.
9. (From 8,7) $w, \rho \models \ulcorner s' \urcorner$.
10. ($1 \Rightarrow 9$) $w, \rho \models \ulcorner \Gamma \urcorner \wedge \Box \forall X. \ulcorner s \urcorner$ implies $w, \rho \models \ulcorner s' \urcorner$.
11. (From 10) $w, \rho \models (\ulcorner \Gamma \urcorner \wedge \Box \forall X. \ulcorner s \urcorner) \supset \ulcorner s' \urcorner$.

E.4 Equivalence of $\ulcorner s \urcorner$ and $\ulcorner [\![s]\!] \urcorner$ in Acyclic Structures

For any ICL$^{\Rightarrow}$ formula s, we know of two ways to embed it into S4$^{\forall}$. One is through the translation $\ulcorner \cdot \urcorner$ from Section 3 which maps ICL$^{\Rightarrow}$ to S4 (and hence implicitly to S4$^{\forall}$, since S4$^{\forall}$ is a superset of S4). The other is to map s to ICL$^{\forall}$ via the translation $[\![\cdot]\!]$ (Section 6) and then map that into S4$^{\forall}$ via $\ulcorner \cdot \urcorner$ (appendix E.3). We now show that the two formulas so obtained ($\ulcorner s \urcorner$ and $\ulcorner [\![s]\!] \urcorner$) are equivalent from the point of view of provability in acyclic Kripke structures.

Lemma 24 *Let* A *and* B *be atomic formulas in S4*$^{\forall}$*. In any* acyclic *Kripke structure* $\mathcal{S} = \langle W, \leq, \rho \rangle$ *of S4*$^{\forall}$*, the following two are equivalent statements*

1. $w, \rho \models \Box \forall X. \Box (\Box (\mathsf{A} \vee \Box X) \supset \Box (\mathsf{B} \vee \Box X))$
2. $w, \rho \models \Box (\mathsf{A} \supset \mathsf{B})$

Proof. (1) $\Rightarrow$ (2). We reason by the method of contradiction. Suppose for the sake of contradiction (1) holds but $w, \rho \not\models \Box (\mathsf{A} \supset \mathsf{B})$. Then in some world $w' \geq w$, it must be the case that $w', \rho \models \mathsf{A}$ and $w', \rho \not\models \mathsf{B}$. Thus $w' \in \rho(\mathsf{A})$ and $w' \notin \rho(\mathsf{B})$.

Let $U = \{w'' \in W \mid w'' \geq w', w'' \neq w'\}$. Consider the following two statements:

(A) $w', \rho[X \mapsto U] \models \Box (\mathsf{A} \vee \Box X)$
(B) $w', \rho[X \mapsto U] \not\models \Box (\mathsf{B} \vee \Box X)$

We claim that (A) and (B) are both true. To show that (A) is true, it suffices to show that for any $w'' \geq w'$, it is the case that $w'', \rho[X \mapsto U] \models \mathsf{A} \vee \Box X$. If $w'' = w'$, then from the fact $w' \in \rho(\mathsf{A})$, it follows that $w'', \rho[X \mapsto U] \models \mathsf{A}$. Hence the statement follows immediately. If $w'' \neq w'$, then $w'' > w'$ (strictly). Then we show that $w'', \rho[X \mapsto U] \models \Box X$. Let $w''' \geq w''$. We need to show that $w''', \rho[X \mapsto U] \models X$. But since $w''' \geq w'' > w'$, it follows that $w''' > w'$. By acyclicity, $w''' \neq w'$. Hence by definition of U, $w''' \in U$. Therefore $w''', \rho[X \mapsto U] \models X$.

To show that (B) is true, we should be able to demonstrate a world $w'' \geq w'$ such that $w'', \rho[X \mapsto U] \not\models \mathsf{B} \vee \Box X$. Choose $w'' = w'$. Clearly, $w', \rho[X \mapsto$

$U] \not\models \mathsf{B}$ because $w' \notin \rho(\mathsf{B})$. Also, since $w' \notin U$, $w', \rho[X \mapsto U] \not\models X$. Hence $w', \rho[X \mapsto U] \not\models \Box X$. Thus $w', \rho[X \mapsto U] \not\models \mathsf{B} \vee \Box X$.

It follows from (A) and (B) that $w', \rho[X \mapsto U] \not\models \Box(\mathsf{A} \vee \Box X) \supset \Box(\mathsf{B} \vee \Box X)$. It is now easy to see that the following hold:

- $w, \rho[X \mapsto U] \not\models \Box(\Box(\mathsf{A} \vee \Box X) \supset \Box(\mathsf{B} \vee \Box X))$
- $w, \rho \not\models \forall X. \Box(\Box(\mathsf{A} \vee \Box X) \supset \Box(\mathsf{B} \vee \Box X))$
- $w, \rho \not\models \Box\forall X. \Box(\Box(\mathsf{A} \vee \Box X) \supset \Box(\mathsf{B} \vee \Box X))$

The last statement is contradictory to our initial assumption. Hence it must be the case that $w, \rho \models \Box(\mathsf{A} \supset \mathsf{B})$.

(2) $\Rightarrow$ (1). Suppose that (2) holds. In order to prove (1), it suffices to show that for any $w_1 \geq w$, $U \subseteq W$ and $w' \geq w_1$, it is the case that $w', \rho[X \mapsto U] \models \Box(\mathsf{A} \vee \Box X) \supset \Box(\mathsf{B} \vee \Box X)$. Assume that

(A) $w', \rho[X \mapsto U] \models \Box(\mathsf{A} \vee \Box X)$

Let $w'' \geq w'$. It suffices to show that $w'', \rho[X \mapsto U] \models \mathsf{B} \vee \Box X$. From (A) it follows that $w'', \rho[X \mapsto U] \models \mathsf{A} \vee \Box X$. Thus $w'', \rho[X \mapsto U] \models \mathsf{A}$ or $w'', \rho[X \mapsto U] \models \Box X$. We case analyze the two possibilities.

Case. $w'', \rho[X \mapsto U] \models \mathsf{A}$. By assumption (2), and $w'' \geq w' \geq w_1 \geq w$, it follows that $w'', \rho \models \mathsf{A} \supset \mathsf{B}$. Since $X \neq \mathsf{A}, \mathsf{B}$, it is implied that $w'', \rho[X \mapsto U] \models \mathsf{A} \supset \mathsf{B}$. Combining with the case assumption, $w'', \rho[X \mapsto U] \models \mathsf{B}$. Hence $w'', \rho[X \mapsto U] \models \mathsf{B} \vee \Box X$.
Case. $w'', \rho[X \mapsto U] \models \Box X$. Clearly then $w'', \rho[X \mapsto U] \models \mathsf{B} \vee \Box X$.

Lemma 25 (Equivalence in Acyclic Structures) *Let s be a ICL$^{\Rightarrow}$ formula and let $\mathcal{S} = \langle W, \leq, \rho \rangle$ be an acyclic S4$^{\forall}$ Kripke structure. Then $\mathcal{S} \models \ulcorner s \urcorner$ if and only if $\mathcal{S} \models \ulcorner [\![s]\!] \urcorner$.*

Proof. We induct on the structure of s to show that for any $w \in W$, $w, \rho \models \ulcorner s \urcorner$ if and only if $w, \rho \models \ulcorner [\![s]\!] \urcorner$. We case analyze the form of s, showing some representative cases only.

Case. $s = p$. In this case, $\ulcorner s \urcorner = \Box p = \ulcorner [\![p]\!] \urcorner$.
Case. $s = s_1 \wedge s_2$. Here $\ulcorner s \urcorner = \ulcorner s_1 \urcorner \wedge \ulcorner s_2 \urcorner$ and $\ulcorner [\![s]\!] \urcorner = \ulcorner [\![s_1]\!] \urcorner \wedge \ulcorner [\![s_2]\!] \urcorner$. Suppose $w, \rho \models \ulcorner s_1 \urcorner \wedge \ulcorner s_2 \urcorner$. We want to show that $w, \rho \models \ulcorner [\![s_1]\!] \urcorner \wedge \ulcorner [\![s_2]\!] \urcorner$. It suffices to show that $w, \rho \models \ulcorner [\![s_i]\!] \urcorner$. By assumption it follows that $w, \rho \models \ulcorner s_i \urcorner$ and hence by i.h., $w, \rho \models \ulcorner [\![s_i]\!] \urcorner$.
The converse is similar.
Case. $s = \mathsf{A}\ \mathsf{says}\ s'$. Then $\ulcorner s \urcorner = \Box(\mathsf{A} \vee \ulcorner s' \urcorner)$ and $\ulcorner [\![s]\!] \urcorner = \Box(\mathsf{A} \vee \ulcorner [\![s']\!] \urcorner)$. Assume that $w, \rho \models \Box(\mathsf{A} \vee \ulcorner s' \urcorner)$. We want to show that $w, \rho \models \Box(\mathsf{A} \vee \ulcorner [\![s']\!] \urcorner)$. Pick any $w' \geq w$. It suffices to show that $w', \rho \models \mathsf{A} \vee \ulcorner [\![s']\!] \urcorner$. By assumption, $w', \rho \models \mathsf{A} \vee \ulcorner s' \urcorner$. If $w', \rho \models \mathsf{A}$, we are done. If $w', \rho \models \ulcorner s' \urcorner$, then by i.h., $w', \rho \models \ulcorner [\![s']\!] \urcorner$. Thus $w', \rho \models \mathsf{A} \vee \ulcorner [\![s']\!] \urcorner$.
Case. $s = \mathsf{A} \Rightarrow \mathsf{B}$. Then $\ulcorner s \urcorner = \Box(\mathsf{A} \supset \mathsf{B})$ and $\ulcorner [\![s]\!] \urcorner = \Box\forall X. \Box(\Box(\mathsf{A} \vee \Box X) \supset \Box(\mathsf{B} \vee \Box X))$. In this case the result follows by Lemma 24.

E.5 Proof of Completeness

We now combine the developments of the previous sections and prove completeness and then Theorem 5.

Lemma 26 (Completeness) *For every ICL*$^{\Rightarrow}$ *formula s, if* $\vdash [\![s]\!]$ *(i.e.,* $\cdot \Longrightarrow [\![s]\!]$*) in ICL*$^{\forall}$ *then* $\vdash s$ *(i.e.,* $\cdot \Longrightarrow s$*) in ICL*$^{\Rightarrow}$*.*

Proof. Suppose $\cdot \Longrightarrow [\![s]\!]$ in ICL$^{\forall}$. By Lemma 23, for any S4$^{\forall}$ Kripke structure $\mathcal{S}$ it is the case that $\mathcal{S} \models \ulcorner [\![s]\!] \urcorner$. In particular, if $\mathcal{S}_A$ is an acyclic S4$^{\forall}$ Kripke structure, then $\mathcal{S}_A \models \ulcorner [\![s]\!] \urcorner$. By Lemma 25, $\mathcal{S}_A \models \ulcorner s \urcorner$. Hence every acyclic S4$^{\forall}$ Kripke structure satisfies $\ulcorner s \urcorner$.

In particular, if $\mathcal{S}$ is an arbitrary S4$^{\forall}$ Kripke structure, then $\mathrm{un}(\mathcal{S}) \models \ulcorner s \urcorner$, since $\mathrm{un}(\mathcal{S})$ is acyclic. Since $\ulcorner s \urcorner$ is quantifier free (by definition of $\ulcorner \cdot \urcorner$), from Lemma 20, $\mathcal{S} \models \ulcorner s \urcorner$. Since $\mathcal{S}$ is arbitrary, it follows that $\models \ulcorner s \urcorner$ in S4$^{\forall}$ Kripke structures. Thus by Lemma 18, $\models \ulcorner s \urcorner$ in S4 Kripke structures.

By Theorem 8, we get $\vdash \ulcorner s \urcorner$ in S4. Finally using Theorem 2 we conclude that $\vdash s$ in ICL$^{\Rightarrow}$.

Proof (Proof of Theorem 5). The soundness part is proved in Lemma 15 and completeness in Lemma 26.