Countermodels from Sequent Calculi in Multi-Modal Logics

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Abstract

A novel countermodel-producing decision procedure that applies to several multi-modal logics, both intuitionistic and classical, is presented. Based on backwards search in labeled sequent calculi, the procedure employs a novel termination condition and countermodel construction. Using the procedure, it is argued that multi-modal variants of several classical and intuitionistic logics including K, T, K4, S4 and their combinations with D are decidable and have the finite model property. At least in the intuitionistic multi-modal case, the decidability results are new. It is further shown that the countermodels produced by the procedure, starting from a set of hypotheses and no goals, characterize the atomic formulas provable from the hypotheses.

1 Introduction

Modal logics are widely used in several fields of Computer Science and their decidability is a subject of deep interest to the academic community. The subject has been investigated through various techniques, notably semantic filtrations [5, 8], semantic tableaux [4, 10, 11], and translation into decidable fragments of first-order logic [2], yet many areas related to decidability of modal logics remain open. Two such areas are: (1) Decidability of multi-modal intuitionistic logics, especially when modalities interact with each other, and (2) Decision procedures based on sequent calculi that can be directly implemented. Both these areas are challenging. Decidability of intuitionistic modal logics is challenging because standard techniques like semantic filtrations and tableaux have not been studied extensively in the intuitionistic setting, whereas sequent calculi are difficult to use for decision procedures in modal logic because of a well-known problem of looping [10, 16, 25], which is exacerbated by the interaction between modalities and intuitionistic implication.

Spanning both these areas, we present a uniform decision procedure for several propositional multi-modal logics (both intuitionistic and classical), based on backwards search in labeled sequent calculi. Our decision procedure is constructive, which means that for any given formula it either produces a derivation which shows that the formula is true in all (Kripke) models or produces a finite set of finite countermodels on all of which the formula is false.

The decision procedure is also general; it applies to any intuitionistic modal logic without possibility (diamond) modalities and any classical modal logic (even with possibility modalities), provided the logic satisfies a specific technical condition, namely the existence of what we call a suitable closure relation or SCR. As examples, we show that the classical and intuitionistic variants of the following multi-modal logics are constructively decidable by our method: K (the basic normal modal logic), T (reflexive frames), K4 (transitive frames), S4 (reflexive and transitive frames) and their combinations with D (serial frames). We further show that several interaction axioms between modalities such as I $((\Box_A \alpha) \to (\Box_B \Box_A \alpha))$ [9, 13], unit $(\alpha \to (\Box_A \alpha))$ [7] and subsumption $((\Box_A \alpha) \to (\Box_B \alpha))$ result in decidable logics. Constructive decidability also implies the finite model property, so our results also show this property for all the logics listed earlier. For multi-modal intuitionistic logics, not only our method, but also the decidability results are new.

Technical approach and challenges. We present our method separately for classical and intuitionistic logics due to minor technical differences between the two. Our method uses the labeled approach to proof theory of modal logic as developed, among others, by [16, 23, 25], and more specifically [16], to produce labeled sequent calculi with strong analyticity properties. We define a multi-modal intuitionistic (classical) logic MMI^{χ} (MM^{χ}) as the set of all formulas that are valid in all intuitionistic (classical) Kripke frames satisfying stipulated conditions, represented as a set χ. Conditions in χ can be arbitrary, but are restricted in two ways: (1) They must be of the form $\forall \vec{x}.((\wedge_{i=1,\ldots,n}\ x_i R_i x_i') \to (xRx'))$, where R, R_i range over the relations of a Kripke frame, and (2) The conditions χ must have a SCR, which is a relation over Kripke frames satisfying some stipulated properties, as discussed later in the paper. Briefly, the existence of the SCR implies that frame relations can be deduced from the conditions χ only in some specific ways. We then build a non-terminating, standard labeled sequent calculus for the logic, which we refine in two steps to obtain a constructive decision procedure that, for a given sequent, either produces a proof of it, or a finite set of finite countermodels for it. Next, we build an extension of our method that works for any logic on which our original method works, extended with seriality. Finally, we prove an interesting property of our decision procedure: The set of countermodels it produces for a given hypotheses without a specific goal completely characterizes the atomic formulas that can be proved from the hypotheses. Thus the set of countermodels produced is, in a sense, complete. We call this property *comprehensiveness*.

The first challenge for our work is to find a general termination condition for backwards search in labeled sequent calculi. Our termination condition is based on containment of the sets of formulas labeling worlds, which we show to be complete so long as the logic has an SCR. This idea is a non-trivial generalization of existing work on logic-specific termination conditions for many uni-modal classical logics [6, 10, 16]. Finding an appropriate definition for SCRs, that is both sufficient to obtain termination and general enough, is the main technical challenge of our work and also its main technical contribution.

The second challenge in our work is to actually build the countermodel when we know that backwards proof search has unsuccessfully terminated. To this end, we observe that a straightforward extension of the model inherent in the sequent at which search terminates (with a few more relations) is actually a countermodel to the sequent. As far as we know, this construction is novel.

Contributions. In summary, our work makes the following contributions to the area of decidability of modal logics:

- It proves, by uniform method, the decidability of the necessitation-only, multi-modal intuitionistic variants of the logics K, T, K4 and S4 and their combinations with the logic D. Our decision procedure produces countermodels, and also establishes the finite model property for these logics. (The corresponding uni-modal intuitionistic logics are already known to be decidable due to the work of Simpson [23] and Hasimoto [15].)

- We provide the first sequent calculus based constructive decision procedure for multi-modal logics. (Sequent calculus based decision procedures for specific uni-modal logics exist in both the classical [16, 24] and intuitionistic [23] settings.)
- At a technical level, we provide a general method for forcing termination in labeled sequent calculi for modal logics and a sufficient condition (the existence of SCRs) under which it works without loss of completeness. We also present a simple method to extract countermodels when search terminates.
- As far as we know, ours is the first decision procedure which produces a set of countermodels that is comprehensive in the sense described above.

Limitations. There are existing undecidability results for modal logics with frame conditions such as symmetry and transitivity [3]. Consequently, we cannot hope for a method that proves decidability for all logics MMI^{χ} or all logics MM^{χ}. Nonetheless, there are some classes of frame conditions which are not known to immediately result in undecidability, but to which our method does not apply (either because they do not fit our definition of χ or because they do not have SCRs):

- Due to an interaction with intuitionistic implication, our method cannot handle possibility modalities in intuitionistic logic, even though it works fine with them in classical logic. This is discussed in Section 4.
- We do not know whether our method can handle "label-producing" conditions like density or confluence. However, it can be easily extended to work with seriality, as discussed in Section 3.6.

Further, like many other methods in this domain, an analysis of our proofs does not necessarily produce good upper bounds on the actual complexity of modal logics.

Organization. Since our constructive decision procedure and its correctness proof are almost identical for intuitionistic and classical modal logics, in the main body of the paper, we present our results only for the intuitionistic case. The case of classical modal logic is briefly discussed in Section 5 and its details are deferred to Appendix B.

In Section 2, we define the syntax, semantics and a standard, but non-terminating labeled sequent calculus for the intuitionistic multi-modal logic MMI^{χ}. We illustrate the known problem of non-termination due to looping in the sequent calculus in Section 2.3. Section 3 presents the main technical work. Starting from an informal introduction, we proceed to a description of the decision procedure (Sections 3.1–3.4), its comprehensiveness (Section 3.5), its extension with seriality (Section 3.6), and its instantiation to several known intuitionistic multi-modal logics (Section 3.7). We discuss limitations of our work and its relation to semantic filtrations in Section 4. Section 5 briefly lists modifications needed to adapt the method to classical logic. Related work is discussed in Section 6 and Section 7 concludes the paper.

2 MMI^{χ}: Multi-Modal Intuitionistic Logic

We start by defining formally the family of intuitionistic multi-modal logics we consider in this paper. The family is parametrized by a set χ of conditions on Kripke frames that must have a specific (standard) form, as described later in this section. The logic obtained by instantiating our definition of syntax and semantics with a specific set χ is called MMI$^\chi$.

Syntax. Let $\mathcal{I} = \{A, B, \ldots\}$ be a finite set of indices for modalities and p denote an atomic formula, drawn from a countable set of such formulas. Then, the syntax of formulas of the logic MMI$^\chi$ is:

$$\text{Formulas} \quad \varphi, \alpha, \beta \quad ::= \quad p \mid \top \mid \bot \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \alpha \to \beta \mid A\ \mathsf{nec}\ \alpha$$

Connectives $\top$, $\bot$, $\wedge$, and $\vee$ have their usual meanings. Implication $\to$ is interpreted intuitionistically. $A\ \mathsf{nec}\ \alpha$ is the necessitation modality of index A. This is commonly written $\Box_A \alpha$, but we prefer the more descriptive notation. Negation is not primitive, but may be defined in a standard way as $\neg\alpha = \alpha \to \bot$. We do not consider possibility modalities in the intuitionistic setting, since they are incompatible with our method (see Section 4).

2.1 Semantics

We provide Kripke (frame) semantics to formulas of MMI$^\chi$ and assume in our presentation that the reader has basic familiarity with this style of semantics. Unlike classical (multi-)modal logic, whose Kripke semantics are standard, several different Kripke semantics for intuitionistic modal logic have been proposed [1, 23, 26]. They differ in the number of relations used, how the relations are related to each other and also how the modalities are interpreted. In what follows, we use semantics that are closest in spirit to those of Wolter et al. [26]. This choice makes the technical development easier.

Definition 2.1 (Kripke model)**.** An intuitionistic model, Kripke model or, simply, model, $\mathcal{M}$ is a tuple $(W, \leq, \{N_A\}_{A \in \mathcal{I}}, h)$ where,

- $(W, \leq)$ is a preorder. Elements of W are called worlds and written x, y, z, w. Since $\leq$ is a preorder, it satisfies the following conditions:

$$\forall x.(x \leq x) \tag{refl}$$
$$\forall x, y, z.(((x \leq y) \wedge (y \leq z)) \to (x \leq z)) \tag{trans}$$

  $\leq^{-1}$ is also written $\geq$.

- Each N_A is a binary relation on W that satisfies the condition $(\leq \circ N_A) \subseteq N_A$, i.e.,[1]

$$\forall x, y, z.(((x \leq y) \wedge (y N_A z)) \to (x N_A z)) \tag{mon-N}$$

- h assigns to each atom p the set of worlds $h(p) \subseteq W$ where p holds. We require h to be monotone w.r.t. $\leq$, i.e., if $x \in h(p)$ and $x \leq y$ then $y \in h(p)$.

A model without the assignment, i.e., the tuple $(W, \leq, \{N_A\}_{A \in \mathcal{I}})$ is also called a *frame* and the conditions on relations above, e.g., (mon-N), are called *frame conditions*.

[1] Wolter et al. [26] require the stronger condition $(\leq \circ N_A \circ \leq) = N_A$, but our condition results in the same set of valid formulas.

The frame conditions χ. In addition to the frame conditions (refl), (trans) and (mon-N), we allow a countable number of additional frame conditions as rules of the following form: $\forall\vec{x}.((\wedge_{i=1,\ldots,n} x_i R_i x'_i) \rightarrow (xRx'))$, where $R_1, \ldots, R_n, R$ are from the set $\{N_A \mid A \in \mathcal{I}\} \cup \{\leq\}$ and all variables x_i, x'_i, x, x' are in $\vec{x}$. A set of such additional frame conditions is denoted χ. MMI^χ is the logic whose valid formulas are exactly those that are valid (in the sense defined below) in frames that satisfy *all* conditions in χ.

Definition 2.2 (Satisfaction)**.** Given a model $\mathcal{M} = (W, \leq, \{N_A\}_{A\in\mathcal{I}}, h)$ and a world $w \in W$, we define the satisfaction relation $\mathcal{M} \models w : \alpha$, read "the world w satisfies formula α in model $\mathcal{M}$" by induction on α as follows:

$\mathcal{M} \models w : p$ iff $w \in h(p)$

$\mathcal{M} \models w : \top$ (unconditionally)

$\mathcal{M} \models w : \alpha \wedge \beta$ iff $\mathcal{M} \models w : \alpha$ and $\mathcal{M} \models w : \beta$

$\mathcal{M} \models w : \alpha \vee \beta$ iff $\mathcal{M} \models w : \alpha$ or $\mathcal{M} \models w : \beta$

$\mathcal{M} \models w : \alpha \rightarrow \beta$ iff for every w' such that $w \leq w'$ and $\mathcal{M} \models w' : \alpha$, we have $\mathcal{M} \models w' : \beta$.

$\mathcal{M} \models w : A\ \mathsf{nec}\ \alpha$ iff for every w' such that wN_Aw', we have $\mathcal{M} \models w' : \alpha$.

We say that $\mathcal{M} \not\models w : \alpha$ if it is not the case that $\mathcal{M} \models w : \alpha$. In particular, for every $\mathcal{M}$ and every w, $\mathcal{M} \not\models w : \bot$.

A formula α is true in a model $\mathcal{M}$, written $\mathcal{M} \models \alpha$, if for every world $w \in \mathcal{M}$, $\mathcal{M} \models w : \alpha$. A formula α is *valid* in MMI^χ, written $\models \alpha$, if $\mathcal{M} \models \alpha$ for every model $\mathcal{M}$ satisfying all conditions in χ.

Valid axioms. For the benefit of readers, we list below some valid axioms and admissible rules that are common to all logics MMI^χ. If a rule/axiom is standard in literature, its common name is mentioned to the extreme right. $\alpha \equiv \beta$ means $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.

(All tautologies of intuitionistic propositional logic are valid in MMI^χ)

$$\frac{\models \alpha}{\models A\ \mathsf{nec}\ \alpha} \qquad \text{(nec)}$$

$$\models (A\ \mathsf{nec}\ (\alpha \rightarrow \beta)) \rightarrow ((A\ \mathsf{nec}\ \alpha) \rightarrow (A\ \mathsf{nec}\ \beta)) \qquad \text{(K)}$$

$$\models ((A\ \mathsf{nec}\ \alpha) \wedge (A\ \mathsf{nec}\ \beta)) \equiv (A\ \mathsf{nec}\ (\alpha \wedge \beta))$$

The frame conditions χ can be used to force additional axioms in a standard way, which has been explored in great detail in literature on correspondence theory [5]. For example,

- The condition $\forall x, y.((xN_Ay) \rightarrow (x \leq y))$ corresponds to the axiom $\alpha \rightarrow (A\ \mathsf{nec}\ \alpha)$, commonly called (unit) and of central importance in lax logic [7].
- The condition $\forall x, y, z.(((xN_Ay) \wedge (yN_Az)) \rightarrow (xN_Az))$ corresponds to the axiom $(A\ \mathsf{nec}\ \alpha) \rightarrow (A\ \mathsf{nec}\ A\ \mathsf{nec}\ \alpha)$, commonly called (4).

The following is a very important, fundamental property of Kripke models of all intuitionistic logics, including ours. It is used to prove our sequent calculus (Section 2.2) sound with respect to our Kripke semantics.

Lemma 2.3 (Monotonicity)**.** *If $\mathcal{M} \models w : \alpha$ and $w \leq w' \in \mathcal{M}$, then $\mathcal{M} \models w' : \alpha$.*

Proof. By induction on α. □

2.2 Seq-MMI$^\chi$: A Labeled Sequent Calculus for MMI$^\chi$

As a first step towards building a constructively complete decision procedure for logics MMI$^\chi$, we build a sound, complete, cut-free sequent calculus for MMI$^\chi$. Following the work of Negri [16], our calculus is presented in what is known as "labeled" style of calculi for modal logics, which means that the calculus proves formulas labeled with symbolic worlds. A labeled formula contains a symbol x, y, z, w, u denoting a world and a formula α, written together as $x : \alpha$. A sequent in our calculus has the form $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$, where

- Σ is a finite set of world symbols appearing in the rest of the sequent. World symbols are also called *labels*.
- $\mathbb{M}$ is a finite multi-set of relations between labels in Σ. Relations have the forms $x \leq y$ and xN_Ay.
- Γ is a finite multi-set of labeled formulas.
- Δ is a finite multi-set of labeled formulas.

The intuition is that if $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$ is valid, then every model with a world set containing at least Σ, satisfying all relations in $\mathbb{M}$ and all labeled formulas in Γ must satisfy at least one labeled formula in Δ. This is formalized in the following definition.

Definition 2.4 (Sequent satisfaction and validity)**.** A model $\mathcal{M}$ and a mapping ρ from elements of Σ to worlds of $\mathcal{M}$ satisfy a (possibly non-provable) sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$, written $\mathcal{M}, \rho \models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$, if one of the following holds:

- There is an $xRy \in \mathbb{M}$ with $R \in \{\leq\} \cup \{N_A \mid A \in \mathcal{I}\}$ such that $\rho(x)\ R\ \rho(y) \notin \mathcal{M}$.
- There is an $x : \alpha \in \Gamma$ such that $\mathcal{M} \not\models \rho(x) : \alpha$.
- There is an $x : \alpha \in \Delta$ such that $\mathcal{M} \models \rho(x) : \alpha$.

A model $\mathcal{M}$ satisfies a sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$, written $\mathcal{M} \models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$, if for every mapping ρ, we have $\mathcal{M}, \rho \models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$. Finally, a sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$ is valid, written $\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$, if for every model $\mathcal{M}$ we have $\mathcal{M} \models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$.

Rules of the sequent calculus. The sequent calculus for MMI$^\chi$ is shown in Figure 1. Following standard approach in labeled calculi, the rules for each connective mimic the (Kripke) semantic definition of the connective. For example, in the rule ($\wedge$R), to prove $x : \alpha \wedge \beta$ in the conclusion, we prove $x : \alpha$ and $x : \beta$ in the premises. The rules ($\to$R) and (necR) introduce fresh worlds into Σ, consistent with the semantic definition (Definition 2.2). As illustrated in Section 2.3, it

Axiom Rules

$$\frac{}{\Sigma; \mathbb{M}, x \leq y; \Gamma, x : p \Rightarrow^{\chi} y : p, \Delta}\,\mathrm{init}$$

Logical Rules

$$\frac{}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} x : \top, \Delta}\,\top\mathrm{R} \qquad \frac{}{\Sigma; \mathbb{M}; \Gamma, x : \bot \Rightarrow^{\chi} \Delta}\,\bot\mathrm{L}$$

$$\frac{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} x : \alpha, x : \alpha \wedge \beta, \Delta \qquad \Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} x : \beta, x : \alpha \wedge \beta, \Delta}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} x : \alpha \wedge \beta, \Delta}\,\wedge\mathrm{R}$$

$$\frac{\Sigma; \mathbb{M}; \Gamma, x : \alpha \wedge \beta, x : \alpha, x : \beta \Rightarrow^{\chi} \Delta}{\Sigma; \mathbb{M}; \Gamma, x : \alpha \wedge \beta \Rightarrow^{\chi} \Delta}\,\wedge\mathrm{L} \qquad \frac{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} x : \alpha, x : \beta, x : \alpha \vee \beta, \Delta}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} x : \alpha \vee \beta, \Delta}\,\vee\mathrm{R}$$

$$\frac{\Sigma; \mathbb{M}; \Gamma, x : \alpha \vee \beta, x : \alpha \Rightarrow^{\chi} \Delta \qquad \Sigma; \mathbb{M}; \Gamma, x : \alpha \vee \beta, x : \beta \Rightarrow^{\chi} \Delta}{\Sigma; \mathbb{M}; \Gamma, x : \alpha \vee \beta \Rightarrow^{\chi} \Delta}\,\vee\mathrm{L}$$

$$\frac{\Sigma, y; \mathbb{M}, x \leq y; \Gamma, y : \alpha \Rightarrow^{\chi} y : \beta, x : \alpha \to \beta, \Delta}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} x : \alpha \to \beta, \Delta}\,\to\mathrm{R}$$

$$\frac{\Sigma; \mathbb{M}, x \leq y; \Gamma, x : \alpha \to \beta \Rightarrow^{\chi} y : \alpha, \Delta \qquad \Sigma; \mathbb{M}, x \leq y; \Gamma, x : \alpha \to \beta, y : \beta \Rightarrow^{\chi} \Delta}{\Sigma; \mathbb{M}, x \leq y; \Gamma, x : \alpha \to \beta \Rightarrow^{\chi} \Delta}\,\to\mathrm{L}$$

$$\frac{\Sigma, y; \mathbb{M}, xN_A y; \Gamma \Rightarrow^{\chi} y : \alpha, x : A \;\mathsf{nec}\; \alpha, \Delta}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} x : A \;\mathsf{nec}\; \alpha, \Delta}\,\mathsf{necR} \qquad \frac{\Sigma; \mathbb{M}, xN_A y; \Gamma, x : A \;\mathsf{nec}\; \alpha, y : \alpha \Rightarrow^{\chi} \Delta}{\Sigma; \mathbb{M}, xN_A y; \Gamma, x : A \;\mathsf{nec}\; \alpha \Rightarrow^{\chi} \Delta}\,\mathsf{necL}$$

Frame Rules

$$\frac{\Sigma, x; \mathbb{M}, x \leq x; \Gamma \Rightarrow^{\chi} \Delta}{\Sigma, x; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta}\,\mathrm{refl} \qquad \frac{\Sigma; \mathbb{M}, x \leq y, y \leq z, x \leq z; \Gamma \Rightarrow^{\chi} \Delta}{\Sigma; \mathbb{M}, x \leq y, y \leq z; \Gamma \Rightarrow^{\chi} \Delta}\,\mathrm{trans}$$

$$\frac{\Sigma; \mathbb{M}, x \leq y, yN_A z, xN_A z; \Gamma \Rightarrow^{\chi} \Delta}{\Sigma; \mathbb{M}, x \leq y, yN_A z; \Gamma \Rightarrow^{\chi} \Delta}\,\mathrm{mon\text{-}N}$$

$$\frac{(\forall \vec{x}.((\wedge_i\ (x_i R_i x_i')) \to (xRx'))) \in \chi \qquad x_i R_i x_i' \in \mathbb{M} \qquad \Sigma; \mathbb{M}, xRx'; \Gamma \Rightarrow^{\chi} \Delta}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta}\,\chi$$

Figure 1: Seq-MMI$^{\chi}$: A labeled sequent calculus for MMI$^{\chi}$

is primarily because of these rules that a backwards derivation in the sequent calculus may not terminate trivially; this motivates most of our technical development.

We say that $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$ if $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$ has a proof in the calculus. The sequent calculus is both sound and complete with respect to the semantics.

Theorem 2.5 (Soundness)**.** *If* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$*, then* $\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$*.*

Proof. Fix an $\mathcal{M}$. It is easily proved by induction on the given derivation of $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$ that for every mapping ρ, $\mathcal{M}, \rho \models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$. ☐

The converse of Theorem 2.5, completeness, also holds. It can be proved using a Henkin-style argument, but we do not need this result in the rest of our development so we do not present the proof here. For those logics MMI$^{\chi}$ to which our method applies, completeness is also a consequence of the completeness of our decision procedure, which we prove later. The following property is critical to the design and correctness of our constructively complete decision procedure.

Theorem 2.6 (Weak subformula property)**.** *If a formula* φ *appears in any proof tree (possibly infinite) obtained by applying the rules of Figure 1 backwards starting from a concluding sequent* $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$*, then* φ *is a subformula of some formula in either* Γ *or* Δ*.*

Proof. By induction on the distance (in the proof tree) of the occurrence of φ from the conclusion $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$. ☐

2.3 Non-Termination in the Sequent Calculus

The main challenge in constructing a decision procedure based on the sequent calculus of Figure 1 is that backwards search in it can loop forever, due to unbounded creation of new worlds in the rules ($\to$R) and (necR). We illustrate this through an example. Suppose $\chi = \{\forall x, y, z.(((x N_A y) \wedge (y N_A z)) \to (x N_A z))\}$, i.e., the relation N_A is transitive. Then the calculus admits the following infinite derivation. The omitted steps marked ($\dagger$) are the same as the steps between the two ($\to$L) rules with the difference that x_1 is used in place of x_0 in all rules. To keep the derivation concise, we drop those formulas from $\mathbb{M}$, Γ and Δ that are no longer needed after each backwards step.

$$\begin{array}{cl}
\vdots & \\
x_0, x_1, x_2; x_0 N_A x_2; x_0 : A\ \mathsf{nec}\ ((A\ \mathsf{nec}\ \alpha) \to \alpha) \to \alpha \Rightarrow^{\chi} x_2 : \alpha & \\ \hline
x_0, x_1, x_2; x_0 N_A x_1, x_1 N_A x_2; x_0 : A\ \mathsf{nec}\ ((A\ \mathsf{nec}\ \alpha) \to \alpha) \to \alpha \Rightarrow^{\chi} x_2 : \alpha & \chi \\
\vdots & \\
(\dagger) & \\
\vdots \qquad \cdots & \\ \hline
x_0, x_1; x_0 N_A x_1, x_1 \le x_1; x_0 : A\ \mathsf{nec}\ ((A\ \mathsf{nec}\ \alpha) \to \alpha) \to \alpha \Rightarrow^{\chi} x_1 : \alpha & \to\mathrm{L} \\ \hline
x_0, x_1; x_0 N_A x_1; x_0 : A\ \mathsf{nec}\ ((A\ \mathsf{nec}\ \alpha) \to \alpha), x_1 : (A\ \mathsf{nec}\ \alpha) \to \alpha \Rightarrow^{\chi} x_1 : \alpha & \mathrm{refl} \\ \hline
x_0, x_1; x_0 N_A x_1; x_0 : A\ \mathsf{nec}\ ((A\ \mathsf{nec}\ \alpha) \to \alpha) \Rightarrow^{\chi} x_1 : \alpha & \mathsf{nec}\mathrm{L} \\ \hline
x_0; \cdot; x_0 : A\ \mathsf{nec}\ ((A\ \mathsf{nec}\ \alpha) \to \alpha), x_0 : (A\ \mathsf{nec}\ \alpha) \to \alpha \Rightarrow^{\chi} x_0 : A\ \mathsf{nec}\ \alpha & \mathsf{nec}\mathrm{R}* \\
\cdots & \\ \hline
x_0; x_0 \le x_0; x_0 : A\ \mathsf{nec}\ ((A\ \mathsf{nec}\ \alpha) \to \alpha), x_0 : (A\ \mathsf{nec}\ \alpha) \to \alpha \Rightarrow^{\chi} x_0 : \alpha & \to\mathrm{L} \\ \hline
x_0; x_0 N_A x_0, x_0 \le x_0; x_0 : A\ \mathsf{nec}\ ((A\ \mathsf{nec}\ \alpha) \to \alpha) \Rightarrow^{\chi} x_0 : \alpha & \mathsf{nec}\mathrm{L}
\end{array}$$

Note that the sequent at the top of the derivation is identical to the premise of the rule marked $*$ with a fresh world x_2 replacing x_1, so this derivation loops forever. The problem is caused by an interaction of the rules ($\to$L), (necR) and our frame condition χ that keeps creating new worlds. Although we do not illustrate them here, similar loops can also be created using the rule ($\to$R) in place of (necR).

The question then is: How do we detect such loops during backwards proof search to obtain a decision procedure? In the rest of this paper we present a general technique that not only detects such loops in a wide variety of logics, but also produces Kripke countermodels witnessing the non-validity of the end-sequent when such loops are detected.

3 Constructively Complete Decision Procedure for MMI^χ

In this section we present our general constructively complete (countermodel producing) decision procedure for several intuitionistic multi-modal logics of the form MMI^χ. Since any application of a rule other than ($\to$R) and (necR) is unnecessary in a backward proof search when the labeled formulas introduced in the premise(s) already exist in Γ and Δ, the use of all rules other than ($\to$R) and (necR) can be bounded easily. So, the main technical challenge is to be able to detect loops, such as the one illustrated in Section 2.3. Although the end-result of our technique, i.e., the decision procedure itself is quite straightforward, building up to it requires some non-standard machinery, which we motivate here by presenting an informal outline of our method.

To control the use of rules ($\to$R) and (necR) in backwards proof search, we generalize a technique from existing work on tableau calculi for classical uni-modal logics [10]. The technique prevents loops by checking for containment of formulas that label a world in those labeling another. We start by observing that in any sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$ obtained during backwards proof search starting from a single goal formula, all worlds in $\mathbb{M}$ lie on a rooted, directed tree, whose edges are relations in $\mathbb{M}$ that were introduced by the rules ($\to$R) and (necR) in earlier steps of the search. We call the reflexive-transitive closure of this tree $\ll$. Next, we define a function $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$ that lists, *approximately*, all formulas labeled by x in Γ and Δ (the exact definition of Sfor depends on χ, and is one of our key technical contributions). This function satisfies a very important, critical property, whose proof requires induction on $\ll$: If there is a world y ($y \neq x$) such that $y \ll x$ and $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x) \subseteq \mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, y)$, then it is *useless* to apply any of the rules ($\to$R) and (necR) on any principal formula labeled by x in the sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$ in backwards proof search. It only remains to show that this condition forces termination. This follows from the fact that $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$ increases monotonically for each x in a backwards proof search and the fact that the number of possible values of Sfor is finite, which, in turn, is a consequence of the weak subformula property (Theorem 2.6).

We further show that if no rule applies backwards to a sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$ (after imposing our termination checks), then we can obtain a *countermodel* to the sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$ by adding an edge $x \leq y$ whenever $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x) \subseteq \mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, y)$. This forms the basis of our countermodel extraction. (As explained in Section 6, this method of extracting countermodels is motivated by Negri's proof of completeness of labeled sequent calculi for uni-modal logic with respect to their Kripke semantics [17]. In that proof, Negri shows how to extract countermodels from failed branches of a non-terminating labeled sequent calculus. However, our construction of the countermodel is different.)

The definition of the function Sfor depends on the conditions χ that define the logic. In our formal development, we define a suitable Sfor for every logic for which there exists what we call a *suitable closure relation* (SCR). Technically, a SCR is a family of relations on frames, which satisfies some stipulated properties. Our entire method applies to any logic MMI^χ for whose χ a SCR exists. This begs the question of how general the existence of a SCR is. As we show in Section 3.7, several (multi-) modal logics with reflexivity, transitivity, and modality interaction conditions, including the multi-modal logics K, S4, T, and I [9, 13] have SCRs. We further show in Section 3.6 that our method can be extended to any logic with a SCR plus the seriality condition on its accessibility relations. This is, in an informal sense, the most we could hope for from a general decision procedure for multi-modal logics because there are negative results for decidability of multi-modal logics with other classes of frame conditions like symmetry (see Section 4). On the negative side, our method does not directly generalize to handle possibility modalities in intuitionistic logic (Section 4). However, the method does work for possibility modalities in classical logic, where possibility can be defined as the DeMorgan dual of necessity (Section 5).

Our technical presentation consists of the following steps:

- We define the term "SCR for χ". Its existence is the only condition that must hold for our method to apply to the logic MMI^χ.
- We define the function Sfor using SCRs. We also define a predicate on sequents, whose elements (sequents) are called *saturated histories*. Roughly, a saturated history is a sequent which satisfies all the termination conditions listed above, i.e., applying any rule other than ($\to$R) and (necR) backwards on the sequent does not add any new labeled formulas, and the application of these two rules is blocked by the containment condition on Sfor. We prove, by construction, the key property of our entire method: If a sequent is a saturated history, then it has a finite countermodel.
- We define an intermediate sequent calculus Seq-$\mathrm{MMI}^\chi_{\mathrm{CM}}$ with judgments $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi_{\mathrm{CM}} \Delta \searrow S$. Here, S is a (possibly empty) finite set of finite countermodels. The correctness property of this calculus is: If $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi_{\mathrm{CM}} \Delta \searrow \{\}$ has a proof, then $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$ and if $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi_{\mathrm{CM}} \Delta \searrow S$ has a proof for $S \neq \{\}$, then every $\mathcal{M} \in S$ satisfies: $\mathcal{M} \not\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$. Backwards search in this calculus does not necessarily terminate because the calculus allows every rule of $\Rightarrow^\chi$, and, in addition, it has a new rule that produces a countermodel when $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history. Consequently, in itself, the calculus is not a decision procedure. However, we find the calculus a useful intermediate step to prove many properties.
- We observe that a specific strategy for proof search in Seq-$\mathrm{MMI}^\chi_{\mathrm{CM}}$ terminates. This strategy is presented as a sequent calculus Seq-$\mathrm{MMI}^\chi_{\mathrm{T}}$, which is a countermodel producing decision procedure.

In the next four subsections, we present the technical details of each of these steps. In particular, the last of these subsections, Section 3.4, describes our decision procedure. In Section 3.5 we state and prove a comprehensiveness property of countermodels generated by our procedure.

3.1 Suitable Closure Relations (SCRs)

We start our technical presentation by defining suitable closure relations (SCRs) for frame conditions χ. Our constructive decision procedure applies to any logic MMI^χ whose χ has a SCR. Call a frame $\mathbb{M}$ *closed* if it is closed under the conditions (refl), (trans), (mon-N) and χ. Given a frame $\mathbb{M}$, let $\overline{\mathbb{M}}$ denote its closure obtained by closing the frame under the conditions (refl), (trans), (mon-N) and χ, obtained as the least fixed point of the application of these rules. Informally, a SCR is a family of relations $(R(A))_{A\in\mathcal{I}}$ that, given any closed frame $\mathbb{M}$ and any extension $\mathbb{M}'$ of it with additional edges of the form $\leq$, characterizes all relations N_A in $\overline{\mathbb{M}'}$ in terms of the relations in $\mathbb{M}$ and the difference $\mathbb{M}' - \mathbb{M}$.

Definition 3.1 (Suitable closure relation (SCR))**.** A family of binary relations $(R(A))_{A\in\mathcal{I}}$ is called a suitable closure relation or SCR for χ if the following hold:

0. Each $R(A)$ is definable in first-order logic in terms of the relations $\leq \cup \{N_A \mid A \in \mathcal{I}\}$.

1. For a finite frame $\mathbb{M}$ and $x, y \in \mathbb{M}$, it can be decided whether $x(R(A))y$ or not.

2. $((R(A))^* \circ N_A) \subseteq N_A$ can be derived from the frame conditions (refl), (trans), (mon-N) and χ.

3. For any *closed* frame $\mathbb{M}$ and any $C \subseteq \{x \leq y \mid x, y \in \mathbb{M}\}$, if $xN_Ay \in \overline{\mathbb{M} \cup C}$, then $x((R(A) \cup C)^* \circ N_A \circ \leq)y$, where all relations $R(A)$ and N_A on the right are in $\mathbb{M}$.

4. For any closed frame $\mathbb{M}$ and any $C \subseteq \{x \leq y \mid x, y \in \mathbb{M}\}$, if $x \leq y \in \overline{\mathbb{M} \cup C}$, then $x(\leq \cup C)^*y$, where all $\leq$ relations on the right are in $\mathbb{M}$.

Observe that condition (4) depends on χ, not on the choice of $R(A)$, but we include it here for uniformity.

SCRs for many different χ are listed in Section 3.7, but we describe one in the following example for illustration.

Example 3.2 (SCR for transitivity)**.** Let $\texttt{trans}(A)$ be the frame condition $\forall x, y, z.(((xN_Ay) \wedge (yN_Az)) \rightarrow (xN_Az))$ and let $\chi = \{\texttt{trans}(A) \mid A \in \mathcal{I}\}$. Then, the relation $R(A) = N_A \cup \leq$ is a SCR for χ. To prove this, we verify each of the conditions (0)–(4) in the definition of SCR. Conditions (0) and (1) are trivially true. (2) is equivalent to $((N_A \cup \leq)^* \circ N_A) \subseteq N_A$, which follows from the frame conditions (mon-N) and χ. To prove (3), suppose that $xN_Ay \in \overline{\mathbb{M} \cup C}$. Then, because the only way to derive a relation N_A is to use either (mon-N) or $\texttt{trans}(A)$, it follows that in $\mathbb{M} \cup C$, we have $x((N_A \cup \leq)^* \circ N_A)y$. So, we also have $x((N_A \cup \leq \cup C)^* \circ N_A)y$, where all $\leq$ and N_A relations are in $\mathbb{M}$, i.e., $x((R(A) \cup C)^* \circ N_A)y$. Finally, since $\leq$ is reflexive, we have: $x((R(A) \cup C)^* \circ N_A \circ \leq)y$, as required. The proof of (4) is similar to that of (3).

3.2 Saturated Histories

Our method applies only to those logics MMI^χ whose χ has a SCR, so in the sequel we fix a set of frame conditions χ and assume there is a SCR $(R(A))_{A\in\mathcal{I}}$ for this χ. Although we present the technical material generically with respect to an abstract χ and SCR, we strongly advise the reader to choose a specific χ and its SCR (for instance, those from Example 3.2), and instantiate our definitions and theorems on them.

A *history* is a tuple $\Sigma; \mathbb{M}; \Gamma; \Delta$ (equivalently, a sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$) such that all labels in $\mathbb{M}$, Γ and Δ occur in Σ. Let $T(\varphi)$ and $F(\varphi)$ be two uninterpreted unary relations. Informally, we read $T(\varphi)$ as "φ should be true" and $F(\varphi)$ as "φ should be false". Given a history $\Sigma; \mathbb{M}; \Gamma; \Delta$ and $x \in \Sigma$, the *signed formulas* of x, written $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$ are defined as follows:

$$
\begin{gathered}
\mathrm{Sfor}(\Sigma;\mathbb{M};\Gamma;\Delta,x) = \\
\{T(\varphi) \mid x : \varphi \in \Gamma\}\cup \\
\{F(\varphi) \mid x : \varphi \in \Delta\}\cup \\
\{T(A\ \mathsf{nec}\ \varphi) \mid \exists y.\ y(R(A))^* x \in \mathbb{M} \text{ and } y : A\ \mathsf{nec}\ \varphi \in \Gamma\}\cup \\
\{T(\varphi \to \psi) \mid \exists y.\ y \leq x \in \mathbb{M} \text{ and } y : \varphi \to \psi \in \Gamma\}\cup \\
\{T(p) \mid \exists y.\ y \leq x \in \mathbb{M} \text{ and } y : p \in \Gamma\}
\end{gathered}
$$

When $\Sigma, \mathbb{M}, \Gamma, \Delta$ are clear from context, we abbreviate $\mathrm{Sfor}(\Sigma;\mathbb{M};\Gamma;\Delta,x)$ to $\mathrm{Sfor}(x)$. We say that $x \preccurlyeq y$ iff $\mathrm{Sfor}(x) \subseteq \mathrm{Sfor}(y)$.

We call a frame $\mathbb{M}$ *tree-like* if it can be derived from a finite tree of the relations $\leq$ and N_A and (possibly partial) closure by frame rules. This tree is called the underlying tree of $\mathbb{M}$ and we say that $x \ll y$ (in $\mathbb{M}$) iff there is a directed path from x to y in the tree underlying $\mathbb{M}$.

The key definition in our method is that of a *saturated history*. Intuitively, this definition characterizes those histories $\Sigma;\mathbb{M};\Gamma;\Delta$ for which we can directly define a countermodel for the sequent $\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi} \Delta$. (The definition of this countermodel is given immediately after the definition of a saturated history.)[2]

Definition 3.3 (Saturated history)**.** A history $\Sigma;\mathbb{M};\Gamma;\Delta$ is called saturated if the following hold:

1. $\mathbb{M}$ is tree-like and saturated with respect to the conditions (refl), (trans), (mon-N) and χ. (In particular, because $\mathbb{M}$ is tree-like, it has a relation $\ll$ defined on it.)
2. If $x : p \in \Gamma$, then there is no y such that $x \leq y \in \mathbb{M}$ and $y : p \in \Delta$.
3. There is no x such that $x : \top \in \Delta$.
4. There is no x such that $x : \bot \in \Gamma$.
5. If $x : \alpha \wedge \beta \in \Gamma$, then $x : \alpha \in \Gamma$ and $x : \beta \in \Gamma$.
6. If $x : \alpha \wedge \beta \in \Delta$, then either $x : \alpha \in \Delta$ or $x : \beta \in \Delta$.
7. If $x : \alpha \vee \beta \in \Gamma$, then either $x : \alpha \in \Gamma$ or $x : \beta \in \Gamma$.
8. If $x : \alpha \vee \beta \in \Delta$, then $x : \alpha \in \Delta$ and $x : \beta \in \Delta$.
9. If $x : \alpha \to \beta \in \Gamma$ and $x \leq y \in \mathbb{M}$, then either $y : \alpha \in \Delta$ or $y : \beta \in \Gamma$.
10. If $x : \alpha \to \beta \in \Delta$, then either:
    (a) There is a y such that $x \leq y \in \mathbb{M}$, $y : \alpha \in \Gamma$ and $y : \beta \in \Delta$ or
    (b) There is a y such that $y \neq x$, $y \ll x$ and $x \preccurlyeq y$.
11. If $x : A\ \mathsf{nec}\ \alpha \in \Gamma$ and $xN_Ay \in \mathbb{M}$, then $y : \alpha \in \Gamma$.
12. If $x : A\ \mathsf{nec}\ \alpha \in \Delta$, then either:
    (a) There is a y such that $xN_Ay \in \mathbb{M}$ and $y : \alpha \in \Delta$ or
    (b) There is a y such that $y \neq x$, $y \ll x$ and $x \preccurlyeq y$.

[2]Saturated histories are labeled generalizations of Hintikka sets.

Definition 3.4 (Countermodel of a saturated history)**.** For a saturated history $\Sigma; \mathbb{M}; \Gamma; \Delta$, the *countermodel* of the history, $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ is defined as follows:

- The worlds of $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ are exactly those in Σ.

- The relations of $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ are $\overline{\mathbb{M} \cup C}$, where $C = \{x \leq y \mid x \preccurlyeq y\}$.

- $h(p) = \{x \mid \exists y. \ (y \leq x \in \mathbb{M}) \wedge (y : p \in \Gamma)\}$.

It is not obvious that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ is a model. It does satisfy all frame conditions. However, we must also show monotonicity: If $x \leq y \in \mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ and $x \in h(p)$, then $y \in h(p)$. The following lemma states that this is the case.

Lemma 3.5. *If $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history, then $CM(\Sigma; \mathbb{M}; \Gamma; \Delta)$ has a monotonic valuation h, i.e., $x \in h(p)$ and $x \leq y \in CM(\Sigma; \mathbb{M}; \Gamma; \Delta)$ imply $y \in h(p)$.*

Proof. Let C be the set $\{x \leq y \mid x \preccurlyeq y\}$. Suppose that $x \leq y \in \mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$, i.e., $x \leq y \in \overline{\mathbb{M} \cup C}$ and $x \in h(p)$. From the latter, there is a z such that $z \leq x \in \mathbb{M}$ and $z : p \in \Gamma$. From the definition of SCR, clause (4) it follows that $x(\leq \cup C)^* y$, where all the relations $\leq$ are in $\mathbb{M}$. Hence, we have a chain $x = x_0(\leq \cup C)x_1 \ldots (\leq \cup C)x_n = y$ where all relations $\leq$ are in $\mathbb{M}$. We induct on i to show that $T(p) \in \mathrm{Sfor}(x_i)$. The result then follows immediately from $T(p) \in \mathrm{Sfor}(x_n) = \mathrm{Sfor}(y)$. See Appendix A, Lemma A.1 for details. □

The next Lemma states the central property of our method. In particular, the Lemma immediately implies that if $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history, then $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ is a countermodel to the sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$.

Lemma 3.6. *The following hold for any saturated history $\Sigma; \mathbb{M}; \Gamma; \Delta$:*

A. If $T(\varphi) \in Sfor(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$, then $CM(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : \varphi$

B. If $F(\varphi) \in Sfor(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$, then $CM(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \varphi$

Proof. We prove both properties simultaneously by *lexicographic* induction, first on φ, and then on the partial (tree-like) order $\ll$ of $\mathbb{M}$. (Note that we cannot induct on either $\mathbb{M}$ or the relation in $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$, because both of these may potentially be cyclic.) Since the context $\Sigma; \mathbb{M}; \Gamma; \Delta$ is fixed here, we abbreviate $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$ to $\mathrm{Sfor}(x)$. We show only one interesting case here; for the remaining cases, see Appendix A, Lemma A.2.

Case. Proof of (A), $\varphi = A \ \mathsf{nec} \ \alpha$. We are given that $T(A \ \mathsf{nec} \ \alpha) \in \mathrm{Sfor}(x)$. We need to show that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : A \ \mathsf{nec} \ \alpha$, i.e., for any y such that $xN_A y$ in the model, we have $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models y : \alpha$. Pick any y such that $xN_A y$ in the model. Because of the definition of SCR, clause (3), we have $x((R(A) \cup C)^* \circ N_A \circ \leq)y$, where the relations $R(A)$ and N_A are in $\mathbb{M}$. So there are $x_0, \ldots, x_n, y'$ such that $x = x_0(R(A) \cup C)x_1 \ldots (R(A) \cup C)x_n N_A y' \leq y$. We now prove, by induction on i, that $T(A \ \mathsf{nec} \ \alpha) \in \mathrm{Sfor}(x_i)$ for each i.

- For $i = 0$, $x_0 = x$ and we are given that $T(A \ \mathsf{nec} \ \alpha) \in \mathrm{Sfor}(x)$.

- For the inductive case, assume that $T(A \ \mathsf{nec} \ \alpha) \in \mathrm{Sfor}(x_i)$ for some i. We show that $T(A \ \mathsf{nec} \ \alpha) \in \mathrm{Sfor}(x_{i+1})$ by case analyzing the relation $x_i(R(A) \cup C)x_{i+1}$.

– $x_i(R(A))x_{i+1} \in \mathbb{M}$: By the i.h., $T(A \text{ nec } \alpha) \in \mathrm{Sfor}(x_i)$ so there is some z such that $z(R(A))^* x_i \in \mathbb{M}$ and $z : A \text{ nec } \alpha \in \Gamma$. Clearly, we have $z(R(A))^* x_{i+1} \in \mathbb{M}$, so $T(A \text{ nec } \alpha) \in \mathrm{Sfor}(x_{i+1})$.
– $(x_i, x_{i+1}) \in C$: Because of the definition of C, $\mathrm{Sfor}(x_{i+1}) \supseteq \mathrm{Sfor}(x_i)$. Thus, $T(A \text{ nec } \alpha) \in \mathrm{Sfor}(x_i)$ immediately implies $T(A \text{ nec } \alpha) \in \mathrm{Sfor}(x_{i+1})$.

Since we just proved that $T(A \text{ nec } \alpha) \in \mathrm{Sfor}(x_i)$, it follows, in particular, that $T(A \text{ nec } \alpha) \in \mathrm{Sfor}(x_n)$. Consequently, there is some z' such that $z'(R(A))^* x_n \in \mathbb{M}$ and $z' : A \text{ nec } \alpha \in \Gamma$. Then, we also have (within $\mathbb{M}$) that: $z'(R(A))^* x'_n N_A y'$. So, by clause (2) of the definition of SCR, $z' N_A y' \in \mathbb{M}$. Hence, by clause (11) of the definition of saturated history, we must have $y' : \alpha \in \Gamma$. Therefore, $T(\alpha) \in \mathrm{Sfor}(y')$ and by the i.h., $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models y' : \alpha$. Since $y' \leq y \in \mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$, by Lemma 2.3, $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models y : \alpha$. ☐

Corollary 3.7 (Existence of countermodel)**.** *If* $\Sigma; \mathbb{M}; \Gamma; \Delta$ *is a saturated history, then* $CM(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$.

Proof. Lemma 3.6 immediately implies that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta), \rho \not\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$, where $\rho : \Sigma \to \Sigma$ is the identity substitution. ☐

3.3 Seq-MMI$^\chi_{\mathrm{CM}}$: Countermodels for MMI$^\chi$

Having defined a saturated history, i.e., a sequent for which a countermodel exists (Corollary 3.7), we now define a sequent calculus Seq-MMI$^\chi_{\mathrm{CM}}$, written $\Rightarrow^\chi_{\mathrm{CM}}$, which uses this fact to emit countermodels from unprovable sequents. Although this calculus is not a decision procedure, we find it a useful step in proving several results, in particular, the results of Section 3.5.

Sequents of Seq-MMI$^\chi_{\mathrm{CM}}$ have the form $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi_{\mathrm{CM}} \Delta \searrow S$, where S is a finite set of finite models. We write $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi_{\mathrm{CM}} \Delta \searrow S)$ if $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi_{\mathrm{CM}} \Delta \searrow S$ has a proof. The meaning of $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi_{\mathrm{CM}} \Delta \searrow S$ depends on S. If $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi_{\mathrm{CM}} \Delta \searrow \{\})$, then $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$ and if $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi_{\mathrm{CM}} \Delta \searrow S)$ with $S \neq \{\}$, then every model $\mathcal{M} \in S$ is a countermodel to $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$ in the sense of (the converse of) Definition 2.4.

The rules of the sequent calculus Seq-MMI$^\chi_{\mathrm{CM}}$ are shown in Figure 2. First, every rule in the ordinary sequent calculus (Figure 1) is modified to have in the conclusion the union of the (counter)models in the premises. This is sound because the rules of the sequent calculus are invertible (i.e., the conclusion of each rule holds iff the premises hold). Second, there is a new rule (CM) that produces the countermodel $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ when $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history.

We emphasize again that this calculus is not necessarily a decision procedure because it includes all rules of $\Rightarrow^\chi$ and hence admits all of the latter's infinite backwards derivations as well.

Theorem 3.8 (Soundness 1)**.** *If* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi_{CM} \Delta \searrow \{\})$*, then* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$.

Proof. By induction on the given derivation of $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi_{\mathrm{CM}} \Delta \searrow \{\}$. Note that the case of rule (CM) does not apply because the set of countermodels in it is non-empty. The proof is straightforward because the rules of $\Rightarrow^\chi_{\mathrm{CM}}$ mimic those of $\Rightarrow^\chi$. ☐

Theorem 3.9 (Soundness 2)**.** *If* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi_{CM} \Delta \searrow S)$*, then for every model* $\mathcal{M} \in S$*,* $\mathcal{M} \not\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$.

Axiom Rules

$$\frac{\Sigma;\mathbb{M};\Gamma;\Delta \text{ is a saturated history}}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow \{\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta)\}}\mathrm{CM} \qquad \frac{}{\Sigma;\mathbb{M},x\leq y;\Gamma,x:p \Rightarrow^{\chi}_{\mathrm{CM}} y:p,\Delta \searrow \{\}}\mathrm{init}$$

Logical Rules

$$\frac{}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} x:\top,\Delta \searrow \{\}}\top\mathrm{R} \qquad \frac{}{\Sigma;\mathbb{M};\Gamma,x:\bot \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow \{\}}\bot\mathrm{L}$$

$$\frac{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} x:\alpha,x:\alpha\wedge\beta,\Delta \searrow S_1 \qquad \Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} x:\beta,x:\alpha\wedge\beta,\Delta \searrow S_2}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} x:\alpha\wedge\beta,\Delta \searrow S_1,S_2}\wedge\mathrm{R}$$

$$\frac{\Sigma;\mathbb{M};\Gamma,x:\alpha\wedge\beta,x:\alpha,x:\beta \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}{\Sigma;\mathbb{M};\Gamma,x:\alpha\wedge\beta \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}\wedge\mathrm{L} \qquad \frac{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} x:\alpha,x:\beta,x:\alpha\vee\beta,\Delta \searrow S}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} x:\alpha\vee\beta,\Delta \searrow S}\vee\mathrm{R}$$

$$\frac{\Sigma;\mathbb{M};\Gamma,x:\alpha\vee\beta,x:\alpha \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S_1 \qquad \Sigma;\mathbb{M};\Gamma,x:\alpha\vee\beta,x:\beta \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S_2}{\Sigma;\mathbb{M};\Gamma,x:\alpha\vee\beta \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S_1,S_2}\vee\mathrm{L}$$

$$\frac{\Sigma,y;\mathbb{M},x\leq y;\Gamma,y:\alpha \Rightarrow^{\chi}_{\mathrm{CM}} y:\beta,x:\alpha\to\beta,\Delta \searrow S}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} x:\alpha\to\beta,\Delta \searrow S}\to\mathrm{R}$$

$$\frac{\Sigma;\mathbb{M},x\leq y;\Gamma,x:\alpha\to\beta \Rightarrow^{\chi}_{\mathrm{CM}} y:\alpha,\Delta \searrow S_1 \qquad \Sigma;\mathbb{M},x\leq y;\Gamma,x:\alpha\to\beta,y:\beta \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S_2}{\Sigma;\mathbb{M},x\leq y;\Gamma,x:\alpha\to\beta \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S_1,S_2}\to\mathrm{L}$$

$$\frac{\Sigma,y;\mathbb{M},xN_Ay;\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} y:\alpha,x:A\ \mathsf{nec}\ \alpha,\Delta \searrow S}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} x:A\ \mathsf{nec}\ \alpha,\Delta \searrow S}\mathsf{necR} \qquad \frac{\Sigma;\mathbb{M},xN_Ay;\Gamma,x:A\ \mathsf{nec}\ \alpha,y:\alpha \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}{\Sigma;\mathbb{M},xN_Ay;\Gamma,x:A\ \mathsf{nec}\ \alpha \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}\mathsf{necL}$$

Frame Rules

$$\frac{\Sigma,x;\mathbb{M},x\leq x;\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}{\Sigma,x;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}\mathrm{refl} \qquad \frac{\Sigma;\mathbb{M},x\leq y,y\leq z,x\leq z;\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}{\Sigma;\mathbb{M},x\leq y,y\leq z;\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}\mathrm{trans}$$

$$\frac{\Sigma;\mathbb{M},x\leq y,yN_Az,xN_Az;\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}{\Sigma;\mathbb{M},x\leq y,yN_Az;\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}\mathrm{mon\text{-}N}$$

$$\frac{(\forall\vec{x}.((\wedge_i\ (x_iR_ix'_i))\to(xRx')))\in\chi \qquad x_iR_ix'_i\in\mathbb{M} \qquad \Sigma;\mathbb{M},xRx';\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}\chi$$

Figure 2: Seq-MMI$^{\chi}_{\mathrm{CM}}$: Countermodel producing sequent calculus for MMI$^{\chi}$

Proof. By induction on the given derivation of $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S$ and case analysis of its last rule. The rules (init), ($\bot$L), and ($\top$R) are vacuous because they have empty S. For all other rules, except (CM), we simply observe that contexts in all major premises are a superset of corresponding contexts in the conclusion and hence we can trivially conclude by induction on one of the premises. The case of rule (CM) is shown below:

Case. $\dfrac{\Sigma; \mathbb{M}; \Gamma; \Delta \text{ is a saturated history}}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow \{\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)\}}\mathrm{CM}$

Here $\mathcal{M} = \mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$. So, the result follows by Corollary 3.7. □

3.4 Seq-MMI$^{\chi}_{\mathrm{T}}$: Termination and Countermodel Extraction for MMI$^{\chi}$

Next, we describe a particular backwards proof search strategy in Seq-MMI$^{\chi}_{\mathrm{CM}}$ that always terminates without losing completeness, thus obtaining a countermodel producing decision procedure for MMI$^{\chi}$. This strategy is described as a calculus Seq-MMI$^{\chi}_{\mathrm{T}}$, with sequents of the form $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S$. Operationally, the rules of the calculus can be interpreted backwards as a decision procedure with inputs Σ, $\mathbb{M}$, Γ, and Δ and output S. For a given Σ, $\mathbb{M}$, Γ, and Δ, $(\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$ is provable iff $S = \{\}$, else every model in S is a countermodel to the sequent.

The rules of the calculus Seq-MMI$^{\chi}_{\mathrm{T}}$ are shown in Figure 3. Each rule in the calculus corresponds to a rule of the same name in Seq-MMI$^{\chi}_{\mathrm{CM}}$ (Figure 2). The only significant difference between the two calculi is that the premise of the rule (CM) in Seq-MMI$^{\chi}_{\mathrm{CM}}$ requires that $\Sigma; \mathbb{M}; \Gamma; \Delta$ be a saturated history, but the rule (CM) applies in Seq-MMI$^{\chi}_{\mathrm{T}}$ only when no other rule applies. To ensure that "no other rule applies" implies that $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history, we spread the negations of the conditions (1) and (5)–(12) from the definition of saturated history to rules other than (CM) as pre-conditions, called *applicability conditions*. Conditions (2), (3) and (4) obviously hold when the rules (init), ($\top$R) and ($\bot$L) do not apply, respectively. Hence, when no rule other than (CM) applies, all 12 conditions of the definition of saturated history must hold, so $\Sigma; \mathbb{M}; \Gamma; \Delta$ must be a saturated history. The conditions are spread to the obvious rules; for example, the negation of condition (5) is applied to the rule ($\wedge$R). In Figure 3, applicability conditions are highlighted using $\boxed{\text{boxes}}$. It only remains to show that the calculus with these applicability conditions does not admit infinite backwards derivations. This follows from a counting argument, as in the proof of Theorem 3.12.

Lemma 3.10 (Correctness of CM)**.** *Let Σ, $\mathbb{M}$, Γ and Δ be such that $\mathbb{M}$ is tree-like and no rule except (CM) applies backwards to $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{T} \Delta \searrow \ldots$. Then, $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history.*

Proof. We verify all conditions in the definition of a saturated history. Each condition corresponds to the negation of premises of one of the rules of Figure 3. □

Lemma 3.11 (Tree-like $\mathbb{M}$)**.** *Let $\mathbb{M}$ be tree-like. Then, the $\mathbb{M}'$ in any sequent $\Sigma'; \mathbb{M}'; \Gamma' \Rightarrow^{\chi}_{T} \Delta' \searrow \ldots$ appearing in a backwards search starting from $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{T} \Delta \searrow \ldots$ is tree-like.*

Proof. By backwards analysis of each rule observing that the $\mathbb{M}$ in the premises of each rule is tree-like if that in the conclusion is. □

Note that the underlying tree of $\mathbb{M}$ in any sequent of a backward proof search starting from a single formula consists of exactly those edges that are introduced in one of the rules ($\to$R) and (necR).

Axiom Rules

$$\frac{\boxed{\text{No other rule applies}}}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow \{\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta)\}}\,\mathrm{CM} \qquad \frac{}{\Sigma;\mathbb{M},x\leq y;\Gamma,x:p \Rightarrow^{\chi}_{\mathrm{T}} y:p,\Delta \searrow \{\}}\,\mathrm{init}$$

Logical Rules

$$\frac{}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} x:\top,\Delta \searrow \{\}}\,\top\mathrm{R} \qquad \frac{}{\Sigma;\mathbb{M};\Gamma,x:\bot \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow \{\}}\,\bot\mathrm{L}$$

$$\frac{\boxed{x:\alpha\notin\Delta \text{ and } x:\beta\notin\Delta} \quad \Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} x:\alpha,x:\alpha\wedge\beta,\Delta \searrow S_1 \quad \Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} x:\beta,x:\alpha\wedge\beta,\Delta \searrow S_2}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} x:\alpha\wedge\beta,\Delta \searrow S_1,S_2}\,\wedge\mathrm{R}$$

$$\frac{\boxed{x:\alpha\notin\Gamma \text{ or } x:\beta\notin\Gamma} \quad \Sigma;\mathbb{M};\Gamma,x:\alpha\wedge\beta,x:\alpha,x:\beta \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}{\Sigma;\mathbb{M};\Gamma,x:\alpha\wedge\beta \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}\,\wedge\mathrm{L}$$

$$\frac{\boxed{x:\alpha\notin\Delta \text{ or } x:\beta\notin\Delta} \quad \Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} x:\alpha,x:\beta,x:\alpha\vee\beta,\Delta \searrow S}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} x:\alpha\vee\beta,\Delta \searrow S}\,\vee\mathrm{R}$$

$$\frac{\boxed{x:\alpha\notin\Gamma \text{ and } x:\beta\notin\Gamma} \quad \Sigma;\mathbb{M};\Gamma,x:\alpha\vee\beta,x:\alpha \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S_1 \quad \Sigma;\mathbb{M};\Gamma,x:\alpha\vee\beta,x:\beta \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S_2}{\Sigma;\mathbb{M};\Gamma,x:\alpha\vee\beta \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S_1,S_2}\,\vee\mathrm{L}$$

$$\frac{\begin{array}{c}\boxed{\forall y\in\Sigma.(x\leq y\in\mathbb{M}) \Rightarrow (y:\alpha\notin\Gamma \text{ or } y:\beta\notin\Delta)}\\ \boxed{\forall y\in\Sigma.(y\ll x) \Rightarrow (x=y \text{ or } x\not\preccurlyeq y)} \quad \Sigma,y;\mathbb{M},x\leq y;\Gamma,y:\alpha \Rightarrow^{\chi}_{\mathrm{T}} y:\beta,x:\alpha\to\beta,\Delta \searrow S\end{array}}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} x:\alpha\to\beta,\Delta \searrow S}\,\to\mathrm{R}$$

$$\frac{\boxed{y:\alpha\notin\Delta \text{ and } y:\beta\notin\Gamma} \quad \Sigma;\mathbb{M},x\leq y;\Gamma,x:\alpha\to\beta \Rightarrow^{\chi}_{\mathrm{T}} y:\alpha,\Delta \searrow S_1 \quad \Sigma;\mathbb{M},x\leq y;\Gamma,x:\alpha\to\beta,y:\beta \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S_2}{\Sigma;\mathbb{M},x\leq y;\Gamma,x:\alpha\to\beta \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S_1,S_2}\,\to\mathrm{L}$$

$$\frac{\begin{array}{c}\boxed{\forall y\in\Sigma.(xN_Ay\in\mathbb{M}) \Rightarrow y:\alpha\notin\Delta}\\ \boxed{\forall y\in\Sigma.(y\ll x) \Rightarrow (x=y \text{ or } x\not\preccurlyeq y)} \quad \Sigma,y;\mathbb{M},xN_Ay;\Gamma \Rightarrow^{\chi}_{\mathrm{T}} y:\alpha,x:A\ \mathsf{nec}\ \alpha,\Delta \searrow S\end{array}}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} x:A\ \mathsf{nec}\ \alpha,\Delta \searrow S}\,\mathsf{necR}$$

$$\frac{\boxed{y:\alpha\notin\Gamma} \quad \Sigma;\mathbb{M},xN_Ay;\Gamma,x:A\ \mathsf{nec}\ \alpha,y:\alpha \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}{\Sigma;\mathbb{M},xN_Ay;\Gamma,x:A\ \mathsf{nec}\ \alpha \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}\,\mathsf{necL}$$

Frame Rules

$$\frac{\boxed{x\leq x\notin\mathbb{M}} \quad \Sigma,x;\mathbb{M},x\leq x;\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}{\Sigma,x;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}\,\mathrm{refl} \qquad \frac{\boxed{x\leq z\notin\mathbb{M}} \quad \Sigma;\mathbb{M},x\leq y,y\leq z,x\leq z;\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}{\Sigma;\mathbb{M},x\leq y,y\leq z;\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}\,\mathrm{trans}$$

$$\frac{\boxed{xN_Az\notin\mathbb{M}} \quad \Sigma;\mathbb{M},x\leq y,yN_Az,xN_Az;\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}{\Sigma;\mathbb{M},x\leq y,yN_Az;\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}\,\mathsf{mon\text{-}N}$$

$$\frac{(\forall\vec{x}.((\wedge_i\ (x_iR_ix_i')) \to (xRx'))) \in \chi \quad x_iR_ix_i'\in\mathbb{M} \quad \boxed{xRx'\notin\mathbb{M}} \quad \Sigma;\mathbb{M},xRx';\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}\,\chi$$

Figure 3: Seq-MMI$^{\chi}_{\mathrm{T}}$: Terminating, countermodel producing sequent calculus for MMI$^{\chi}$. Applicability conditions are written in $\boxed{\text{boxes}}$. Wherever mentioned, the relation $\preccurlyeq$ is the equivalence relation of the contexts $\Sigma;\mathbb{M};\Gamma;\Delta$ in the conclusion of the rule. Similarly, $\ll$ is the order of the underlying tree of $\mathbb{M}$.

Theorem 3.12 (Termination)**.** *The following hold:*

1. *Any backwards search in Seq-MMI$^{\chi}_{T}$ starting from a sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{T} \Delta$ with $\mathbb{M}$ tree-like terminates.*

2. *For any $\Sigma; \mathbb{M}; \Gamma; \Delta$ with $\mathbb{M}$ tree-like, there is an S such that $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{T} \Delta \searrow S)$ and such an S can be finitely computed.*

Proof. Proof of (1): Suppose, for the sake of contradiction, that there is an infinite backward proof starting from $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow \ldots$. Since the proof is finitely branching (every rule has a bounded number of premises), it must have an infinite path. Observe that Γ, Δ are monotonic backwards, so the applicability conditions in the rules prevent application of the same rule on the same principal *labeled formula* more than once in any branch. Since there are only a finite number of formulas that can appear in any search (weak subformula property, Theorem 2.6), it follows that in the infinite path there must be an infinite number of labels. Let T be the underlying tree of this entire path (i.e., the underlying tree of the union of $\mathbb{M}$ for each sequent on this path). Since the tree is finitely branching (because we cannot apply rules ($\to$R) and (necR) to the same label infinitely often), it must have an infinite path. Let this path be $x_0 \ll x_1 \ll \ldots$. Let S_i be the value of $\mathrm{Sfor}(x_i)$ when either of the rules ($\to$R) and (necR) is applied to create x_{i+1}. Note that for $i < j$, $S_i \not\supseteq S_j$, because if $S_i \supseteq S_j$, then at the time that x_{j+1} is created, $\mathrm{Sfor}(x_i) \supseteq S_i \supseteq S_j = \mathrm{Sfor}(x_j)$, so the application of the rules ($\to$R) and (necR) on x_j would be blocked, so x_{j+1} could not have been created. Hence, for $i < j$, $S_i \not\supseteq S_j$. Call this fact (A). (The reader may note that the deduction $\mathrm{Sfor}(x_i) \supseteq S_i$ two sentences ago relies on the fact that $\mathrm{Sfor}(x)$ increases monotonically as we move backwards in a derivation.)

If Φ is the set of all subformulas of the original sequent we start from, then by Theorem 2.6, each $S_i \subseteq \{T(\alpha) \mid \alpha \in \Phi\} \cup \{F(\alpha) \mid \alpha \in \Phi\}$. Note that the right hand side is a finite set, so its subsets form a finite partial order under set inclusion. Call this partial order P. Since P is finite, it has a finite number of chains and since the sequence $S_1, S_2, \ldots$ is infinite, at least one infinite subsequence R of $S_1, S_2, \ldots$ must contain elements from only a single chain in P. Consider any two elements $S_i, S_j \in P$ with $i < j$. Since P is a chain, we must have either $S_i \supseteq S_j$ or $S_i \subsetneq S_j$. The former is ruled out fact (A). So $S_i \subsetneq S_j$. Hence, we have $S_1 \subsetneq S_2 \subsetneq S_3 \ldots$, so the chain P contains an infinite ascending sequence, which is a contradiction because P is finite.

Proof of (2): Follows immediately from (1), Lemma 3.11, and the observation that all applicability conditions are finitely computable. The latter follows from condition (1) of the definition of SCRs. ∎

Note that Theorem 3.12(2) does not stipulate that the S be unique. Indeed, depending on the order in which the rules of the calculus $\Rightarrow^{\chi}_{\mathrm{T}}$ are applied to a given sequent, S may be different. However, the fact that at least one such S exists and can be computed is enough to get decidability for MMI$^{\chi}$.

Lemma 3.13 (Simulation)**.** *If $\mathbb{M}$ is tree-like and $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{T} \Delta \searrow S)$, then $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{CM} \Delta \searrow S)$.*

Proof. By induction on the given derivation of $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S$. The case of rule (CM) follows from Lemma 3.10. The rest of the cases are immediate from the i.h. The only fact to take care of is that the tree-like property holds for each i.h. application. This follows from Lemma 3.11. ∎

Theorem 3.14 (Decidability)**.** *For a tree-like* $\mathbb{M}$*, suppose that* S *is such that* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{T} \Delta \searrow S)$ *(such an* S *must exist and can be computed using Theorem 3.12). Then:*

1. *If* $S = \{\}$*, then* $\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$*.*
2. *If* $S \neq \{\}$*, then every model* $\mathcal{M}$ *in* S *is a countermodel to the sequent, i.e.,* $\mathcal{M} \not\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$*.*

Proof. By Lemma 3.13, we have that $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S)$. Now, (1) follows from Theorems 3.8 and 2.5 and (2) follows from Theorem 3.9. □

Corollary 3.15 (Decidability and finite model property)**.** *If a SCR exists for* χ*, then MMI*$^{\chi}$ *is decidable, has the finite model property and has a constructive decision procedure.*

Proof. Immediate from Theorem 3.14. □

3.5 Comprehensiveness of Seq-MMI$^{\chi}_{\mathrm{CM}}$ and Seq-MMI$^{\chi}_{\mathrm{T}}$ Countermodels

Countermodels generated by Seq-MMI$^{\chi}_{\mathrm{CM}}$ (and Seq-MMI$^{\chi}_{\mathrm{T}}$) have an interesting property: If $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S)$, then $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} x : p, \Delta \searrow \{\})$ if and only if $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$. Thus, if we can produce a set of countermodels S by running without an actual goal (like $x : p$), then the set of atoms that are actually true are exactly those that are in the intersection of the valuation of all models in the set S. Further, because the result applies to derivations in Seq-MMI$^{\chi}_{\mathrm{CM}}$, it also applies to derivations in Seq-MMI$^{\chi}_{\mathrm{T}}$ due to Lemma 3.13 and the latter can be used to actually produce the set S. We call this result comprehensiveness and prove it below.

Theorem 3.16 (Comprehensiveness)**.** *Suppose* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{CM} \Delta \searrow S)$*. Then* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{CM} x : p, \Delta \searrow \{\})$ *iff* $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$*.*

Proof. See Appendix A, Theorem A.5. □

Corollary 3.17 (Comprehensiveness in Seq-MMI$^{\chi}_{\mathrm{T}}$)**.** *Suppose* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{T} \Delta \searrow S)$*. Then,* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} x : p, \Delta)$ *iff* $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$*.*

Proof. From $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S)$ we derive $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S)$ using Lemma 3.13. The result then follows from Theorem 3.16. □

3.6 Adding Seriality

In this section we show that if χ has a SCR, then our method applies not only to the logic MMI$^{\chi}$ (Corollary 3.15), but also to the logic which, in addition, forces *seriality* with respect to some of its relations N_A. Seriality for index A is the condition $\forall x.\exists y.(x N_A y)$. This corresponds to the axiom $\neg(A \mathsf{\ nec\ } \bot)$, also called D in literature [5]. Note that seriality does not fit our definition of χ because frame conditions in χ cannot contain existentials, so it cannot be handled in the method described so far. Consequently, we must modify our method slightly to include seriality as a frame condition. The only new challenge is to control creation of worlds due to the seriality condition during backwards search; for this we use an approach similar to that for controlling the use of rules ($\to$R) and (necR). Proofs do not change significantly.

Suppose we wish to make relations N_A for $A \in \mathcal{J} \subseteq \mathcal{I}$ serial. We first add the following rule to our sequent calculus Seq-MMI$^\chi$ (Figure 1) for every $A \in \mathcal{J}$:

$$\frac{\Sigma, x, y; \mathbb{M}, xN_Ay; \Gamma \Rightarrow^\chi \Delta}{\Sigma, x; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta}\mathrm{D}$$

Next, we change clause (1) of the definition of saturated history *not* to require closure under this new frame condition, which would cause infinite models, but instead new conditions based on $\preccurlyeq$:

> (1) $\mathbb{M}$ is tree-like and saturated with respect to the conditions (refl), (trans), (mon-N) and χ. In addition, at least one of the following must hold for each $x \in \Sigma$ and each index $A \in \mathcal{J}$:
>
> (a) There is a $y \in \Sigma$ such that $xN_Ay \in \mathbb{M}$, or
>
> (b) There is a $y \in \Sigma$ such that $y \neq x$, $y \ll x$ and $x \preccurlyeq y$.

With this new clause (1), we can show by induction on $\ll$ that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ is closed under seriality for $A \in \mathcal{J}$, hence it is a model of our (modified) logic. Next, we add the following rule for every $A \in \mathcal{J}$ to the terminating calculus Seq-MMI$^\chi_{\mathrm{T}}$ and a corresponding rule without the applicability conditions to Seq-MMI$^\chi_{\mathrm{CM}}$.

$$\frac{\boxed{\forall y \in \Sigma.\ (xN_Ay \notin \mathbb{M})} \qquad \boxed{\forall y \in \Sigma.(y \ll x) \Rightarrow (y = x \text{ or } x \not\preccurlyeq y)} \qquad \Sigma, x, y; \mathbb{M}, xN_Ay; \Gamma \Rightarrow^\chi_{\mathrm{T}} \Delta \searrow S}{\Sigma, x; \mathbb{M}; \Gamma \Rightarrow^\chi_{\mathrm{T}} \Delta \searrow S}\mathrm{D}$$

With these changes, our entire development works with only minor changes to the proofs (interestingly, the proof of Lemma 3.6 does not change).

Theorem 3.18 (Constructive decidability with seriality)**.** *Let D contain seriality conditions for some set of indices and let the frame conditions χ have a SCR. Then the logic MMI$^{\chi,D}$ is constructively decidable by our method.*

3.7 Constructive Decidability for Common Intuitionistic Logics

In this section, we list some common sets of frame conditions with their SCRs, thus showing that the intuitionistic logics corresponding to each of them is constructively decidable by our method. Unfortunately, SCRs are not modular: We cannot combine the SCRs for frame conditions χ_1 and χ_2 to get a SCR for a modal logic $\chi_1 \cup \chi_2$. As a result, we must explicitly construct a SCR for every modal logic of interest.

Figure 4 lists some common intuitionistic logics, their frame conditions (χ), the corresponding axioms they admit and the corresponding SCRs. In cases where the name of the logic is not common, we have cited the source of the logic. We note two things: (1) This list is not exhaustive, but merely representative, and (2) Our method also applies to any of these logics combined with seriality from Section 3.6 due to Theorem 3.18.

Theorem 3.19 (Decidability of Common Logics)**.** *The intuitionistic logics shown in Figure 4 have the SCRs also shown in that figure. Consequently, all these logics (and their combination with the seriality condition from Section 3.6) are constructively decidable by our method.*

Logic	Frame conditions χ	Additional Axioms	SCR
K	$\{\}$	–	$R(A) = (\leq)$
T	$\forall A, x.\ xN_Ax$	$(A\ \mathsf{nec}\ \alpha) \rightarrow \alpha$	$R(A) = (\leq)$
K4	$\forall A, x, y, z.\ ((xN_Ay) \wedge (yN_Az)) \rightarrow (xN_Az)$	$(A\ \mathsf{nec}\ \alpha) \rightarrow (A\ \mathsf{nec}\ A\ \mathsf{nec}\ \alpha)$	$R(A) = (N_A \cup \leq)$
S4	Conditions of K4 and T	Axioms of K4 and T	$R(A) = (N_A \cup \leq)$
I [9, 13]	$\forall A, B, x, y, z.\ ((xN_By) \wedge (yN_Az)) \rightarrow (xN_Az)$	$(A\ \mathsf{nec}\ \alpha) \rightarrow (B\ \mathsf{nec}\ A\ \mathsf{nec}\ \alpha)$	$R(A) = ((\cup_{B\in\mathcal{I}} N_B) \cup \leq)$
unit [7]	$\forall A, x, y.\ (xN_Ay) \rightarrow (x \leq y)$	$\alpha \rightarrow (A\ \mathsf{nec}\ \alpha)$	$R(A) = (\leq)$
–	$\forall x, y.\ (xN_Ay) \rightarrow (xN_By)$	$(B\ \mathsf{nec}\ \alpha) \rightarrow (A\ \mathsf{nec}\ \alpha)$	$R(A) = (\leq)$

Figure 4: SCRs for some multi-modal intuitionistic logics. All these logics are constructively decidable by our method.

4 Discussion

This section discusses some loose ends: The connection between our technique and the technique of semantic filtrations, and some broad limitations of our work.

Relation to semantic filtration. Semantic filtrations [5] are a technique for establishing the finite model property of modal logics. The key idea is to show the existence of an accessibility relation (called a filtration) on the finite model obtained by collapsing worlds of any model that satisfy the same set of formulas. The relation must satisfy some specific conditions. Often the accessibility relation constructed has a definition similar to our SCR relations $R(A)$ and, superficially, the two techniques may look similar. However, a careful examination reveals differences. Primarily, filtrations are semantic techniques that manipulate Kripke models whereas our method is purely syntactic and SCRs only work with sets of formulas generated during a specific backwards search. A consequence of this difference is that, for any of the logics considered in this paper, we have not been able to find a suitable filtration on the obvious model whose worlds are equivalence classes of $\preccurlyeq \cap \succcurlyeq$. In particular, it seems extremely difficult to satisfy what is known as the "back condition", which is required of a filtration. We also note that there is no standard definition of filtrations for intuitionistic modal logics. We know of only one work in this domain, and that work is also limited to the uni-modal case [15] (the author of the paper notes in the conclusion that generalizing to the multi-modal case is not trivial).

Complexity bounds. Even though our method proves the finite model property and provides a constructive decision procedure for a wide-variety of multi-modal logics, even in the intuitionistic case, it does not provide tight upper bounds on the complexity of the logics. For example, based on the result of [15], we expect that the intuitionistic modal logic K is decidable in doubly exponential time, but an analysis of Theorem 3.12 yields a bound that is at best quadruply exponential. This inability to produce accurate complexity bounds may be partly attributed to the method's generality (it also applies to multi-modal logics) and partly to the fact that it constructs a comprehensive set of countermodels (Section 3.5). We do not know of any existing complexity bound for producing such comprehensive sets of countermodels.

Intuitionistic possibility modalities. Our method does not work when intuitionistic possibility modalities $A\ \mathsf{pos}\ \alpha$ (commonly written $\Diamond_A\varphi$) are included in the logic. The reason for this is subtle, but primarily stems from the fact that the binary relation P_A used to define the possibility modality must satisfy $(\geq \circ P_A) \subseteq P_A$ (as compared to $(\leq \circ N_A) \subseteq N_A$ for necessitation) [26]. Consequently, in the definition of saturated history, the clause for A pos α analogous to clause (12) reads:

If $x : A\ \mathsf{pos}\ \alpha \in \Gamma$, then either:

(a) There is a y such that $xP_Ay \in \mathbb{M}$ and $y : \alpha \in \Gamma$ or

(b) There is a y such that $y \neq x$, $y \ll x$ and $y \preccurlyeq x$.

Observe that in part (b) above, we have the condition $y \preccurlyeq x$ instead of $x \preccurlyeq y$ in clause (12b). This difference is enough to break the termination property (Theorem 3.12); in particular, the fact called (A) in its proof cannot be established with possibility modalities.

In classical logic, where the relation $\leq$ is absent, possibility modalities are easily handled by our method. In fact, they need no special treatment as they are just the DeMorgan duals of necessitation (Appendix B).

Symmetry condition. To the best of our knowledge, there is no SCR for the symmetry condition $(\forall x, y.(xN_Ay) \to (yN_Ax))$, the cornerstone of the modal logic S5. Consequently, our method cannot be used to prove decidability for this logic, in either the uni-modal or multi-modal setting. This is not surprising because Baldoni has shown that, in general, multi-modal logics with symmetry conditions are undecidable [3]. So, a general method like ours is unlikely to be able to handle these conditions. Nonetheless it is dissatisfying that our method is unable to handle even *uni*-modal S5, which is known to be decidable.

Other label producing conditions. We showed in Section 3.6 that our method is compatible with the seriality frame condition. A question is whether it is also compatible with other frame conditions that produce labels, like seriality does. For example, can it be extended to handle density: $\forall x, y.((xN_Ay) \to \exists z.((xN_Az) \wedge (zN_Ay)))$? Although we have not investigated this question in detail, there seems to be no obvious method to extend our technique to include such conditions in general. Nonetheless, it may be possible to combine our method with work on decidability for layered modal logics [11], that are derived from a specific subclass of label-producing conditions. We leave this combination to future work.

5 Classical Logic

Although we developed our decision procedure with intuitionistic modal logic in mind, it can be modified to apply to classical multi-modal logics as well. The overall approach of using SCRs and the structure of the calculi remains the same. However, because Kripke frames in classical logic do not require the preorder $\leq$, we must change the definition of $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ (Definition 3.4) that relies on $\leq$. This is not difficult: Instead of adding $x \leq y$ when $x \preccurlyeq y$, we add the relation xN_Az when yN_Az and $x \preccurlyeq y$. Appendix B describes in detail our method as it applies to classical logics, together will all relevant proofs. In classical logic, the possibility modality of index A can be defined as the DeMorgan dual of the necessitation modality of the same index, so on classical logic, our method applies to possibility modalities as well.

6 Related Work

The applicability conditions of rules ($\to$R) and (necR) in Figure 3, based on the relation $\preccurlyeq$, are inspired by the work of Gasquet et al. [10] in which tableaux-based decision procedures are given for classical uni-modal logics with the following frame conditions: Transitivity, reflexivity, symmetry, Euclideanness, seriality and confluence. Our method is based on labeled sequent calculi and it applies to both classical and intuitionistic modal logics with any number of modalities. As a consequence, we had to develop new proof techniques to establish our results, particularly in the intuitionistic setting.

Our method of extracting countermodels is inspired by Negri's proof of completeness of labeled sequent calculi for uni-modal logic with respect to their Kripke semantics [17][18, Chapter 11]. In that proof, it is shown how to extract (possibly infinite) countermodels from non-terminating branches of a failed proof search taking the union of all $\mathbb{M}$ occurring along the branch. Here, instead, countermodels are built in the context of a decision procedure and *finite* countermodels are built by adding additional edges based on the relation $\preccurlyeq$ and saturating with respect to the frame conditions.

Boretti and Negri [6] develop a countermodel producing decision procedure similar to ours for a Priorean linear time fixed point calculus (a variant of linear time temporal logic, LTL), which also includes two rules like seriality. They also use a notion of saturation and construct countermodels from histories. The main difference between this work and [6] is that this work handles general frame conditions and, additionally, intuitionistic connectives. Boretti and Negri also discuss previous tableaux-style approaches to the generation of countermodels for LTL, such as [22].

Countermodel producing sequent calculi, also known in the literature as "refutation calculi", have been given for intuitionistic logic, bi-intuitionistic logic, and provability logics [12, 14, 19] and in a way closer to the present paper's approach in [20]. One of the peculiarities of our method in relation to previous work is that the countermodel construction is made part of the calculus itself.

Gasquet and Said [11] introduce a technique called dynamic filtration to establish complexity bounds for the satisfiability problem of classical *layered modal logics* (LMLs), i.e., "logics characterized by semantic properties only stating the existence of possible worlds that are in some sense further than the other". Typically, such logics include confluence-like conditions, but they do not include transitivity-like conditions. Our work provides constructive decision procedures for a different and disjoint class of logics to which the techniques in [11] do not apply. In fact, with the exception of seriality, none of the frame conditions considered in this paper fall in the class of LMLs. Moreover, because LMLs cannot be applied with transitivity conditions it is not clear whether the techniques in [11] are suitable in the intuitionistic setting.

Simpson [23] presents decision procedures based on labeled sequent calculi for the intuitionistic uni-modal logics K, D, T and B together with S5. He leaves open the decidability of intuitionistic S4, K4 and KD4. Our method shows that the necessitation-only fragments of all three logics are decidable, not only in the uni-modal case, but also in the multi-modal case and, further, that the logics have constructive decision procedures.

Schmidt and Tishkovsky [21] present a general method for synthesizing sound and complete tableaux calculi given a semantic filtration [5] for the underlying classical modal logic. Although semantic filtrations are a powerful and general technique, their definition is not clear for many intuitionistic and multi-modal logics, so our method handles several logics that cannot be handled by Schmidt and Tishkovsky. To the best of our knowledge, the only work on filtrations for intuitionistic logics is limited to the uni-modal case [15] (the author of the paper notes in the conclusion that generalizing to the multi-modal case is not trivial). Filtration-based methods are also technically different from our syntactic approach. Whereas filtrations manipulate Kripke models, our method is purely syntactic and SCRs only work with sets of formulas generated during a specific backwards search. A consequence of this difference is that, for any of the logics considered in this paper, we have not been able to find a suitable filtration on the obvious model whose worlds are equivalence classes of $\preccurlyeq \cap \succcurlyeq$. In particular, it seems extremely difficult to satisfy the "back condition" of a filtration.

Alechina and Shakatov [2] present a general technique to prove decidability of intuitionistic (multi)-modal logics by embedding the relational definition of the semantics into Monadic Second Order Logic (MSO). As noted by the authors themselves and unlike our method, this approach does not give good decision procedures since it proceeds by reduction to satisfiability of formulas in SkS (monadic second-order theory of trees with constant branching factor k), which has non-elementary complexity. Moreover, the method applies to a logic only if its frame conditions can be expressed as acyclic closure conditions in MSO; this makes the method inapplicable to logics with frame conditions mentioning more than one modal relation, e.g., the logic (I) from Figure 4.

Negri [16] and Viganó [24] provide sound, complete and terminating labeled sequent calculi for uni-modal classical logics K, T, K4 and S4. Out method extends these results for a wider class of modal logics including axiom D, multiple modalities and intuitionistic logics.

Goré and Nguyen [13] present *non-labeled* tableaux calculi for seven types of classical multi-modal logics to reason about epistemic states of agents in distributed systems. The introduced tableaux require formulas of the logic to be first translated into a clausal form. We observe that one of the axioms presented in [13] is positive introspection for beliefs and corresponds to axiom (I) in Figure 4. To the best of our knowledge, [13] is the only work to provide decision procedures for logics including axiom (I).

7 Conclusion and Future Work

We have presented a sequent calculus-based, constructive decision procedure for several multi-modal logics, both intuitionistic and classical. Besides a novel construction of countermodels and a novel termination condition, we show, for the first time, that several standard intuitionistic multi-modal logics without diamonds, as well as several logics with interactions between modalities, are decidable. We also show that our procedure has an interesting, novel comprehensiveness property.

Although we believe that our work is a significant step in using sequent calculi (especially, labeled sequent calculi) in constructive decision procedures for modal logics, a lot still needs to be done. First, we believe that our method can be extended to label-producing frame conditions more general than seriality. In particular, it should be possible to extend the technique of Section 3.6 to all layered modal logics of Gasquet and Said [11], which we discussed in Section 6. Second, we would like to either extend our method, or find a new one that can handle possibility modalities in intuitionistic logic.

Another direction of research is to exploit the embedding of intuitionistic modal logics into classical bi-modal logics to establish decidability results for the former, a direction pursued in [26] but with semantic rather than syntactic methods.

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A Proofs from Section 3

Lemma A.1 (Lemma 3.5)**.** *If $\Sigma;\mathbb{M};\Gamma;\Delta$ is a saturated history, then $CM(\Sigma;\mathbb{M};\Gamma;\Delta)$ has a monotonic valuation h, i.e., $x \in h(p)$ and $x \leq y \in CM(\Sigma;\mathbb{M};\Gamma;\Delta)$ imply $y \in h(p)$.*

Proof. Let C be the set $\{x \leq y \mid x \preccurlyeq y\}$. Suppose that $x \leq y \in \mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta)$, i.e., $x \leq y \in \overline{\mathbb{M} \cup C}$ and $x \in h(p)$. From the latter, there is a z such that $z \leq x \in \mathbb{M}$ and $z : p \in \Gamma$. From the definition of SCR, clause (4) it follows that $x(\leq \cup C)^* y$, where all the relations $\leq$ are in $\mathbb{M}$. Hence, we have a chain $x = x_0(\leq \cup C)x_1 \ldots (\leq \cup C)x_n = y$ where all relations $\leq$ are in $\mathbb{M}$. We induct on i to show that $T(p) \in \mathrm{Sfor}(x_i)$.

- For $i = 0$, $x_0 = x$ and we know that $z : p \in \Gamma$ and $z \leq x \in \mathbb{M}$. It follows from definition of Sfor that $T(p) \in \mathrm{Sfor}(x)$, as required.
- For the induction step, assume that $T(p) \in \mathrm{Sfor}(x_i)$. We prove that $T(p) \in \mathrm{Sfor}(x_{i+1})$. We consider two possible cases on the relation $x_i(\leq \cup C)x_{i+1}$.

- $x_i \le x_{i+1} \in \mathbb{M}$. Because $T(p) \in \mathrm{Sfor}(x_i)$, there is a z' such that $z' \le x_i \in \mathbb{M}$ and $z' : p \in \Gamma$. Hence, also $z' \le x_{i+1} \in \mathbb{M}$. So $T(p) \in \mathrm{Sfor}(x_{i+1})$.
- $(x_i, x_{i+1}) \in C$. Because of the definition of C, $\mathrm{Sfor}(x_i) \subseteq \mathrm{Sfor}(x_{i+1})$, so $T(p) \in \mathrm{Sfor}(x_i)$ immediately implies $T(p) \in \mathrm{Sfor}(x_{i+1})$.

This completes the inductive proof that $T(p) \in \mathrm{Sfor}(x_i)$. In particular, $T(p) \in \mathrm{Sfor}(x_n)$. By definition of Sfor, there is a z' such that $z' \le x_n \in \mathbb{M}$ and $z' : p \in \Gamma$. This immediately implies $x_n \in h(p)$, i.e., $y \in h(p)$, as required. □

Lemma A.2 (Lemma 3.6). *The following hold for any saturated history* $\Sigma; \mathbb{M}; \Gamma; \Delta$*:*

A. *If* $T(\varphi) \in \mathit{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$*, then* $\mathit{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : \varphi$

B. *If* $F(\varphi) \in \mathit{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$*, then* $\mathit{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \varphi$

Proof. We prove both properties simultaneously by *lexicographic* induction, first on φ, and then on the partial (tree-like) order $\ll$ of $\mathbb{M}$. (Note that we cannot induct on either $\mathbb{M}$ or the relation in $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$, because both of these may potentially be cyclic.) Since the context $\Sigma; \mathbb{M}; \Gamma; \Delta$ is fixed here, we abbreviate $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$ to $\mathrm{Sfor}(x)$. Let C be the set $\{(x, y) \mid x \preccurlyeq y\}$.

Proof of A.

Case. $\varphi = p$. We are given that $T(p) \in \mathrm{Sfor}(x)$ and want to prove that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : p$. Since $T(p) \in \mathrm{Sfor}(x)$, we know from definition of the function Sfor that there is a y with $y \le x \in \mathbb{M}$ and $y : p \in \Gamma$. Since $y \le x \in \mathbb{M}$, we know from definition of $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ that $x \in h(p)$. Hence, by definition of $\models$, we have $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : p$.

Case. $\varphi = \top$. Here, $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : \top$ is trivial by the definition of $\models$.

Case. $\varphi = \bot$. Then the pre-condition $T(\bot) \in \mathrm{Sfor}(x)$ or, equivalently, $x : \bot \in \Delta$ is impossible by clause (3) of the definition of saturated history. So this case is vacuous.

Case. $\varphi = \alpha \wedge \beta$. We are given that $T(\alpha \wedge \beta) \in \mathrm{Sfor}(x)$ or, equivalently, that $x : \alpha \wedge \beta \in \Gamma$. By clause (5) of the definition of saturated history, $x : \alpha \in \Gamma$ and $x : \beta \in \Gamma$. Hence, $T(\alpha) \in \mathrm{Sfor}(x)$ and $T(\beta) \in \mathrm{Sfor}(x)$. By the i.h., $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : \alpha$ and $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : \beta$. Hence, $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : \alpha \wedge \beta$, as required.

Case. $\varphi = \alpha \vee \beta$. We are given that $T(\alpha \vee \beta) \in \mathrm{Sfor}(x)$ or, equivalently, that $x : \alpha \vee \beta \in \Gamma$. By clause (7) of the definition of saturated history, either $x : \alpha \in \Gamma$ or $x : \beta \in \Gamma$. Hence, either $T(\alpha) \in \mathrm{Sfor}(x)$ or $T(\beta) \in \mathrm{Sfor}(x)$. By the i.h., either $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : \alpha$ or $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : \beta$. In either case, $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : \alpha \wedge \beta$, as required.

Case. $\varphi = \alpha \to \beta$. We are given that $T(\alpha \to \beta) \in \mathrm{Sfor}(x)$. We need to show that for any y such that $x \le y$ in the model and $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models y : \alpha$, we have $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models y : \beta$. Pick any y such that $x \le y$ in the model and $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models y : \alpha$. From the definition of SCR, clause (4), it follows that $x(\le \cup C)^* y$, where the $\le$ relations are in $\mathbb{M}$. Hence, there is a chain $x = x_0(\le \cup C)x_1 \ldots (\le \cup C)x_n = y$, where the $\le$ relations are in $\mathbb{M}$. We induct on i to prove that $T(\alpha \to \beta) \in \mathrm{Sfor}(x_i)$ for each i.

- For $i = 0$, $x_0 = x$ and we are given that $T(\alpha \to \beta) \in \mathrm{Sfor}(x)$, so we are done.
- For the inductive case, assume that $T(\alpha \to \beta) \in \mathrm{Sfor}(x_i)$ for some i. We show that $T(\alpha \to \beta) \in \mathrm{Sfor}(x_{i+1})$. We consider two possible cases on the relation $x_i(\leq \cup C)x_{i+1}$:
  - $x_i \leq x_{i+1} \in \mathbb{M}$: From the i.h., we know that $T(\alpha \to \beta) \in \mathrm{Sfor}(x_i)$. Hence, there is a z' such that $z' \leq x_i \in \mathbb{M}$ and $z' : \alpha \to \beta \in \Gamma$. Clearly, $z' \leq x_{i+1} \in \mathbb{M}$, so $T(\alpha \to \beta) \in \mathrm{Sfor}(x_{i+1})$.
  - $(x_i, x_{i+1}) \in C$: Because of the definition of C, $\mathrm{Sfor}(x_i) \subseteq \mathrm{Sfor}(x_{i+1})$, so $T(\alpha \to \beta) \in \mathrm{Sfor}(x_i)$ immediately implies $T(\alpha \to \beta) \in \mathrm{Sfor}(x_{i+1})$.

This completes the inductive proof. It follows, in particular, that $T(\alpha \to \beta) \in \mathrm{Sfor}(x_n)$. Consequently, there is some z' such that $z' \leq x_n = y \in \mathbb{M}$ and $z' : \alpha \to \beta \in \Gamma$. Hence, by clause (9) of the definition of saturated history, we must have either $y : \alpha \in \Delta$ or $y : \beta \in \Gamma$. The former implies, by the i.h., that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models y : \alpha$, which contradicts our assumption that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models y : \alpha$. So, we must have $y : \beta \in \Gamma$. This implies $T(\beta) \in \mathrm{Sfor}(y)$ and hence, by the i.h., that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models y : \beta$.

Case. $\varphi = A \ \mathsf{nec}\ \alpha$. We are given that $T(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x)$. We need to show that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : A \ \mathsf{nec}\ \alpha$, i.e., for any y such that xN_Ay in the model, we have $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models y : \alpha$. Pick any y such that xN_Ay in the model. Because of the definition of SCR, clause (3), we have $x((R(A) \cup C)^* \circ N_A \circ \leq)y$, where the relations $R(A)$ and N_A are in $\mathbb{M}$. So there are $x_0, \ldots, x_n, y'$ such that $x = x_0(R(A) \cup C)x_1 \ldots (R(A) \cup C)x_n N_A y' \leq y$. We now prove, by induction on i, that $T(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x_i)$ for each i.

- For $i = 0$, $x_0 = x$ and we are given that $T(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x)$.
- For the inductive case, assume that $T(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x_i)$ for some i. We show that $T(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x_{i+1})$ by case analyzing the relation $x_i(R(A) \cup C)x_{i+1}$.
  - $x_i(R(A))x_{i+1} \in \mathbb{M}$: By the i.h., $T(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x_i)$ so there is some z such that $z(R(A))^*x_i \in \mathbb{M}$ and $z : A \ \mathsf{nec}\ \alpha \in \Gamma$. Clearly, we have $z(R(A))^*x_{i+1} \in \mathbb{M}$, so $T(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x_{i+1})$.
  - $(x_i, x_{i+1}) \in C$: Because of the definition of C, $\mathrm{Sfor}(x_{i+1}) \supseteq \mathrm{Sfor}(x_i)$. Thus, $T(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x_i)$ immediately implies $T(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x_{i+1})$.

Since we just proved that $T(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x_i)$, it follows, in particular, that $T(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x_n)$. Consequently, there is some z' such that $z'(R(A))^*x_n \in \mathbb{M}$ and $z' : A \ \mathsf{nec}\ \alpha \in \Gamma$. Then, we also have (within $\mathbb{M}$) that: $z'(R(A))^*x_n N_A y'$. So, by clause (2) of the definition of SCR, $z' N_A y' \in \mathbb{M}$. Hence, by clause (11) of the definition of saturated history, we must have $y' : \alpha \in \Gamma$. Therefore, $T(\alpha) \in \mathrm{Sfor}(y')$ and by the i.h., $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models y' : \alpha$. Since $y' \leq y \in \mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$, by Lemma 2.3, $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models y : \alpha$.

Proof of B.

Case. $\varphi = p$. We are given that $F(p) \in \mathrm{Sfor}(x)$ or, equivalently, that $x : p \in \Delta$. Suppose, for the sake of contradiction, that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : p$. Then, $x \in h(p)$ and hence, from the construction of the countermodel, there is a z such that $z \leq x \in \mathbb{M}$ and $z : p \in \Gamma$. This immediately contradicts clause (2) of the definition of saturated history because we have $z \leq x \in \mathbb{M}$, $z : p \in \Gamma$ and $x : p \in \Delta$. Hence we must have $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : p$.

Case. $\varphi = \top$. Then the pre-condition $F(\top) \in \mathrm{Sfor}(x)$ or, equivalently, $x : \top \in \Delta$ is impossible by clause (3) of the definition of saturated history. So this case is vacuous.

Case. $\varphi = \bot$. Here, $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \bot$ is trivial by the definition of $\models$.

Case. $\varphi = \alpha \wedge \beta$. Suppose $F(\alpha \wedge \beta) \in \mathrm{Sfor}(x)$. Then, $x : \alpha \wedge \beta \in \Delta$. Hence, by clause (6) of the definition of saturated history, either $x : \alpha \in \Delta$ or $x : \beta \in \Delta$. Therefore, either $F(\alpha) \in \mathrm{Sfor}(x)$ or $F(\beta) \in \mathrm{Sfor}(x)$. By i.h., either $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \alpha$ or $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \beta$. In either case, $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \alpha \wedge \beta$.

Case. $\varphi = \alpha \vee \beta$. Suppose $F(\alpha \vee \beta) \in \mathrm{Sfor}(x)$. Then, $x : \alpha \vee \beta \in \Delta$. Hence, by clause (8) of the definition of saturated history, $x : \alpha \in \Delta$ and $x : \beta \in \Delta$. Therefore, $F(\alpha) \in \mathrm{Sfor}(x)$ and $F(\beta) \in \mathrm{Sfor}(x)$. By i.h., $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \alpha$ and $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \beta$. By definition of $\models$, we have $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \alpha \vee \beta$.

Case. $\varphi = \alpha \rightarrow \beta$. Suppose $F(\alpha \rightarrow \beta) \in \mathrm{Sfor}(x)$. This implies, by definition of Sfor, that $x : \alpha \rightarrow \beta \in \Delta$. By clause (10) of the definition of saturated history, we have that either:

1. There is a y such that $x \leq y \in \mathbb{M}$, $y : \alpha \in \Gamma$ and $y : \beta \in \Delta$ or
2. There is a y such that $y \neq x$, $y \ll x$ and $x \preccurlyeq y$.

If (a) holds, then by the i.h., $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models y : \alpha$ and $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models y : \beta$. Further, $x \leq y$, so $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \alpha \rightarrow \beta$.

If (b) holds, then since $x \preccurlyeq y$, $F(\alpha \rightarrow \beta) \in \mathrm{Sfor}(y)$. By the i.h. on the world y, which is strictly smaller than x in the relation $\ll$ (since $y \neq x$), it follows that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models y : \alpha \rightarrow \beta$. Note that in $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$, $x \leq y$. So, by Lemma 2.3, $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \alpha \rightarrow \beta$, as required.

Case. $\varphi = A\ \mathsf{nec}\ \alpha$. Suppose $F(A\ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x)$. This implies, by definition of Sfor that $x : A\ \mathsf{nec}\ \alpha \in \Delta$. By clause (12) of the definition of saturated history, we have that either:

(a) There is a y such that $xN_Ay \in \mathbb{M}$ and $y : \alpha \in \Delta$ or

(b) There is a y such that $y \neq x$, $y \ll x$ and $x \preccurlyeq y$.

If (a) holds, then by the i.h., $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models y : \alpha$. Since xN_Ay, it immediately follows that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : A\ \mathsf{nec}\ \alpha$.

If (b) holds, then since $x \preccurlyeq y$, $F(A\ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(y)$. By the i.h. on the world y, which is strictly smaller in the order $\ll$ (since $x \neq y$), it follows that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models y : A\ \mathsf{nec}\ \alpha$. Since in $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ we have $x \leq y$, Lemma 2.3 immediately implies $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : A\ \mathsf{nec}\ \alpha$, as required. ☐

Lemma A.3 (Comprehensiveness 1)**.** *Suppose* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{CM} \Delta \searrow S)$. *Suppose* x *and* p *are such that* $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$. *Then,* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{CM} x : p, \Delta \searrow \{\})$.

Proof. By induction on the given derivation of $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\text{CM}} \Delta \searrow S$ and case analysis of its last rule (the rules are listed in Figure 2).

Case. $\dfrac{\Sigma; \mathbb{M}; \Gamma; \Delta \text{ is a saturated history}}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\text{CM}} \Delta \searrow \{\text{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)\}}\text{CM}$

Here $S = \{\text{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)\}$. The given condition $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$ implies (by definition of CM) that there is a z such that $z \leq x$ and $z : p \in \Gamma$. Therefore, by rule (init), $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\text{CM}} x : p, \Delta \searrow \{\})$, as required.

Case. $\dfrac{}{\Sigma; \mathbb{M}, y' \leq y; \Gamma, y' : q \Rightarrow^{\chi}_{\text{CM}} y : q, \Delta \searrow \{\}}\text{init}$

By rule (init), we have $\vdash (\Sigma; \mathbb{M}, y' \leq y; \Gamma, y' : q \Rightarrow^{\chi}_{\text{CM}} x : p, y : q, \Delta \searrow \{\})$, which is what we need to prove.

Case. $\dfrac{}{\Sigma; \mathbb{M}; \Gamma, y : \bot \Rightarrow^{\chi}_{\text{CM}} \Delta \searrow \{\}}\bot\text{L}$

By rule ($\bot$L), $\vdash (\Sigma; \mathbb{M}; \Gamma, y : \bot \Rightarrow^{\chi}_{\text{CM}} x : p, \Delta \searrow \{\})$, as required.

Case. $\dfrac{}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\text{CM}} y : \top, \Delta \searrow \{\}}\top\text{R}$

By rule ($\top$R), $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\text{CM}} x : p, y : \top, \Delta \searrow \{\})$, as required.

Case. $\dfrac{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\text{CM}} y : \alpha, y : \alpha \wedge \beta, \Delta \searrow S_1 \qquad \Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\text{CM}} y : \beta, y : \alpha \wedge \beta, \Delta \searrow S_2}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\text{CM}} y : \alpha \wedge \beta, \Delta \searrow S_1, S_2}\wedge\text{R}$

Here, $S = S_1, S_2$. We are given that $\forall \mathcal{M} \in (S_1, S_2).\ \mathcal{M} \models x : p$.

1. $\forall \mathcal{M} \in S_1.\ \mathcal{M} \models x : p$ (From assumption $\forall \mathcal{M} \in (S_1, S_2).\ \mathcal{M} \models x : p$)
2. $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\text{CM}} x : p, y : \alpha, y : \alpha \wedge \beta, \Delta \searrow \{\})$ (i.h. on 1st premise and (1))
3. $\forall \mathcal{M} \in S_2.\ \mathcal{M} \models x : p$ (From assumption $\forall \mathcal{M} \in (S_1, S_2).\ \mathcal{M} \models x : p$)
4. $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\text{CM}} x : p, y : \beta, y : \alpha \wedge \beta, \Delta \searrow \{\})$ (i.h. on 2nd premise and (2))
5. $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\text{CM}} x : p, y : \alpha \wedge \beta, \Delta \searrow \{\})$ (Rule ($\wedge$R) on 2,4)

Case. All other cases are similar to the case of ($\wedge$R) above: We apply the i.h. to the premises and reapply the rule. □

Lemma A.4 (Comprehensiveness 2)**.** *Suppose* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{CM} \Delta \searrow S)$*. Suppose* x *and* p *are such that* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{CM} x : p, \Delta \searrow \{\})$*. Then,* $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$*.*

Proof. Suppose $\mathcal{M} \in S$. From Theorem 3.9, we know that (1) $\forall w, w' \in \Sigma.\ (wRw' \in \mathbb{M}) \Rightarrow (wRw' \in \mathcal{M})$, (2) $\forall (w : \varphi) \in \Gamma.\ \mathcal{M} \models w : \varphi$ and (3) $\forall (w : \varphi) \in \Delta.\ \mathcal{M} \not\models w : \varphi$. By Theorem 3.8 applied to the assumption $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\text{CM}} x : p, \Delta \searrow \{\})$, we know that $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} x : p, \Delta)$. Applying Theorem 2.5, we get that $\mathcal{M}, \rho \models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} x : p, \Delta)$ for every ρ and, in particular, for $\rho(x) = x$. Using (1)–(3) and the definition of $\models$ on sequents, we immediately get $\mathcal{M} \models x : p$, as required. □

Theorem A.5 (Comprehensivenss, Theorem 3.16)**.** *Suppose* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{CM} \Delta \searrow S)$. *Then* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{CM} x : p, \Delta \searrow \{\})$ *iff* $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$.

Proof. Lemmas A.3 and A.4 each state one direction of this theorem. $\square$

B Constructive Decidability for Classical Multi-Modal Logic

This section re-develops our method for classical logic instead of intuitionistic logic. To eliminate confusion between names the classical logic with frame conditions χ is written MM^{χ}, not MMI^{χ}.

Syntax. The syntax of the formulas of the logic MM^{χ} is shown below. $\mathcal{I} = \{A, B, \ldots\}$ is a finite set of indices for modalities and p denotes an atomic formula, drawn from a countable set of such formulas.

$$\text{Formulas} \quad \varphi, \alpha, \beta \quad ::= \quad p \mid \top \mid \alpha \wedge \beta \mid \neg\alpha \mid A \ \mathsf{nec}\ \alpha$$

Other standard connectives not listed above can be defined: $\bot = \neg\top$, $\alpha \vee \beta = \neg((\neg\alpha) \wedge (\neg\beta))$, $\alpha \to \beta = (\neg\alpha) \vee \beta$ and $A\ \mathsf{pos}\ \alpha = \neg(A\ \mathsf{nec}\ (\neg\alpha))$. $A\ \mathsf{pos}\ \alpha$ is the possibility modality of index A.

B.1 Semantics

We provide Kripke (frame) semantics to formulas of MM^{χ} and assume in our presentation that the reader has basic familiarity with this style of semantics.

Definition B.1 (Kripke model)**.** A classical model, Kripke model or, simply, model, $\mathcal{M}$ is a tuple $(W, \{N_A\}_{A \in \mathcal{I}}, h)$ where,

- W is a set, whose elements x, y, z, w are called worlds.
- Each N_A is a binary relation on W
- h assigns to each atom p the set of worlds $h(p) \subseteq W$ where p holds.

A model without the assignment, i.e., the tuple $(W, \{N_A\}_{A \in \mathcal{I}})$ is also called a *frame*.

The frame conditions χ. We allow a countable number of additional frame conditions denoted by rules of the following form: $\forall \vec{x}.((\wedge_{i=1,\ldots,n}\ x_i R_i x'_i) \to (xRx'))$, where $R_1, \ldots, R_n, R$ are from the set $\{N_A \mid A \in \mathcal{I}\}$ and all variables x_i, x'_i, x, x' are in $\vec{x}$. A set of such additional frame conditions is denoted χ. MM^{χ} is the logic whose valid formulas are exactly those that are valid (in the sense defined below) in frames that satisfy all conditions in χ.

Definition B.2 (Satisfaction)**.** Given a model $\mathcal{M} = (W, \{N_A\}_{A \in \mathcal{I}}, h)$ and a world $w \in W$, we define the satisfaction relation $\mathcal{M} \models w : \alpha$, read "the world w satisfies formula α in model $\mathcal{M}$" by induction on α as follows:

$\mathcal{M} \models w : p$ iff $w \in h(p)$

$\mathcal{M} \models w : \top$ (unconditionally)

$\mathcal{M} \models w : \alpha \wedge \beta$ iff $\mathcal{M} \models w : \alpha$ and $\mathcal{M} \models w : \beta$

Axiom Rules

$$\frac{}{\Sigma; \mathbb{M}; \Gamma, x : p \rightarrow^{\chi} x : p, \Delta}\text{init}$$

Logical Rules

$$\frac{}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} x : \top, \Delta}\top\text{R} \qquad \frac{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} x : \alpha, x : \alpha \wedge \beta, \Delta \qquad \Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} x : \beta, x : \alpha \wedge \beta, \Delta}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} x : \alpha \wedge \beta, \Delta}\wedge\text{R}$$

$$\frac{\Sigma; \mathbb{M}; \Gamma, x : \alpha \wedge \beta, x : \alpha, x : \beta \rightarrow^{\chi} \Delta}{\Sigma; \mathbb{M}; \Gamma, x : \alpha \wedge \beta \rightarrow^{\chi} \Delta}\wedge\text{L} \qquad \frac{\Sigma; \mathbb{M}; \Gamma, x : \alpha \rightarrow^{\chi} x : \neg\alpha, \Delta}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} x : \neg\alpha, \Delta}\neg\text{R} \qquad \frac{\Sigma; \mathbb{M}; \Gamma, x : \neg\alpha \rightarrow^{\chi} x : \alpha, \Delta}{\Sigma; \mathbb{M}; \Gamma, x : \neg\alpha \rightarrow^{\chi} \Delta}\neg\text{L}$$

$$\frac{\Sigma, y; \mathbb{M}, xN_Ay; \Gamma \rightarrow^{\chi} y : \alpha, x : A \,\mathsf{nec}\, \alpha\Delta}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} x : A \,\mathsf{nec}\, \alpha, \Delta}\mathsf{necR} \qquad \frac{\Sigma; \mathbb{M}, xN_Ay; \Gamma, x : A \,\mathsf{nec}\, \alpha, y : \alpha \rightarrow^{\chi} \Delta}{\Sigma; \mathbb{M}, xN_Ay; \Gamma, x : A \,\mathsf{nec}\, \alpha \rightarrow^{\chi} \Delta}\mathsf{necL}$$

Frame Rules

$$\frac{(\forall\vec{x}.((\wedge_i\ (x_iR_ix'_i)) \rightarrow (xRx'))) \in \chi \qquad x_iR_ix'_i \in \mathbb{M} \qquad \Sigma; \mathbb{M}, xRx'; \Gamma \rightarrow^{\chi} \Delta}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta}\chi$$

Figure 5: Seq-MM$^{\chi}$: A labeled sequent calculus for MM$^{\chi}$

$\mathcal{M} \models w : \neg\alpha$ iff $\mathcal{M} \not\models w : \alpha$

$\mathcal{M} \models w : A \,\mathsf{nec}\, \alpha$ iff for every w' such that wN_Aw', we have $\mathcal{M} \models w' : \alpha$.

Note that for every $\mathcal{M}$ and every w, $\mathcal{M} \not\models w : \bot$.

A formula α is true in a model $\mathcal{M}$, written $\mathcal{M} \models \alpha$, if for every world $w \in \mathcal{M}$, $\mathcal{M} \models w : \alpha$. A formula α is *valid* in MM$^{\chi}$, written $\models \alpha$, if $\mathcal{M} \models \alpha$ for every model $\mathcal{M}$ satisfying all conditions in χ.

B.2 Seq-MM$^{\chi}$: A Labeled Sequent Calculus for MM$^{\chi}$

As a first step towards building a constructively complete decision procedure for logics MM$^{\chi}$, we build a sound, complete, cut-free sequent calculus for MM$^{\chi}$. Our calculus is presented in the so-called "labeled" style of calculi for modal logics, which means that the calculus proves formulas labeled with symbolic worlds. A labeled formula contains a symbol x, y, z, w, u denoting a world and a formula α, written together as $x : \alpha$. A sequent in our calculus has the form $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta$, where

- Σ is a finite set of world symbols appearing in the rest of the sequent. World symbols are also called *labels*.
- $\mathbb{M}$ is a finite set of relations between labels in Σ. Relations have the forms $x \leq y$ and xN_Ay.
- Γ is a finite set of labeled formulas.
- Δ is a finite set of labeled formulas.

The intuition is that if $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta$ is valid, then every model with a world set containing at least Σ, satisfying all relations in $\mathbb{M}$ and all labeled formulas in Γ must satisfy at least one labeled formula in Δ. This is formalized in the following definition.

Definition B.3 (Sequent satisfaction and validity)**.** A model $\mathcal{M}$ and a mapping ρ from elements of Σ to worlds of $\mathcal{M}$ satisfy a (possibly non-provable) sequent $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta$, written $\mathcal{M}, \rho \models (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$, if one of the following holds:

- There is an $xRy \in \mathbb{M}$ with $R \in \{N_A \mid A \in \mathcal{I}\}$ such that $\rho(x)\ R\ \rho(y) \notin \mathcal{M}$.
- There is an $x : \alpha \in \Gamma$ such that $\mathcal{M} \not\models \rho(x) : \alpha$.
- There is an $x : \alpha \in \Delta$ such that $\mathcal{M} \models \rho(x) : \alpha$.

A model $\mathcal{M}$ satisfies a sequent $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta$, written $\mathcal{M} \models (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$, if for every mapping ρ, we have $\mathcal{M}, \rho \models (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$. Finally, a sequent $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta$ is valid, written $\models (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$ if for every model $\mathcal{M}$, we have $\mathcal{M} \models (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$.

Rules of the sequent calculus. The sequent calculus for MM^{χ} is shown in Figure 5. Following standard approach in labeled calculi, the rules for each connective mimic the (Kripke) semantic definition of the connective. For example, in the rule ($\wedge$R), to prove $x : \alpha \wedge \beta$ in the conclusion, we prove $x : \alpha$ and $x : \beta$ in the premises. The rule (necR) introduces fresh worlds into Σ, consistent with the semantic definition (Definition B.2).

Theorem B.4 (Soundness)**.** *If* $\vdash (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$*, then* $\models (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$.

Proof. Fix an $\mathcal{M}$. It is easily proved by induction on the given derivation of $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta$ that for every mapping ρ, $\mathcal{M}, \rho \models (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$. □

The converse of Theorem B.4, completeness, also holds. It can be proved using a Henkin-style argument, but we do not need this result in the rest of our development so we do not present the proof here. For those logics MM^{χ} to which our method applies, completeness is also a consequence of the completeness of our decision procedure, which we prove later. The following property is critical to the design and correctness of our constructively complete decision procedure.

Theorem B.5 (Weak subformula property)**.** *If a formula φ appears in any proof tree (possibly infinite) obtained by applying the rules of Figure 5 backwards starting from a concluding sequent* $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta$*, then φ is a subformula of some formula in either Γ or Δ.*

Proof. By induction on the distance (in the proof tree) of the occurrence of φ from the conclusion $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta$. □

B.3 Suitable Closure Relations (SCRs)

We start the technical presentation of our constructively complete decision procedure by defining suitable closure relations (SCRs) for frame conditions χ. Our constructive decision procedure applies to any logic MM^{χ} whose χ has a SCR. Call a frame $\mathbb{M}$ *closed* if it is closed under the frame conditions χ. Let C denote a set of pairs of worlds in a frame $\mathbb{M}$. Define $\mathbb{M}(C)$ as the set obtained by closing $\mathbb{M}$ simultaneously under the frame conditions χ and the condition $\forall x, y, z.\ ((x, y) \in C \wedge yN_Az) \rightarrow xN_Az$.

Definition B.6 (Suitable closure relation (SCR)). A family of binary relations $(R(A))_{A\in\mathcal{I}}$ is called a suitable closure relation or SCR for χ if the following hold:

0. Each $R(A)$ is defined in terms of the relations $\{N_A \mid A \in \mathcal{I}\}$.
1. For a finite frame $\mathbb{M}$ and $x, y \in \mathbb{M}$, it can be decided whether $x(R(A))y$ or not.
2. $((R(A))^* \circ N_A) \subseteq N_A$ can be derived from the frame conditions χ.
3. For any *closed* frame $\mathbb{M}$ and any $C \subseteq \{(x, y) \mid x, y \in \mathbb{M}\}$, if $xN_Ay \in \mathbb{M}(C)$, then $x((R(A) \cup C)^* \circ N_A)y$, where all relations $R(A)$ and N_A on the right are in $\mathbb{M}$.

SCRs for many different χ are listed in Section B.9, but we describe one in the following example for illustration.

Example B.7 (SCR for transitivity). Let $\texttt{trans}(A)$ be the frame condition $\forall x, y, z.(((xN_Ay) \wedge (yN_Az)) \to (xN_Az))$ and let $\chi = \{\texttt{trans}(A) \mid A \in \mathcal{I}\}$. Then, the relation $R(A) = N_A$ is a SCR for χ. To prove this, we verify each of the conditions (0)–(3) in the definition of SCR. Conditions (0) and (1) are trivially true. (2) is equivalent to $(N_A^* \circ N_A) \subseteq N_A$, which follows from the frame conditions χ. To prove (3), suppose that $xN_Ay \in \mathbb{M}(C)$. Then, we prove that $x((N_A \cup C)^* \circ N_A)y$ in $\mathbb{M}$ by induction on the recursive process that derives $xN_Ay \in \mathbb{M}(C)$. There are three ways to derive $xN_Ay \in \mathbb{M}(C)$: (a) xN_Ay exists in $\mathbb{M}$ (base case), (b) $(x, z) \in C$ and $zN_Ay \in \mathbb{M}(C)$, or (c) $xN_Az \in \mathbb{M}(C)$ and $zN_Ay \in \mathbb{M}$. In case (a), we trivially have $x((N_A \cup C)^* \circ N_A)y$ in $\mathbb{M}$ by choosing 0 steps for the $(N_A \cup C)^*$. In case (b), by the i.h. we get $z((N_A \cup C)^* \circ N_A)y$ in $\mathbb{M}$ and, hence, from $(x, z) \in C$, that $x((N_A \cup C)^* \circ N_A)y$ in $\mathbb{M}$. In case (c), the i.h. yields both $x((N_A \cup C)^* \circ N_A)z$ and $z((N_A \cup C)^* \circ N_A)y$ in $\mathbb{M}$. This immediately implies $x(((N_A \cup C)^* \circ N_A) \circ ((N_A \cup C)^* \circ N_A))y$ in $\mathbb{M}$ and, therefore, $x((N_A \cup C)^* \circ N_A)y$ in $\mathbb{M}$.

B.4 Saturated Histories

Our method applies only to those logics MM^χ, whose χ has a SCR, so in the sequel we fix a set of frame conditions χ and assume there is a SCR $(R(A))_{A\in\mathcal{I}}$ for this χ. Although we present the technical material generically with respect to an abstract χ and SCR, we strongly advise the reader to choose a specific χ and its SCR (for instance, those from Example B.7), and instantiate our definitions and theorems on them.

A *history* is a tuple $\Sigma; \mathbb{M}; \Gamma; \Delta$ (equivalently, a sequent $\Sigma; \mathbb{M}; \Gamma \to^\chi \Delta$) such that all labels in $\mathbb{M}$, Γ and Δ occur in Σ. Let $T(\varphi)$ and $F(\varphi)$ be two uninterpreted unary relations. Informally, we read $T(\varphi)$ as "φ should be true" and $F(\varphi)$ as "φ should be false". Given a history $\Sigma; \mathbb{M}; \Gamma; \Delta$ and $x \in \Sigma$, the *signed formulas* of x, written $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$ are defined as follows:

$$\begin{gathered}\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x) = \\ \{T(\varphi) \mid x : \varphi \in \Gamma\} \cup \\ \{F(\varphi) \mid x : \varphi \in \Delta\} \cup \\ \{T(A \ \mathsf{nec}\ \varphi) \mid \exists y.\ y(R(A))^* x \in \mathbb{M} \text{ and } y : A \ \mathsf{nec}\ \varphi \in \Gamma\}\end{gathered}$$

When $\Sigma, \mathbb{M}, \Gamma, \Delta$ are clear from context, we abbreviate $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$ to $\mathrm{Sfor}(x)$. We say that $x \preccurlyeq y$ iff $\mathrm{Sfor}(x) \subseteq \mathrm{Sfor}(y)$.

We call a frame $\mathbb{M}$ *tree-like* if it can be derived from a finite tree of the relations N_A and (possibly partial) closure by frame rules. This tree is called the underlying tree of $\mathbb{M}$ and we say that $x \ll y$ (in $\mathbb{M}$) iff there is a directed path from x to y in the tree underlying $\mathbb{M}$.

The key definition in our method is that of a *saturated history*. Intuitively, this definition characterizes those histories $\Sigma;\mathbb{M};\Gamma;\Delta$ for which we can directly define a countermodel for the sequent $\Sigma;\mathbb{M};\Gamma \rightarrow^{\chi} \Delta$. (The definition of this countermodel is given soon after the definition of a saturated history.)

Definition B.8 (Saturated history). A history $\Sigma;\mathbb{M};\Gamma;\Delta$ is called saturated if the following hold:

1. $\mathbb{M}$ is tree-like and saturated with respect to the frame conditions χ. (In particular, because $\mathbb{M}$ is tree-like, it has a relation $\ll$ defined on it.)

2. If $x : p \in \Gamma$, then $x : p \notin \Delta$.

3. There is no x such that $x : \top \in \Delta$.

4. If $x : \alpha \wedge \beta \in \Gamma$, then $x : \alpha \in \Gamma$ and $x : \beta \in \Gamma$.

5. If $x : \alpha \wedge \beta \in \Delta$, then either $x : \alpha \in \Delta$ or $x : \beta \in \Delta$.

6. If $x : \neg\alpha \in \Gamma$, then $x : \alpha \in \Delta$.

7. If $x : \neg\alpha \in \Delta$, then $x : \alpha \in \Gamma$.

8. If $x : A\ \mathsf{nec}\ \alpha \in \Gamma$ and $xN_Ay \in \mathbb{M}$, then $y : \alpha \in \Gamma$.

9. If $x : A\ \mathsf{nec}\ \alpha \in \Delta$, then either:
   (a) There is a y such that $xN_Ay \in \mathbb{M}$ and $y : \alpha \in \Delta$ or
   (b) There is a y such that $y \neq x$, $y \ll x$ and $x \preccurlyeq y$.

Definition B.9 (Countermodel of a saturated history). For a saturated history $\Sigma;\mathbb{M};\Gamma;\Delta$, the *countermodel* of the history, $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta)$ is defined as follows:

- The worlds of $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta)$ are exactly those in Σ.

- The relations of $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta)$ are $\mathbb{M}(C)$, where $C = \{(x, y) \mid x \preccurlyeq y\}$.

- $h(p) = \{x \mid x : p \in \Gamma\}$.

The next Lemma states the central property of our method. In particular, the Lemma immediately implies that if $\Sigma;\mathbb{M};\Gamma;\Delta$ is a saturated history, then $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta)$ is a countermodel to the sequent $\Sigma;\mathbb{M};\Gamma \rightarrow^{\chi} \Delta$.

Lemma B.10. *The following hold for any saturated history* $\Sigma;\mathbb{M};\Gamma;\Delta$*:*

A. *If* $T(\varphi) \in$ *Sfor*$(\Sigma;\mathbb{M};\Gamma;\Delta, x)$*, then CM*$(\Sigma;\mathbb{M};\Gamma;\Delta) \models x : \varphi$

B. *If* $F(\varphi) \in$ *Sfor*$(\Sigma;\mathbb{M};\Gamma;\Delta, x)$*, then CM*$(\Sigma;\mathbb{M};\Gamma;\Delta) \not\models x : \varphi$

Proof. We prove both properties simultaneously by *lexicographic* induction, first on φ, and then on the partial (tree-like) order $\ll$ of $\mathbb{M}$. (Note that we cannot induct on either $\mathbb{M}$ or the relation in $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta)$, because both of these may potentially be cyclic.) Since the context $\Sigma;\mathbb{M};\Gamma;\Delta$ is fixed here, we abbreviate $\mathrm{Sfor}(\Sigma;\mathbb{M};\Gamma;\Delta, x)$ to $\mathrm{Sfor}(x)$. Let C be the set $\{(x, y) \mid x \preccurlyeq y\}$.

Proof of A.

Case. $\varphi = p$. We are given that $T(p) \in \mathrm{Sfor}(x)$ and want to prove that $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \models x : p$. Since $T(p) \in \mathrm{Sfor}(x)$, we know from definition of the function Sfor that $x : p \in \Gamma$. Therefore, $x \in h(p)$. So, $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \models x : p$.

Case. $\varphi = \top$. Here, $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \models x : \top$ is trivial by the definition of $\models$.

Case. $\varphi = \alpha \wedge \beta$. We are given that $T(\alpha \wedge \beta) \in \mathrm{Sfor}(x)$ or, equivalently, that $x : \alpha \wedge \beta \in \Gamma$. By clause (4) of the definition of saturated history, $x : \alpha \in \Gamma$ and $x : \beta \in \Gamma$. Hence, $T(\alpha) \in \mathrm{Sfor}(x)$ and $T(\beta) \in \mathrm{Sfor}(x)$. By the i.h., $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \models x : \alpha$ and $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \models x : \beta$. Hence, $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \models x : \alpha \wedge \beta$, as required.

Case. $\varphi = \neg\alpha$. We are given that $T(\neg\alpha) \in \mathrm{Sfor}(x)$ and want to show that $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \models x : \neg\alpha$. Since $T(\neg\alpha) \in \mathrm{Sfor}(x)$, we also have $x : \neg\alpha \in \Gamma$, so by clause (6) of the definition of saturated history, $x : \alpha \in \Delta$. So, $F(\alpha) \in \mathrm{Sfor}(x)$ and by the i.h., $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \not\models x : \alpha$. This implies $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \models x : \neg\alpha$ by definition of $\models$.

Case. $\varphi = A \mathsf{\ nec\ } \alpha$. We are given that $T(A \mathsf{\ nec\ } \alpha) \in \mathrm{Sfor}(x)$. We need to show that $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \models x : A \mathsf{\ nec\ } \alpha$, i.e., for any y such that xN_Ay in the model, we have $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \models y : \alpha$. Pick any y such that xN_Ay in the model. Because of the definition of SCR, clause (3), we have $x((R(A) \cup C)^* \circ N_A)y$, where the relations $R(A)$ and N_A are in $\mathbb{M}$. So there are $x_0, \ldots, x_n$ such that $x = x_0(R(A) \cup C)x_1 \ldots (R(A) \cup C)x_nN_Ay$. We now prove, by induction on i, that $T(A \mathsf{\ nec\ } \alpha) \in \mathrm{Sfor}(x_i)$ for each i.

- For $i = 0$, $x_0 = x$ and we are given that $T(A \mathsf{\ nec\ } \alpha) \in \mathrm{Sfor}(x)$.
- For the inductive case, assume that $T(A \mathsf{\ nec\ } \alpha) \in \mathrm{Sfor}(x_i)$ for some i. We show that $T(A \mathsf{\ nec\ } \alpha) \in \mathrm{Sfor}(x_{i+1})$ by case analyzing the relation $x_i(R(A) \cup C)x_{i+1}$.
  - $x_i(R(A))x_{i+1} \in \mathbb{M}$: By the i.h., $T(A \mathsf{\ nec\ } \alpha) \in \mathrm{Sfor}(x_i)$ so there is some z such that $z(R(A))^*x_i \in \mathbb{M}$ and $z : A \mathsf{\ nec\ } \alpha \in \Gamma$. Clearly, we have $z(R(A))^*x_{i+1} \in \mathbb{M}$, so $T(A \mathsf{\ nec\ } \alpha) \in \mathrm{Sfor}(x_{i+1})$.
  - $(x_i, x_{i+1}) \in C$: Because of the definition of C, $\mathrm{Sfor}(x_{i+1}) \supseteq \mathrm{Sfor}(x_i)$. Thus, $T(A \mathsf{\ nec\ } \alpha) \in \mathrm{Sfor}(x_i)$ immediately implies $T(A \mathsf{\ nec\ } \alpha) \in \mathrm{Sfor}(x_{i+1})$.

Since we just proved that $T(A \mathsf{\ nec\ } \alpha) \in \mathrm{Sfor}(x_i)$, it follows, in particular, that $T(A \mathsf{\ nec\ } \alpha) \in \mathrm{Sfor}(x_n)$. Consequently, there is some z' such that $z'(R(A))^*x_n \in \mathbb{M}$ and $z' : A \mathsf{\ nec\ } \alpha \in \Gamma$. Then, we also have (within $\mathbb{M}$) that: $z'(R(A))^*x'_nN_Ay$. So, by clause (2) of the definition of SCR, $z'N_Ay \in \mathbb{M}$. Hence, by clause (7) of the definition of saturated history, we must have $y : \alpha \in \Gamma$. Therefore, $T(\alpha) \in \mathrm{Sfor}(y)$ and by the i.h., $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \models y : \alpha$.

Proof of B.

Case. $\varphi = p$. We are given that $F(p) \in \mathrm{Sfor}(x)$ or, equivalently, that $x : p \in \Delta$. Suppose, for the sake of contradiction, that $\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta) \models x : p$. Then, $x \in h(p)$ and hence, $x : p \in \Gamma$. This immediately contradicts clause (2) of the definition of saturated history because we have $x : p \in \Gamma$

and $x : p \in \Delta$. Hence we must have $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : p$.

Case. $\varphi = \top$. Then the pre-condition $F(\top) \in \mathrm{Sfor}(x)$ or, equivalently, $x : \top \in \Delta$ is impossible by clause (3) of the definition of saturated history. So this case is vacuous.

Case. $\varphi = \alpha \wedge \beta$. Suppose $F(\alpha \wedge \beta) \in \mathrm{Sfor}(x)$. Then, $x : \alpha \wedge \beta \in \Delta$. Hence, by clause (5) of the definition of saturated history, either $x : \alpha \in \Delta$ or $x : \beta \in \Delta$. Therefore, either $F(\alpha) \in \mathrm{Sfor}(x)$ or $F(\beta) \in \mathrm{Sfor}(x)$. By i.h., either $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \alpha$ or $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \beta$. In either case, $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \alpha \wedge \beta$.

Case. $\varphi = \neg\alpha$. We are given that $F(\neg\alpha) \in \mathrm{Sfor}(x)$ and want to show that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \neg\alpha$. Since $F(\neg\alpha) \in \mathrm{Sfor}(x)$, we also have $x : \neg\alpha \in \Delta$, so by clause (7) of the definition of saturated history, $x : \alpha \in \Gamma$. So, $T(\alpha) \in \mathrm{Sfor}(x)$ and by the i.h., $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : \alpha$. This implies $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \neg\alpha$ by definition of $\models$.

Case. $\varphi = A \ \mathsf{nec}\ \alpha$. Suppose $F(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(x)$. This implies, by definition of Sfor that $x : A \ \mathsf{nec}\ \alpha \in \Delta$. By clause (9) of the definition of saturated history, we have that either:

(a) There is a y such that $xN_Ay \in \mathbb{M}$ and $y : \alpha \in \Delta$ or

(b) There is a y such that $y \neq x$, $y \ll x$ and $x \preccurlyeq y$.

If (a) holds, then by the i.h., $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models y : \alpha$. Since xN_Ay, it immediately follows that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : A \ \mathsf{nec}\ \alpha$.

If (b) holds, then since $x \preccurlyeq y$, $F(A \ \mathsf{nec}\ \alpha) \in \mathrm{Sfor}(y)$. By the i.h. on the world y, which is strictly smaller in the order $\ll$ (since $x \neq y$), it follows that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models y : A \ \mathsf{nec}\ \alpha$. So, there is a world z such that yN_Az (in $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$) and $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models z : \alpha$. Note that in $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ we also have xN_Ay (because $x \preccurlyeq y$ and yN_Az). So, by definition of $\models$, we have $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : A \ \mathsf{nec}\ \alpha$, as required. □

Corollary B.11 (Existence of countermodel)**.** *If* $\Sigma; \mathbb{M}; \Gamma; \Delta$ *is a saturated history, then CM*$(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$.

Proof. Lemma B.10 immediately implies that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta), \rho \not\models (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$, where $\rho : \Sigma \rightarrow \Sigma$ is the identity substitution. □

B.5 Seq-MM$^{\chi}_{\mathrm{CM}}$: Countermodels for MM$^{\chi}$

Having defined a saturated history, i.e., a sequent for which a countermodel exists (Corollary B.11), we now define a sequent calculus Seq-MM$^{\chi}_{\mathrm{CM}}$, written $\rightarrow^{\chi}_{\mathrm{CM}}$, which uses this fact to emit countermodels from unprovable sequents. Although this calculus is not a decision procedure, we find it a useful step in proving several results, in particular, the results of Section B.7.

Sequents of Seq-MM$^{\chi}_{\mathrm{CM}}$ have the form $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S$, where S is a finite set of finite models. We write $\vdash (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S)$ if $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S$ has a proof. The meaning of $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S$ depends on S. If $\vdash (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow \{\})$, then $\vdash (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$ and if $\vdash (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S)$ with $S \neq \{\}$, then every model $\mathcal{M} \in S$ is a countermodel to $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta$ in the sense of (the converse of) Definition B.3.

Axiom Rules

$$\frac{\Sigma; \mathbb{M}; \Gamma; \Delta \text{ is a saturated history}}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow \{\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)\}}\mathrm{CM} \qquad \frac{}{\Sigma; \mathbb{M}; \Gamma, x : p \rightarrow^{\chi}_{\mathrm{CM}} x : p, \Delta \searrow \{\}}\mathrm{init}$$

Logical Rules

$$\frac{}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} x : \top, \Delta \searrow \{\}}\top\mathrm{R}$$

$$\frac{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} x : \alpha, x : \alpha \wedge \beta, \Delta \searrow S_1 \qquad \Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} x : \beta, x : \alpha \wedge \beta, \Delta \searrow S_2}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} x : \alpha \wedge \beta, \Delta \searrow S_1, S_2}\wedge\mathrm{R}$$

$$\frac{\Sigma; \mathbb{M}; \Gamma, x : \alpha \wedge \beta, x : \alpha, x : \beta \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}{\Sigma; \mathbb{M}; \Gamma, x : \alpha \wedge \beta \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}\wedge\mathrm{L} \qquad \frac{\Sigma; \mathbb{M}; \Gamma, x : \alpha \rightarrow^{\chi} x : \neg\alpha, \Delta \searrow S}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} x : \neg\alpha, \Delta \searrow S}\neg\mathrm{R}$$

$$\frac{\Sigma; \mathbb{M}; \Gamma, x : \neg\alpha \rightarrow^{\chi} x : \alpha, \Delta \searrow S}{\Sigma; \mathbb{M}; \Gamma, x : \neg\alpha \rightarrow^{\chi} \Delta \searrow S}\neg\mathrm{L} \qquad \frac{\Sigma, y; \mathbb{M}, xN_Ay; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} y : \alpha, x : A \;\mathsf{nec}\; \alpha, \Delta \searrow S}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} x : A \;\mathsf{nec}\; \alpha, \Delta \searrow S}\mathsf{necR}$$

$$\frac{\Sigma; \mathbb{M}, xN_Ay; \Gamma, x : A \;\mathsf{nec}\; \alpha, y : \alpha \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}{\Sigma; \mathbb{M}, xN_Ay; \Gamma, x : A \;\mathsf{nec}\; \alpha \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}\mathsf{necL}$$

Frame Rules

$$\frac{(\forall \vec{x}.((\wedge_i\ (x_i R_i x_i')) \rightarrow (xRx'))) \in \chi \qquad x_i R_i x_i' \in \mathbb{M} \qquad \Sigma; \mathbb{M}, xRx'; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}\chi$$

Figure 6: Seq-MM$^{\chi}_{\mathrm{CM}}$: Countermodel producing sequent calculus for MM$^{\chi}$

The rules of the sequent calculus Seq-MM$^{\chi}_{\mathrm{CM}}$ are shown in Figure 6. First, every rule in the ordinary sequent calculus (Figure 5) is modified to have in the conclusion the union of the (counter)models in the premises. This is sound because the rules of the sequent calculus are invertible (i.e., the conclusion of each rule holds iff the premises hold). Second, there is a new rule (CM) that produces the countermodel $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ when $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history.

We emphasize again that this calculus is not necessarily a decision procedure because it includes all rules of $\rightarrow^{\chi}$ and hence admits all of the latter's infinite backwards derivations as well.

Theorem B.12 (Soundness 1)**.** *If* $\vdash (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{CM} \Delta \searrow \{\})$, *then* $\vdash (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$.

Proof. By induction on the given derivation of $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow \{\}$. Note that the case of rule (CM) does not apply because the set of countermodels in it is non-empty. The proof is straightforward because the rules of $\rightarrow^{\chi}_{\mathrm{CM}}$ mimic those of $\rightarrow^{\chi}$. □

Theorem B.13 (Soundness 2)**.** *If* $\vdash (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{CM} \Delta \searrow S)$, *then for every model* $\mathcal{M} \in S$, $\mathcal{M} \not\models (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$.

Proof. By induction on the given derivation of $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S$ and case analysis of its last rule. The rules (init), ($\perp$L), and ($\top$R) are vacuous because they have empty S. For all other rules, except (CM), we simply observe that contexts in all major premises are a superset of corresponding contexts in the conclusion and hence we can trivially conclude by induction on one of the premises. The case of rule (CM) is shown below:

Case. $\dfrac{\Sigma; \mathbb{M}; \Gamma; \Delta \text{ is a saturated history}}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow \{\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)\}}\mathrm{CM}$

Here $\mathcal{M} = \mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$. So, the result follows by Corollary B.11. □

B.6 Seq-MM$^{\chi}_{\mathrm{T}}$: Termination and Countermodel Extraction for MM$^{\chi}$

Next, we describe a particular backwards proof search strategy in Seq-MM$^{\chi}_{\mathrm{CM}}$ that always terminates without losing completeness, thus obtaining a countermodel producing decision procedure for MM$^{\chi}$. This strategy is described as a calculus Seq-MM$^{\chi}_{\mathrm{T}}$, with sequents of the form $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S$. Operationally, the rules of the calculus can be interpreted backwards as a decision procedure with inputs Σ, $\mathbb{M}$, Γ, and Δ and output S. For a given Σ, $\mathbb{M}$, Γ, and Δ, $(\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$ is provable iff $S = \{\}$, else every model in S is a countermodel to the sequent.

The rules of the calculus Seq-MM$^{\chi}_{\mathrm{T}}$ are shown in Figure 7. Each rule in the calculus corresponds to a rule of the same name in Seq-MM$^{\chi}_{\mathrm{CM}}$ (Figure 6). The only significant difference between the two calculi is that the premise of the rule (CM) in Seq-MM$^{\chi}_{\mathrm{CM}}$ requires that $\Sigma; \mathbb{M}; \Gamma; \Delta$ be a saturated history, but the rule (CM) applies in Seq-MM$^{\chi}_{\mathrm{T}}$ only when no other rule applies. To ensure that "no other rule applies" implies that $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history, we spread the negations of the conditions (1) and (4)–(9) from the definition of saturated history as pre-conditions, called *applicability conditions*, to the other rules. Conditions (2) and (3) obviously hold when the rules (init) and ($\top$R) do not apply, respectively. Hence, when no rule other than (CM) applies, all 9 conditions of the definition of saturated history must hold, so $\Sigma; \mathbb{M}; \Gamma; \Delta$ must be a saturated history. The conditions are spread to the obvious rules; for example, the negation of condition (4) is applied to the rule ($\wedge$R). In Figure 7, applicability conditions are highlighted using $\boxed{\text{boxes}}$. It only remains to show that the calculus with these applicability conditions does not admit infinite backwards derivations. This follows from a straightforward combinatorial argument in Theorem B.16.

Lemma B.14 (Correctness of CM)**.** *Let Σ, $\mathbb{M}$, Γ and Δ be such that $\mathbb{M}$ is tree-like and no rule except (CM) applies backwards to $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{T} \Delta \searrow \ldots$. Then, $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history.*

Proof. We verify all conditions in the definition of a saturated history. Each condition corresponds to the negation of premises of one of the rules of Figure 7. □

Lemma B.15 (Tree-like $\mathbb{M}$)**.** *Let $\mathbb{M}$ be tree-like. Then, the $\mathbb{M}'$ in any sequent $\Sigma'; \mathbb{M}'; \Gamma' \rightarrow^{\chi}_{T} \Delta' \searrow \ldots$ appearing in a backwards search starting from $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{T} \Delta \searrow \ldots$ is tree-like.*

Proof. By backwards analysis of each rule observing that the $\mathbb{M}$ in the premises of each rule is tree-like if that in the conclusion is. □

Note that the underlying tree of $\mathbb{M}$ in any sequent of a backward proof search starting from a single formula consists of exactly those edges that are introduced in the rule (necR).

Theorem B.16 (Termination)**.** *The following hold:*

1. *Any backwards derivation in Seq-MM$^{\chi}_{T}$ starting from a sequent $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{T} \Delta$ with $\mathbb{M}$ tree-like terminates.*

$\boxed{\text{Axiom Rules}}$

$$\dfrac{\boxed{\text{No other rule applies}}}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow \{\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)\}}\,\mathrm{CM} \qquad \dfrac{}{\Sigma; \mathbb{M}; \Gamma, x : p \rightarrow^{\chi}_{\mathrm{T}} x : p, \Delta \searrow \{\}}\,\mathrm{init}$$

$\boxed{\text{Logical Rules}}$

$$\dfrac{}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{T}} x : \top, \Delta \searrow \{\}}\,\top\mathrm{R}$$

$$\dfrac{\boxed{x : \alpha \notin \Delta \text{ and } x : \beta \notin \Delta} \qquad \Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{T}} x : \alpha, x : \alpha \wedge \beta, \Delta \searrow S_1 \qquad \Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{T}} x : \beta, x : \alpha \wedge \beta, \Delta \searrow S_2}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{T}} x : \alpha \wedge \beta, \Delta \searrow S_1, S_2}\,\wedge\mathrm{R}$$

$$\dfrac{\boxed{x : \alpha \notin \Gamma \text{ or } x : \beta \notin \Gamma} \qquad \Sigma; \mathbb{M}; \Gamma, x : \alpha \wedge \beta, x : \alpha, x : \beta \rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}{\Sigma; \mathbb{M}; \Gamma, x : \alpha \wedge \beta \rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}\,\wedge\mathrm{L}$$

$$\dfrac{\boxed{x : \alpha \notin \Gamma} \qquad \Sigma; \mathbb{M}; \Gamma, x : \alpha \rightarrow^{\chi} x : \neg\alpha, \Delta \searrow S}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} x : \neg\alpha, \Delta \searrow S}\,\neg\mathrm{R} \qquad \dfrac{\boxed{x : \alpha \notin \Delta} \qquad \Sigma; \mathbb{M}; \Gamma, x : \neg\alpha \rightarrow^{\chi} x : \alpha, \Delta \searrow S}{\Sigma; \mathbb{M}; \Gamma, x : \neg\alpha \rightarrow^{\chi} \Delta \searrow S}\,\neg\mathrm{L}$$

$$\dfrac{\begin{array}{c}\boxed{\forall y \in \Sigma.(xN_A y \in \mathbb{M}) \Rightarrow y : \alpha \notin \Delta} \\ \boxed{\forall y \in \Sigma.(y \ll x) \Rightarrow (x = y \text{ or } x \not\preccurlyeq y)} \qquad \Sigma, y; \mathbb{M}, xN_A y; \Gamma \rightarrow^{\chi}_{\mathrm{T}} y : \alpha, x : A \;\mathsf{nec}\; \alpha, \Delta \searrow S\end{array}}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{T}} x : A \;\mathsf{nec}\; \alpha, \Delta \searrow S}\,\mathsf{necR}$$

$$\dfrac{\boxed{y : \alpha \notin \Gamma} \qquad \Sigma; \mathbb{M}, xN_A y; \Gamma, x : A \;\mathsf{nec}\; \alpha, y : \alpha \rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}{\Sigma; \mathbb{M}, xN_A y; \Gamma, x : A \;\mathsf{nec}\; \alpha \rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}\,\mathsf{necL}$$

$\boxed{\text{Frame Rules}}$

$$\dfrac{(\forall \vec{x}.((\wedge_i\ (x_i R_i x_i')) \rightarrow (xRx'))) \in \chi \qquad x_i R_i x_i' \in \mathbb{M} \qquad \boxed{xRx' \notin \mathbb{M}} \qquad \Sigma; \mathbb{M}, xRx'; \Gamma \rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}{\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}\,\chi$$

Figure 7: Seq-MM$^{\chi}_{\mathrm{T}}$: Terminating, countermodel producing sequent calculus for MM$^{\chi}$. Applicability conditions are written in $\boxed{\text{boxes}}$. Wherever mentioned, the relation $\preccurlyeq$ is the equivalence relation of the contexts $\Sigma; \mathbb{M}; \Gamma; \Delta$ in the conclusion of the rule. Similarly, $\ll$ is the order of the underlying tree of $\mathbb{M}$.

2. *For any* $\Sigma; \mathbb{M}; \Gamma; \Delta$ *with* $\mathbb{M}$ *tree-like, there is an* S *such that* $\vdash (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{T} \Delta \searrow S)$ *and such an* S *can be finitely computed.*

Proof. Proof of (1): Suppose, for the sake of contradiction, that there is an infinite backward proof starting from $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{T}} \Delta$. Since the proof is finitely branching (every rule has a bounded number of premises), it must have an infinite path. Observe that Γ, Δ are monotonic backwards, so the applicability conditions in the rules prevent application of the same rule on the same principal *labeled formula* more than once in any branch. Since there are only a finite number of formulas that can appear in any search (weak subformula property, Theorem B.5), it follows that in the infinite path there must be an infinite number of labels. Let T be the underlying tree of this entire path (i.e., the underlying tree of the union of $\mathbb{M}$ for each sequent on this path). Since the tree is finitely branching (because we cannot apply (necR) to the same label infinitely often), it must have an infinite path. Let this path be $x_0 \ll x_1 \ll \ldots$. Let S_i be the value of $\mathrm{Sfor}(x_i)$ when the rule (necR) is applied to create x_{i+1}. Note that for $i < j$, $S_i \not\supseteq S_j$, because if $S_i \supseteq S_j$, then at the time that x_{j+1} is created, $\mathrm{Sfor}(x_i) \supseteq S_i \supseteq S_j = \mathrm{Sfor}(x_j)$, so the application of the rule (necR) on x_j would be blocked, so x_{j+1} could not have been created. Hence, $i < j$, $S_i \not\supseteq S_j$. Call this fact (A). (The reader may note that the deduction $\mathrm{Sfor}(x_i) \supseteq S_i$ two sentences ago relies on the fact that $\mathrm{Sfor}(x)$ increases monotonically as we move backwards in a derivation.)

If Φ is the set of all subformulas of the original sequent we start from, then by Theorem B.5, each $S_i \subseteq \{T(\alpha) \mid \alpha \in \Phi\} \cup \{F(\alpha) \mid \alpha \in \Phi\}$. Note that the right hand side is a finite set, so its subsets form a finite partial order under set inclusion. Call this partial order P. Since P is finite, it has a finite number of chains and since the sequence $S_1, S_2, \ldots$ is infinite, at least one infinite subsequence R of $S_1, S_2, \ldots$ must contain elements from only a single chain in P. On any two elements S_i and S_j of R with $i < j$, fact (A) forces $S_i \subsetneq S_j$. This is a contradiction because R is a strictly ascending, infinite chain in a finite partial order P.

Proof of (2): Follows immediately from (1), Lemma B.15, and the observation that all applicability conditions are finitely computable. The latter follows from condition (1) of the definition of SCRs. □

Note that Theorem B.16(2) does not stipulate that the S be unique. Indeed, depending on the order in which the rules of the calculus $\rightarrow^{\chi}_{\mathrm{T}}$ are applied to a given sequent, S may be different. However, the fact that at least one such S exists and can be computed is enough to get decidability for MM^{χ}.

Lemma B.17 (Simulation)**.** *If* $\mathbb{M}$ *is tree-like and* $\vdash (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{T} \Delta \searrow S)$*, then* $\vdash (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{CM} \Delta \searrow S)$*.*

Proof. By induction on the given derivation of $\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S$. The case of rule (CM) follows from Lemma B.14. The rest of the cases are immediate from the i.h. The only fact to take care of is that the tree-like property holds for each i.h. application. This follows from Lemma B.15. □

Theorem B.18 (Decidability)**.** *For a tree-like* $\mathbb{M}$*, suppose that* S *is such that* $\vdash (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi}_{T} \Delta \searrow S)$ *(such an* S *must exist and can be computed using Theorem B.16). Then:*

1. *If* $S = \{\}$*, then* $\models (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$*.*

2. *If* $S \neq \{\}$*, then every model* $\mathcal{M}$ *in* S *is a countermodel to the sequent, i.e.,* $\mathcal{M} \not\models (\Sigma; \mathbb{M}; \Gamma \rightarrow^{\chi} \Delta)$*.*

Proof. By Lemma B.17, we have that $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{CM}} \Delta \searrow S)$. Now, (1) follows from Theorems B.12 and B.4 and (2) follows from Theorem B.13. $\square$

Corollary B.19 (Decidability and finite model property)**.** *If a SCR exists for χ, then MM^{χ} is decidable, has the finite model property and has a constructive decision procedure.*

Proof. Immediate from Theorem B.18. $\square$

B.7 Comprehensiveness of Seq-MM$^{\chi}_{\mathbf{CM}}$ and Seq-MM$^{\chi}_{\mathbf{T}}$ Countermodels

Countermodels generated by Seq-MM$^{\chi}_{\mathrm{CM}}$ (and Seq-MM$^{\chi}_{\mathrm{T}}$) have an interesting property: If $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{CM}} \Delta \searrow S)$, then $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{CM}} x : p, \Delta \searrow \{\})$ if and only if $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$. Thus, if we can produce a set of countermodels S by running without an actual goal (like $x : p$), then the set of atoms that are actually true are exactly those that are in the intersection of the valuation of all models in the set S. Further, because the result applies to derivations in Seq-MM$^{\chi}_{\mathrm{CM}}$, it also applies to derivations in Seq-MM$^{\chi}_{\mathrm{T}}$ due to Lemma B.17 and the latter can be used to actually produce the set S. We prove this result below.

Lemma B.20 (Comprehensiveness 1)**.** *Suppose $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{CM} \Delta \searrow S)$. Suppose x and p are such that $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$. Then, $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{CM} x : p, \Delta \searrow \{\})$.*

Proof. By induction on the given derivation of $\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{CM}} \Delta \searrow S$ and case analysis of its last rule (the rules are listed in Figure 6).

Case. $$\frac{\Sigma; \mathbb{M}; \Gamma; \Delta \text{ is a saturated history}}{\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{CM}} \Delta \searrow \{\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)\}}\,\mathrm{CM}$$

Here $S = \{\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)\}$. The given condition $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$ implies (by definition of CM) that $x : p \in \Gamma$. Therefore, by rule (init), $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{CM}} x : p, \Delta \searrow \{\})$, as required.

Case. $$\frac{}{\Sigma; \mathbb{M}; \Gamma, y : q \to^{\chi}_{\mathrm{CM}} y : q, \Delta \searrow \{\}}\,\mathrm{init}$$

By rule (init), we have $\vdash (\Sigma; \mathbb{M}; \Gamma, y : q \to^{\chi}_{\mathrm{CM}} x : p, y : q, \Delta \searrow \{\})$, which is what we need to prove.

Case. $$\frac{}{\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{CM}} y : \top, \Delta \searrow \{\}}\,\top\mathrm{R}$$

By rule ($\top$R), $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{CM}} x : p, y : \top, \Delta \searrow \{\})$, as required.

Case. $$\frac{\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{CM}} y : \alpha, y : \alpha \wedge \beta, \Delta \searrow S_1 \qquad \Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{CM}} y : \beta, y : \alpha \wedge \beta, \Delta \searrow S_2}{\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{CM}} y : \alpha \wedge \beta, \Delta \searrow S_1, S_2}\,\wedge\mathrm{R}$$

Here, $S = S_1, S_2$. We are given that $\forall \mathcal{M} \in (S_1, S_2).\ \mathcal{M} \models x : p$.

1. $\forall \mathcal{M} \in S_1.\ \mathcal{M} \models x : p$ (From assumption $\forall \mathcal{M} \in (S_1, S_2).\ \mathcal{M} \models x : p$)
2. $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{CM}} x : p, y : \alpha, y : \alpha \wedge \beta, \Delta \searrow \{\})$ (i.h. on 1st premise and (1))
3. $\forall \mathcal{M} \in S_2.\ \mathcal{M} \models x : p$ (From assumption $\forall \mathcal{M} \in (S_1, S_2).\ \mathcal{M} \models x : p$)

4. $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\text{CM}} x : p, y : \beta, y : \alpha \wedge \beta, \Delta \searrow \{\})$ (i.h. on 2nd premise and (2))

5. $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\text{CM}} x : p, y : \alpha \wedge \beta, \Delta \searrow \{\})$ (Rule ($\wedge$R) on 2,4)

Case. All other cases are similar to the case of ($\wedge$R) above: We apply the i.h. to the premises and reapply the rule. □

Lemma B.21 (Comprehensiveness 2)**.** *Suppose* $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{CM} \Delta \searrow S)$*. Suppose* x *and* p *are such that* $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{CM} x : p, \Delta \searrow \{\})$*. Then,* $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$*.*

Proof. Suppose $\mathcal{M} \in S$. From Theorem B.13, we know that (1) $\forall w, w' \in \Sigma.\ (wRw' \in \mathbb{M}) \Rightarrow (wRw' \in \mathcal{M})$, (2) $\forall (w : \varphi) \in \Gamma.\ \mathcal{M} \models w : \varphi$ and (3) $\forall (w : \varphi) \in \Delta.\ \mathcal{M} \not\models w : \varphi$. By Theorem B.12 applied to the assumption $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\text{CM}} x : p, \Delta \searrow \{\})$, we know that $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi} x : p, \Delta)$. Applying Theorem B.4, we get that $\mathcal{M}, \rho \models (\Sigma; \mathbb{M}; \Gamma \to^{\chi} x : p, \Delta)$ for every ρ and, in particular, for $\rho(x) = x$. Using (1)–(3) and the definition of $\models$ on sequents, we immediately get $\mathcal{M} \models x : p$, as required. □

Theorem B.22 (Comprehensiveness)**.** *Suppose* $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{CM} \Delta \searrow S)$*. Then* $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{CM} x : p, \Delta \searrow \{\})$ *iff* $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$*.*

Proof. Lemmas B.20 and B.21 each state one direction of this theorem. □

Corollary B.23 (Comprehensiveness in Seq-MM$^{\chi}_{\text{T}}$)**.** *Suppose* $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{T} \Delta \searrow S)$*. Then,* $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi} x : p, \Delta)$ *iff* $\forall \mathcal{M} \in S.\ \mathcal{M} \models x : p$*.*

Proof. From $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\text{T}} \Delta \searrow S)$ we derive $\vdash (\Sigma; \mathbb{M}; \Gamma \to^{\chi}_{\text{CM}} \Delta \searrow S)$ using Lemma B.17. The result then follows from Theorem B.22. □

B.8 Adding Seriality

In this section, we show that if χ has a SCR, then our method applies not only to the logic MM$^{\chi}$ (Corollary B.19), but also to the logic whose valid formulas are exactly those that are true in all frames satisfying χ as well as a condition known as *seriality*: $\forall A, x. \exists y.(xN_A y)$. This standard condition corresponds to the axiom $\neg(A\ \mathsf{nec}\ \bot)$, also called (D) in literature [5]. Note that seriality does not fit our definition of χ because frame conditions in χ cannot contain existentials, so it cannot be handled in the method described so far. Consequently, we must modify our method slightly to include seriality as a frame condition. The only new challenge is to control creation of worlds due to the seriality condition during backwards search; for this we use an approach similar to that for controlling the use of the rule (necR). Proofs not related to termination do not change significantly.

To accommodate the seriality condition in our method, we must first add the following rule to our sequent calculus Seq-MM$^{\chi}$ (Figure 5):

$$\frac{\Sigma, x, y; \mathbb{M}, xP_A y, xN_A y; \Gamma \to^{\chi} \Delta}{\Sigma, x; \mathbb{M}; \Gamma \to^{\chi} \Delta}\text{D}$$

The resulting calculus is sound and complete with respect to the semantics of the modal logic MM$^{\chi}$ with the additional frame condition (D). Next, we change clause (1) of the definition of saturated history *not* to require closure under this new frame condition, which would cause infinite models, but instead new conditions based on $\preccurlyeq$:

Logic	Frame conditions χ	Additional Axioms	SCR
K	$\{\}$	–	$R(A) = \{\}$
T	$\forall A, x.\ xN_Ax$	$(A \text{ nec } \alpha) \to \alpha$	$R(A) = \{\}$
K4	$\forall A, x, y, z.\ ((xN_Ay) \wedge (yN_Az)) \to (xN_Az)$	$(A \text{ nec } \alpha) \to (A \text{ nec } A \text{ nec } \alpha)$	$R(A) = N_A$
S4	Conditions of K4 and T	Axioms of K4 and T	$R(A) = N_A$
–	$\forall x, y.\ (xN_Ay) \to (xN_By)$	$(B \text{ nec } \alpha) \to (A \text{ nec } \alpha)$	$R(A) = \{\}$

Figure 8: SCRs for some multi-modal classical logics. All these logics are constructively decidable by our method.

1. $\mathbb{M}$ is tree-like and saturated with respect to the rules (refl), (trans), (mon-N) and χ. In addition, at least one of the following must hold for each $x \in \Sigma$ and each index $A \in \mathcal{I}$:
   (a) There is a $y \in \Sigma$ such that $xN_Ay \in \mathbb{M}$
   (b) There is a $y \in \Sigma$ such that $y \neq x$, $y \ll x$ and $x \preccurlyeq y$.

With this new clause (1), we can show by induction on $\ll$ that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ is closed under (D), hence it is a model of our (modified) logic. Next, we add the following rule to the terminating calculus Seq-MM$^{\chi}_{\mathrm{T}}$ and a corresponding rule without the applicability conditions to Seq-MM$^{\chi}_{\mathrm{CM}}$.

$$\frac{\boxed{\forall y \in \Sigma.\ (xN_Ay \notin \mathbb{M})} \quad \boxed{\forall y \in \Sigma. (y \ll x) \Rightarrow (y = x \text{ or } x \not\preccurlyeq y)} \quad \Sigma, x, y; \mathbb{M}, xP_Ay, xN_Ay; \Gamma \to^{\chi}_{\mathrm{T}} \Delta \searrow S}{\Sigma, x; \mathbb{M}; \Gamma \to^{\chi}_{\mathrm{T}} \Delta \searrow S}\,\mathrm{D}$$

With these changes, our entire development works with only two minor changes to the proofs (interestingly, the proof of Lemma B.10 does not change): (1) We must change the termination argument in Theorem B.16 to also include the rule (D). This is trivial. (2) We must add a case for rule (D) to every proof that inducts on sequent derivations, but every such case is trivial.

Theorem B.24 (Constructive decidability with seriality)**.** *Suppose the frame conditions χ have a SCR. Then the logic $MM^{\chi,D}$ is constructively decidable by our method.*

B.9 Constructive Decidability for Common Logics

In Figure 8, we list some common sets of frame conditions with their SCRs, thus showing that the classical logics corresponding to each of them is constructively decidable by our method. We note two things: (1) This list is not exhaustive, but merely representative, and (2) Our method also applies to any of these logics combined with seriality from Section B.8 due to Theorem B.24.

Theorem B.25 (Decidability of Common Logics)**.** *The classical logics shown in Figure 8 have the SCRs also shown in that figure. Consequently, all these logics (and their combination with the seriality condition from Section B.8) are constructively decidable by our method.*