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Testing Read-Once Functions over the Elementary Basis

We prove a universal upper bound on checking test length for read-once functions over the elementary basis. We also identify the exact value of the corresponding Shannon function for the basis of conjunction and disjunction.

Key words: checking test, read-once function, Shannon function, elementary basis.

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A checking test problem for read-once functions, also known as *testing with respect to read-once alternatives*, was set up by A. A. Voronenko in [1]. For a read-once function $f(x_1, \ldots, x_n)$ depending essentially an all its variables, it is required to construct a *test*—a set of vectors that distinguishes f from all other read-once functions depending on the same variables. This setting, unlike a number of related problems, is not degenerate and has polynomial solutions for a wide range of Boolean bases. In this paper, the *elementary basis*, consisting of conjunction, disjunction and negation, is considered. Minimal test *length* (the number of vectors contained in it) for a function f is denoted by $T(f)$, and the value of the corresponding *Shannon function* is denoted by $T(n)$. In [2], the inequality $T(n) < 7n/2$ is obtained. The main goal of this paper is to obtain the following bound.

Theorem 1. *Any read-once function $f(x_1, \ldots, x_n)$ over $\{\wedge, \vee, \neg\}$ has a test with respect to read-once alternatives of length less or equal to $2n+1$:*

$$T(n) \leqslant 2n + 1.$$

For an n-ary disjunction, the lower bound $T(x_1 \vee \ldots \vee x_n) \geqslant n+1$ is obvious. S. E. Bubnov formulated a conjecture that the equality $T(f) = n+1$ holds for all read-once functions f having exactly n essential variables. Note that, e. g., in the case of the basis of all two-variable functions, the known universal upper bound of $\binom{n}{2} + n + 1$ coincides with the length of a unique irredundant test for the disjunction [3]. However, for individual read-once functions over this basis, stronger upper bounds (including linear ones) are known [4].

Remark. For the elementary basis, a failed attempt to prove the universal upper bound of $n+1$ is contained in [5]. For instance, for the function $f = (x_1 \vee x_2) \wedge (x_3 \vee x_4)$, the algorithm of that paper mistakenly suggests the set of vectors $(0010), (0110), (1000), (1001), (1010)$ as

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a test. Nevertheless, one can check that f has a test of length 5, which consists of vectors (0011), (1100), (1010), (0110), (1101). Thus S. E. Bubnov's conjecture has been neither proved nor disproved.

Let us proceed to the main results of this paper. Let $f(x_1, \ldots, x_n)$ be a read-once function depending essentially on all its variables. Construct auxiliary sets of vectors $A_0(f)$ and $A_1(f)$ using the following inductive description:

1. Suppose that $f = x_i$. Then put $A_0(f) = \{(0)\}$ and $A_1(f) = \{(1)\}$.
2. Suppose that $f = f_1 \wedge f_2 \wedge \ldots \wedge f_s$, where each f_i is either a variable or a disjunction of monotone read-once functions. (For $f = f_1 \vee f_2 \vee \ldots \vee f_s$, the construction is dual.) Let the order of variables of f be given by sequential enumeration of all variables of f_1, and then ones of $f_2, \ldots, f_s$. Groups of variables corresponding to different functions f_i will be separated from each other by vertical bars. Take an arbitrary vector α_i from each set $A_1(f_i)$ and include the vector
$$\alpha = (\, \alpha_1 \mid \alpha_2 \mid \ldots \mid \alpha_s \,)$$
in the set $A_1(f)$. Also add to this set all vectors obtained from α by replacing any subvector α_i with an arbitrary subvector α_i' from the same $A_1(f_i)$. Furthermore, let $A_0(f)$ be the set of all vectors of form
$$\beta = (\, \mathbf{1} \mid \ldots \mid \mathbf{1} \mid \beta_i \mid \mathbf{1} \mid \ldots \mid \mathbf{1} \,),$$
where $\beta_i \in A_0(f_i)$ for $i = 1, \ldots, s$ and symbols $\mathbf{1}$ denote subvectors consisting of ones.
3. Suppose that f is obtained from f' by substituting $\overline{x}_i$ for x_i in a formula expressing f'. In this case, obtain vectors of sets $A_0(f)$ and $A_1(f)$ from the corresponding vectors for f' by inverting the component representing x_i.

Note that for any read-once function f, transformations in the last item of the list above are unambiguous.

Claim 1. *For any monotone read-once function $f(x_1, \ldots, x_n)$, the sets $A_0(f)$ and $A_1(f)$ consist of upper zeros and lower ones of f, respectively.*

Claim 2. *If x_i is an essential variable of a monotone read-once function f, then there exists a vector in $A_0(f)$ with $x_i = 0$ and a vector in $A_1(f)$ with $x_i = 1$.*

Lemma 1. *For any read-once function f, the set $A_0(f) \cup A_1(f)$ contains exactly $n + 1$ vectors, where n is the number of essential variables of f.*

Proof. It follows from Claim 1 that for any read-once function f, the sets $A_0(f)$ and $A_1(f)$ are disjoint, so the cardinality of their union is equal to the sum of their cardinalities. Now use induction over n. For $n = 1$, the desired is straightforward. Suppose that $n \geqslant 2$. Assume without loss of generality that $f = f_1 \wedge \ldots \wedge f_s$, where $s \geqslant 2$. Then by construction of $A_0(f)$ and $A_1(f)$, it holds that

$$\begin{aligned} |A_0(f)| + |A_1(f)| - 1 &= \sum_{i=1}^{s} |A_0(f_i)| + \left(1 + \sum_{i=1}^{s} (|A_1(f_i)| - 1)\right) - 1 \\ &= \sum_{i=1}^{s} (|A_0(f_i)| + |A_1(f_i)| - 1), \end{aligned}$$

which gives the desired by the inductive assumption.

Lemma 2 (main lemma). *Let f be a read-once function over $\{\wedge, \vee\}$ depending essentially on all its variables. Then the set $A_0(f) \cup A_1(f)$ is a test for f with respect to read-once alternatives over $\{\wedge, \vee\}$.*

Proof. Use induction on the depth of a read-once formula expressing f. For $n = 1$, the set $A_0(f) \cup A_1(f)$ contains all possible input vectors, so there is nothing to prove. Suppose that $n \geqslant 2$. Assume without loss of generality that $f = f_1 \wedge f_2 \wedge \ldots \wedge f_s$, where $s \geqslant 2$ and all f_i are variables and disjunctions of read-once functions. We now reconstruct (learn) f using its values on vectors from $A_0(f) \cup A_1(f)$ and exploiting the fact that it is read-once over $\{\wedge, \vee\}$. We have:

$$\begin{array}{llll} & f(\,\alpha_1' \mid \alpha_2 \mid \ldots \mid \alpha_s\,) = 1, & \alpha_1' \in A_1(f_1), & \text{(input vectors of } f \text{ are from } A_1(f)) \\ & f(\,\beta_1 \mid \mathbf{1} \mid \ldots \mid \mathbf{1}\,) = 0, & \beta_1 \in A_0(f_1), & \text{(input vectors of } f \text{ are from } A_0(f)) \\ \Rightarrow & f(\,\beta_1 \mid \alpha_2 \mid \ldots \mid \alpha_s\,) = 0, & \beta_1 \in A_0(f_1), & \text{(by monotonicity of } f) \\ \Rightarrow & f(\,- \mid \alpha_2 \mid \ldots \mid \alpha_s\,) \equiv f_1. & & \text{(by the inductive assumption)} \end{array}$$

Arguing as above, we prove that f has projections equal to $f_2, \ldots, f_s$ on the corresponding subcubes.

Observe that for any two variables x_k, x_m, the projection $h(x_k, x_m)$ of f depending essentially on both x_k and x_m is the same for any possible choice of constants substituted for f's other variables. In order to reconstruct f, it is sufficient to reconstruct a graph $G_\wedge$ defined as follows. The set of vertices of this graph is the set of essential variables of f, and an edge (x_k, x_m) is present if and only if f has a projection equal to $x_k \wedge x_m$. This technique dates back to the paper [6] by V. A. Gurvich and is used, e. g., in [2] and [7]. When applied to the problem of testing read-once functions, it is most extensively developed in [4]. In our case, it is sufficient to prove the connectivity of each subgraph of $G_\wedge$ induced by a set of vertices—variables of projections f_i and f_j (then, since the complement of such a subgraph is always disconnected, all missing information about edges is obtained automatically). Furthermore, according to the argument above, any distinct induced subgraph for each f_i can be regarded as already known. It either contains only one vertex, or is disconnected and has connected components that correspond to the summands in the read-once formula for f_i.

Consider a subgraph for f_1 and f_2. Without loss of generality, suppose that the vector α contained in the test has the form

$$\alpha = (\,\alpha_{1,1}\,\mathbf{0}\ldots\mathbf{0} \mid \alpha_{2,1}\,\mathbf{0}\ldots\mathbf{0} \mid \alpha_3 \mid \ldots \mid \alpha_s\,).$$

Here and below, vectors' components are split into groups according to read-once expressions $f = f_1 \wedge f_2 \wedge \ldots \wedge f_s$ and $f_i = f_{i,1} \vee \ldots \vee f_{i,n_i}$. Naturally, symbols $\mathbf{0}$ denote subvectors consisting only of zeros. Known information about f allows us to determine that the value of f is equal to 0 on any vector differing from α in one component that equals to 1 in $\alpha_{1,1}$ or $\alpha_{2,1}$ (by Claim 1, vectors $\alpha_{1,1}$ and $\alpha_{2,1}$ are lower ones of $f_{1,1}$ and $f_{2,1}$, respectively), so for at least one pair of variables x_k and x_m of projections $f_{1,1}$ and $f_{2,1}$, there exists a projection of f equal to $x_k \wedge x_m$. We now reconstruct one edge between variables of projections $f_{1,1}$ and $f_{2,j}$, for all possible j, and one between variables of $f_{1,i}$ and $f_{2,1}$, for all possible i. This is sufficient to prove the connectivity of the considered subgraph.

Without loss of generality, consider only one pair $f_{1,1}$ and $f_{2,2}$. The form of vector α indicated above implies that the set $A_1(f)$ contains at least one vector of the form

$$(\,\alpha_{1,1}\,\mathbf{0}\ldots\mathbf{0} \mid \mathbf{0}\,\alpha_{2,2}\,\mathbf{0}\ldots\mathbf{0} \mid \alpha_3 \mid \ldots \mid \alpha_s\,), \qquad \alpha_{2,2} \in A_1(f_{2,2}),$$

so

$$\begin{aligned} & f(\,\alpha_{1,1}\,\mathbf{0}\ldots\mathbf{0} \mid \mathbf{0}\,\alpha_{2,2}\,\mathbf{0}\ldots\mathbf{0} \mid \alpha_3 \mid \ldots \mid \alpha_s\,) = 1 && \text{(input vector of } f \text{ is from } A_1(f)\text{)} \\ \Rightarrow\ & f(\gamma_{1,1}) = 1, \quad \gamma_{1,1} = (\,\alpha_{1,1}\,\mathbf{0}\ldots\mathbf{0} \mid \mathbf{0}\,\alpha_{2,2}\,\mathbf{0}\ldots\mathbf{0} \mid \mathbf{1} \mid \ldots \mid \mathbf{1}\,). && \text{(by monotonicity of } f\text{)} \end{aligned}$$

Furthermore,

$$\begin{aligned} & f(\,\beta_1 \mid \mathbf{1} \mid \ldots \mid \mathbf{1}\,) = 0, && \beta_1 \in A_0(f_1), && \text{(input vectors of } f \text{ are from } A_0(f)\text{)} \\ & f(\,\alpha_1' \mid \alpha_2 \mid \ldots \mid \alpha_s\,) = 1, && \alpha_1' \in A_1(f_1), && \text{(input vectors of } f \text{ are from } A_1(f)\text{)} \\ \Rightarrow\ & f(\,\alpha_1' \mid \mathbf{1} \mid \ldots \mid \mathbf{1}\,) = 1, && \alpha_1' \in A_1(f_1), && \text{(by monotonicity of } f\text{)} \\ \Rightarrow\ & f(\,- \mid \mathbf{1} \mid \ldots \mid \mathbf{1}\,) \equiv f_1. && && \text{(by the inductive assumption)} \end{aligned}$$

By Claim 1, for each subvector $\alpha_{1,1}'$ obtained from $\alpha_{1,1}$ by changing a 1 to 0 in any its component, it holds that

$$\begin{aligned} & f(\,\alpha_{1,1}'\,\mathbf{0}\ldots\mathbf{0} \mid \mathbf{1} \mid \ldots \mid \mathbf{1}\,) = 0 \\ \Rightarrow\ & f(\gamma_{0,1}) = 0, \quad \gamma_{0,1} = (\,\alpha_{1,1}'\,\mathbf{0}\ldots\mathbf{0} \mid \mathbf{0}\,\alpha_{2,2}\,\mathbf{0}\ldots\mathbf{0} \mid \mathbf{1} \mid \ldots \mid \mathbf{1}\,). && \text{(by monotonicity of } f\text{)} \end{aligned}$$

We also have

$$
\begin{array}{llll}
& f(\,\alpha_{1,1}\,\mathbf{0}\ldots\mathbf{0}\mid\alpha_2'\mid\alpha_3\mid\ldots\mid\alpha_s\,)=1, & \alpha_2'\in A_1(f_2), & \text{(input vectors of } f \text{ are from } A_1(f))\\
\Rightarrow & f(\,\alpha_{1,1}\,\mathbf{0}\ldots\mathbf{0}\mid\alpha_2'\mid\mathbf{1}\mid\ldots\mid\mathbf{1}\,)=1, & \alpha_2'\in A_1(f_2), & \text{(by monotonicity of } f)\\
& f(\,\mathbf{1}\mid\beta_2\mid\mathbf{1}\mid\ldots\mid\mathbf{1}\,)=0, & \beta_2\in A_0(f_2), & \text{(input vectors of } f \text{ are from } A_0(f))\\
\Rightarrow & f(\,\alpha_{1,1}\,\mathbf{0}\ldots\mathbf{0}\mid\beta_2\mid\mathbf{1}\mid\ldots\mid\mathbf{1}\,)=0, & \beta_2\in A_0(f_2), & \text{(by monotonicity of } f)\\
\Rightarrow & f(\,\alpha_{1,1}\,\mathbf{0}\ldots\mathbf{0}\mid-\mid\mathbf{1}\mid\ldots\mid\mathbf{1}\,)\equiv f_2 & & \text{(by the inductive assumption)}\\
\Rightarrow & f(\gamma_{1,0})=0,\quad \gamma_{1,0}=(\,\alpha_{1,1}\,\mathbf{0}\ldots\mathbf{0}\mid\mathbf{0}\,\alpha_{2,2}'\,\mathbf{0}\ldots\mathbf{0}\mid\mathbf{1}\mid\ldots\mid\mathbf{1}\,), & & \text{(by monotonicity of } f)
\end{array}
$$

where $\alpha_{2,2}'$ is obtained from $\alpha_{2,2}$ by changing any 1 to 0.

The values of f on the three vectors $\gamma_{1,1}$, $\gamma_{0,1}$ and $\gamma_{1,0}$ prove that one of its projections is a conjunction of two variables of $f_{1,1}$ and $f_{2,2}$. According to the argument above, this proves the connectivity of the induced subgraph for f_1 and f_2. A special case when at least one of these functions has exactly one essential variable, is analyzed in a similar manner. This concludes the proof.

Theorem 2. *Let $f(x_1,\ldots,x_n)$ be a read-once function over $\{\wedge,\vee\}$. Then f has a test with respect to read-once alternatives over $\{\wedge,\vee\}$ of length $n+1$.*

This theorem identifies the exact value of Shannon function for test length with respect to read-once alternatives over the basis of conjunction and disjunction. The statement of the theorem implies that n-ary disjunction is the hardest (in the context of checking test problem) read-once function of n variables over this basis. Note that for this basis, there exist read-once functions that are easier to test than the disjunction. For instance, in [7] a subsequence of functions is constructed such that its function depending on n variables has a checking test of length less than $3\sqrt{n}$.

We now return to the elementary basis.

Proof of Theorem 1. Assume without loss of generality that f is monotone and all its variables are essential. By Claim 2, for each variable x_i there exists a vector $\delta_i\in A_1(f)$ with $x_i=1$. By Claim 1, for the vector δ_i' differing from δ_i only in $x_i=0$, it holds that $f(\delta_i')=0$, whereas $f(\delta_i)=1$. Let $A_2(f)$ consist of vectors δ_i' for all i. Then the values of f on vectors from $A_0(f)\cup A_1(f)\cup A_2(f)$, firstly, prove that f is monotone in all its variables (or, equivalently, read-once over $\{\wedge,\vee\}$) and, secondly, distinguish f from all other read-once functions over $\{\wedge,\vee\}$ that depend on the same variables. It follows that this set is a test for f with respect to read-once alternatives over the basis $\{\wedge,\vee,\neg\}$. By Lemma 1, the length of this test is at most $(n+1)+n=2n+1$. This completes the proof.

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