Regular Separators for VASS Coverability Languages

33. Theorietag “Automaten und Formale Sprachen”, Kaiserslautern

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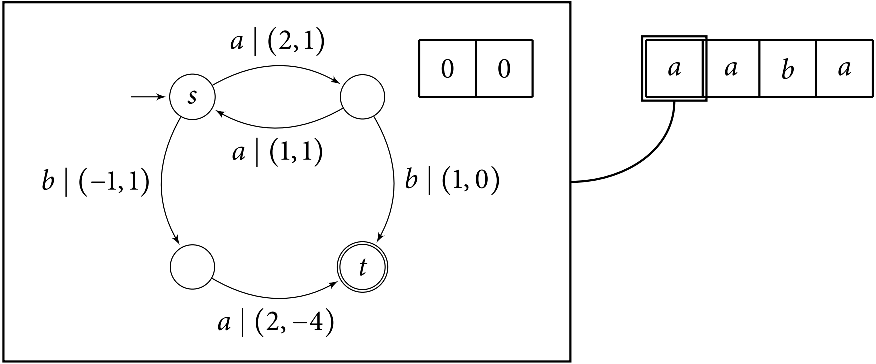
October 4, 2023

Vector Addition Systems with States (VASS)



- $L(\mathfrak{V}) = \{w \in \Sigma^* \mid \exists \vec{v} \in \mathbb{N}^d : (s, \vec{0}) \xrightarrow{w}_{\mathfrak{V}} (t, \vec{v}) \geq (t, \vec{0})\}$

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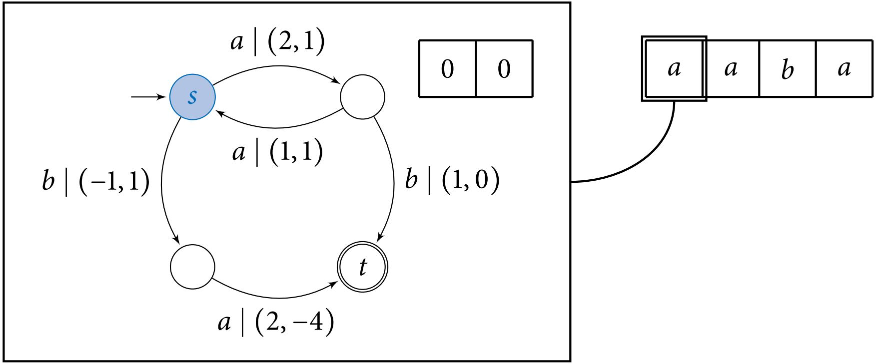

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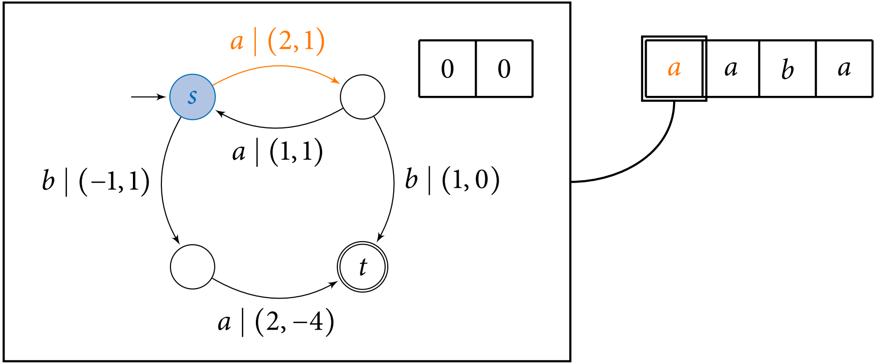

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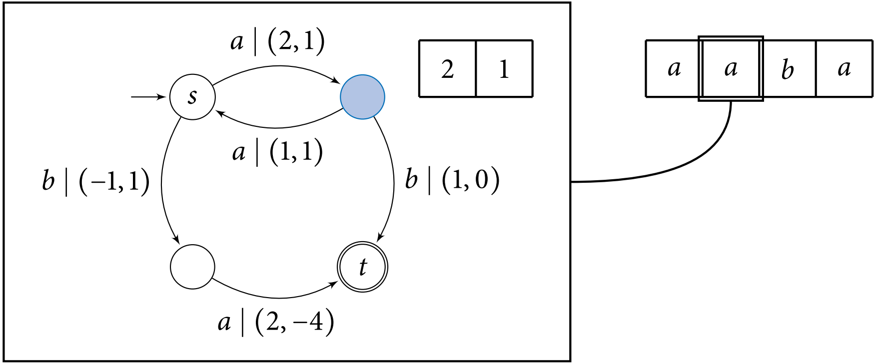

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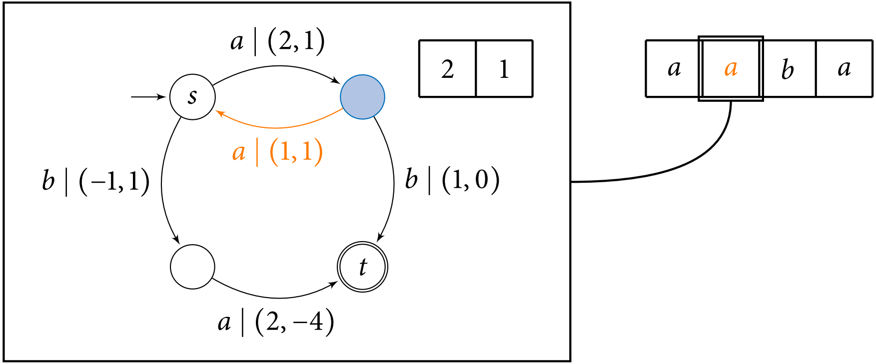

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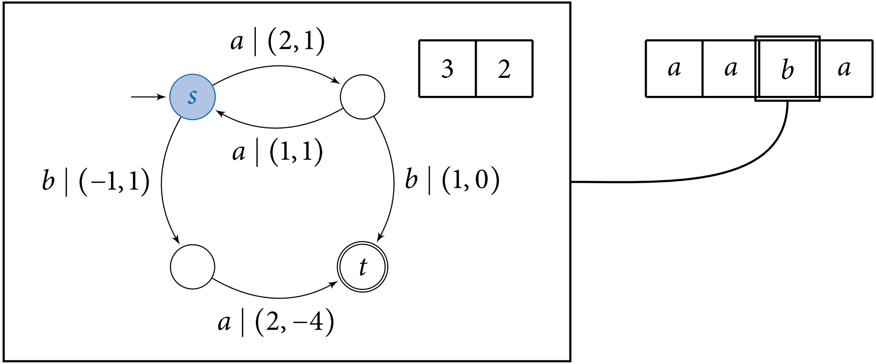

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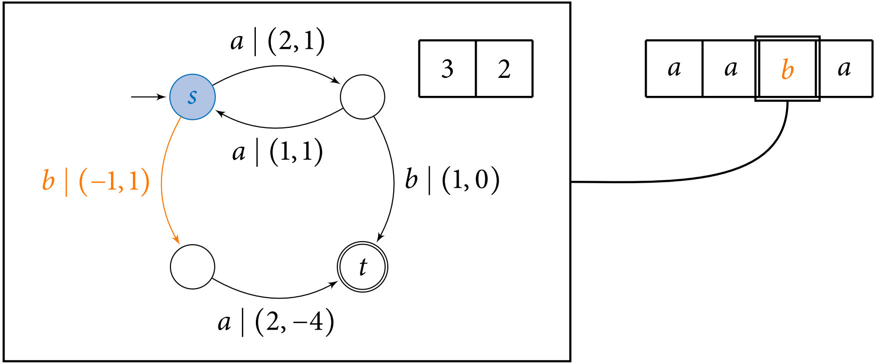

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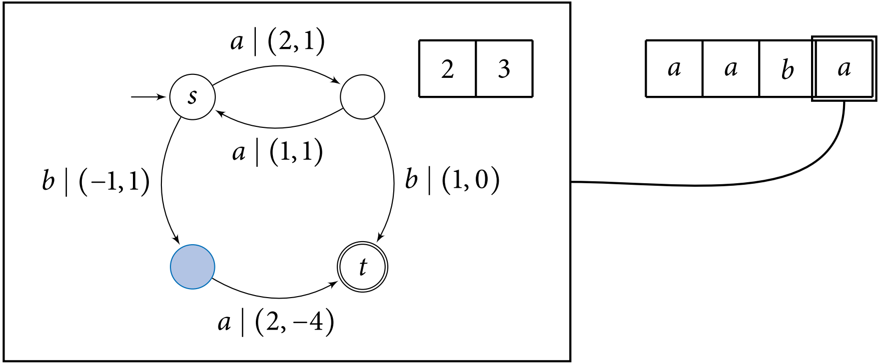

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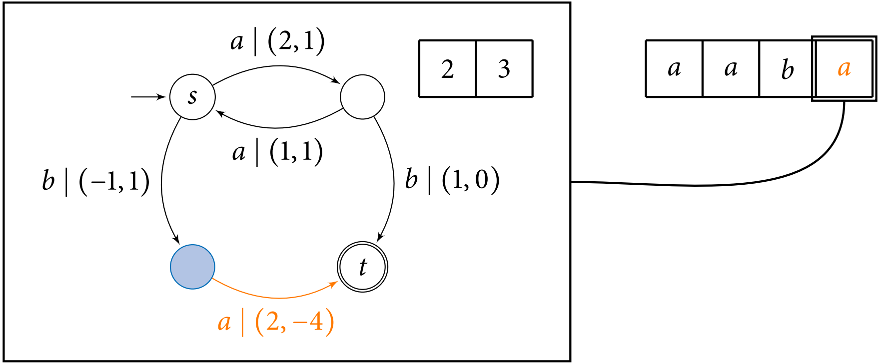

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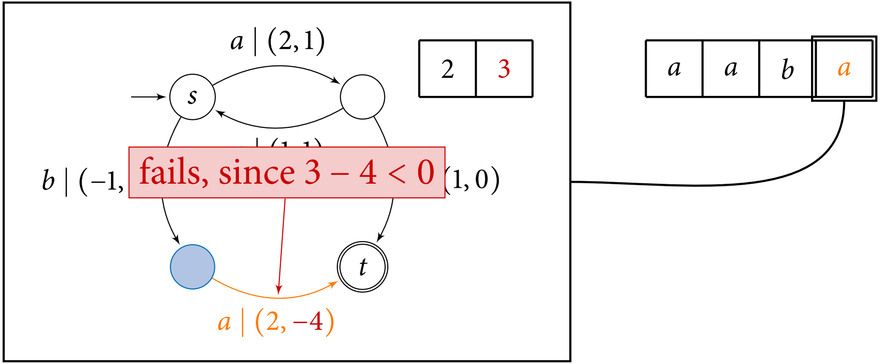

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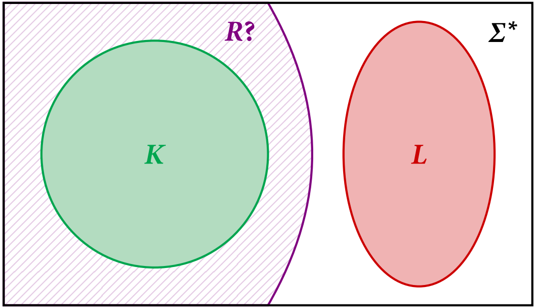
Regular Separability (1)

Problem

- Given two languages $K, L \subseteq \Sigma^*$.
- Is there a regular language $R \subseteq \Sigma^*$ with $K \subseteq R$ and $L \cap R = \varnothing$?



- Note: Regular Separability ≠ Disjointness!

Regular Separability (2)

Theorem (Czerwiński et al. @ CONCUR 2018)

Let $\mathfrak{V}$ and $\mathfrak{W}$ be two VASS. Then $L(\mathfrak{V})$ and $L(\mathfrak{W})$ are regular separable if, and only if, $L(\mathfrak{V}) \cap L(\mathfrak{W}) = \emptyset$.

Question

What is the size of a regular separator of $L(\mathfrak{V})$ and $L(\mathfrak{W})$?

- Czerwiński et al.: doubly exp. lower bound & triply exp. upper bound

Theorem (Main Theorem)

Let $\mathfrak{V}$ and $\mathfrak{W}$ be two VASS with $\leq n$ states and updates of norm $\leq m$. If $L(\mathfrak{V}) \cap L(\mathfrak{W}) = \emptyset$ then there is an separating NFA with at most $(n+m)^{2^{\mathrm{poly}(d)}}$ many states.

Proof (1): Reduce to Counter Instructions

- $\Gamma_d = \{\mathbf{a_i}, \overline{\mathbf{a_i}} \mid 1 \le i \le d\}$
  - $\mathbf{a_i}$ increase counter i by 1
  - $\overline{\mathbf{a_i}}$ decrease counter i by 1
- $C_d = \{w \in \Gamma_d^* \mid \forall$ prefixes v of $w, 1 \le i \le d: |v|_{\mathbf{a_i}} \ge |v|_{\overline{\mathbf{a_i}}}\}$

Lemma (Jantzen 1979)

$L \subseteq \Sigma^*$ is a VASS coverability language iff there is a rational transduction T with $L = T(C_d)$.

Corollary

Let $\mathfrak{V}$ and $\mathfrak{W}$ be two VASS and T be a rational transduction with $L(\mathfrak{W}) = T(C_d)$. Then $L(\mathfrak{V})$ is regularly separable from $L(\mathfrak{W})$ iff $T^{-1}(L(\mathfrak{V}))$ is regularly separable from C_d.

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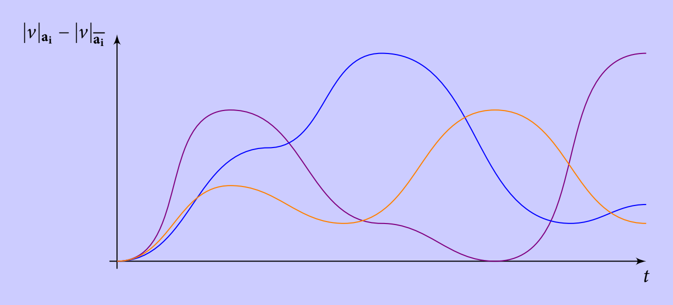
Lemma (Ja…tzen 1979)

$L \subseteq \Sigma^*$ is a VASS … *with $L = T(C_d)$.*

Corollary

Let $\mathfrak{V}$ and $\mathfrak{W}$ be … (C_d). *Then*
$L(\mathfrak{V})$ is regularly … *from C_d.*



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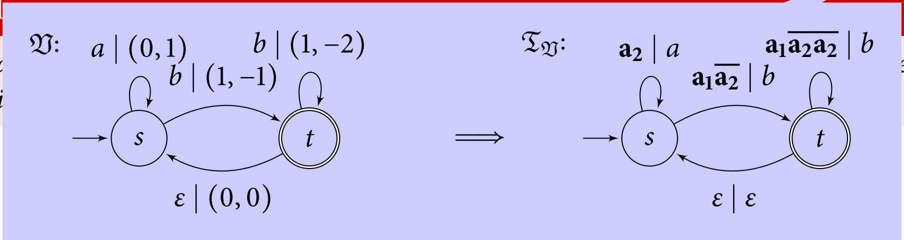

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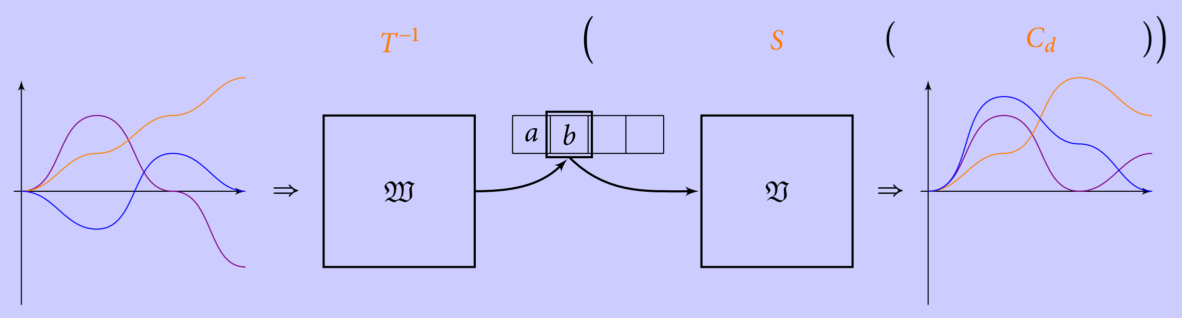

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Proof (2): Basic Separators

- For $k \in \mathbb{N}$ let $B_k \subseteq \Gamma_d^*$ be the following language: $w \in B_k$ iff there is $1 \leq i \leq d$ with
  - there is a prefix v of w with $|v|_{\mathbf{a_i}} < |v|_{\overline{\mathbf{a_i}}}$ and
  - each proper prefix u of v satisfies $0 \leq |u|_{\mathbf{a_i}} - |u|_{\overline{\mathbf{a_i}}} \leq k$
- B_k is accepted by a DFA of size $O(k^d)$.

Theorem (Czerwiński & Zetzsche @ LICS 2020)

Let $\mathfrak{V}$ and $\mathfrak{W}$ be two VASS with $L(\mathfrak{V}) \cap L(\mathfrak{W}) = \emptyset$ and let T be a rational transduction with $L(\mathfrak{W}) = T(C_d)$. Then B_k is a regular separator of $T^{-1}(L(\mathfrak{V}))$ and C_d for a $k \in \mathbb{N}$.

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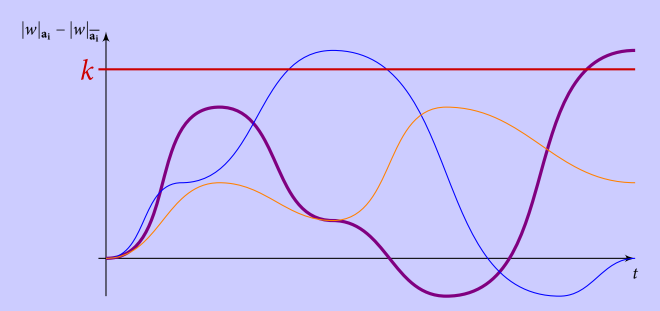
$L(\mathfrak{W}) = T(C_d)$. … *$k \in \mathbb{N}$.*



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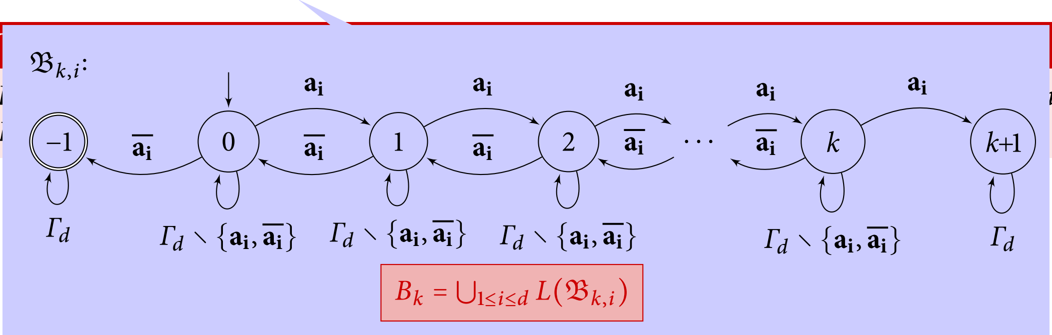

Proof (3): Covering

Theorem (Rackoff 1978)

Let $\mathfrak{V}$ be a VASS, c be a configuration of $\mathfrak{V}$, and a vector $\vec{v} \in \mathbb{N}^d$ with $c \rightarrow^*_{\mathfrak{V}} (t, \vec{v}) \geq (t, \vec{0})$.
Then there is $0 \leq \ell \leq \underbrace{(n+m)^{2^{\mathrm{poly}(d)}}}_{=:\mathrm{Rackoff}(\mathfrak{V})}$ and $\vec{w} \in \mathbb{N}^d$ with $c \rightarrow^{\ell}_{\mathfrak{V}} (t, \vec{w}) \geq (t, \vec{0})$.
Here, n is the number of states in $\mathfrak{V}$ and m is the norm of the counter updates in $\mathfrak{V}$.

Theorem

Let $\mathfrak{V}$ and $\mathfrak{W}$ be two VASS with $L(\mathfrak{V}) \cap L(\mathfrak{W}) = \emptyset$ and let T be a rational transduction with $L(\mathfrak{W}) = T(C_d)$. Then $B_{\mathrm{Rackoff}(\mathfrak{V}\times\mathfrak{W})}$ is a regular separator of $T^{-1}(L(\mathfrak{V}))$ and C_d.

- Finally, $T(B_{\mathrm{Rackoff}(\mathfrak{V}\times\mathfrak{W})})$ is a regular separator of $L(\mathfrak{V})$ and $L(\mathfrak{W})$. □

Conclusion

		NFAs		DFAs	
		unary	binary	unary	binary
d as input		2-exp.		3-exp.	
d fixed	$d \geq 2$	poly.	exp.	exp.	2-exp.
	$d = 1$	poly.	exp.	exp.	exp.

Thank you!