Regular Separators for VASS Coverability Languages

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Vector Addition Systems with States (VASS)

\[ L(\mathcal{V}) = \{ w \in \Sigma^* | \exists \vec{v} \in \mathbb{N}^d : (s, \vec{0}) \xrightarrow{\mathcal{V}} (t, \vec{v}) \geq (t, \vec{0}) \} \]
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Regular Separability (1)

Problem

- Given two languages $K, L \subseteq \Sigma^*$.
- Is there a regular language $R \subseteq \Sigma^*$ with $K \subseteq R$ and $L \cap R = \emptyset$?

Note: Regular Separability ≠ Disjointness!
Theorem (Czerwiński et al. @ CONCUR 2018)

Let $\mathcal{V}$ and $\mathcal{W}$ be two VASS. Then $L(\mathcal{V})$ and $L(\mathcal{W})$ are regular separable if, and only if, $L(\mathcal{V}) \cap L(\mathcal{W}) = \emptyset$.

Question

What is the size of a regular separator of $L(\mathcal{V})$ and $L(\mathcal{W})$?

- Czerwiński et al.: doubly exp. lower bound & triply exp. upper bound

Theorem (Main Theorem)

Let $\mathcal{V}$ and $\mathcal{W}$ be two VASS with $\leq n$ states and updates of norm $\leq m$. If $L(\mathcal{V}) \cap L(\mathcal{W}) = \emptyset$ then there is an separating NFA with at most $(n + m)^{2^{\text{poly}(d)}}$ many states.
Proof (1): Reduce to Counter Instructions

\[ \Gamma_d = \{ a_i, \overline{a_i} \mid 1 \leq i \leq d \} \]

- \( a_i \) increase counter \( i \) by 1
- \( \overline{a_i} \) decrease counter \( i \) by 1

\[ C_d = \{ w \in \Gamma_d^* \mid \forall \text{ prefixes } v \text{ of } w, 1 \leq i \leq d : |v|_{a_i} \geq |v|_{\overline{a_i}} \} \]

Lemma (Jantzen 1979)

\( L \subseteq \Sigma^* \) is a VASS coverability language iff there is a rational transduction \( T \) with \( L = T(C_d) \).

Corollary

Let \( \mathcal{W} \) and \( \mathcal{W} \) be two VASS and \( T \) be a rational transduction with \( L(\mathcal{W}) = T(C_d) \). Then \( L(\mathcal{W}) \) is regularly separable from \( L(\mathcal{W}) \) iff \( T^{-1}(L(\mathcal{W})) \) is regularly separable from \( C_d \).
Proof (1): Reduce to Counter Instructions

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\( L \subseteq \Sigma^* \) is a VASS coverability language iff there is a rational transduction \( T \) with \( L = T(C_d) \).

Corollary

Let \( \mathfrak{V} \) and \( \mathfrak{W} \) be two VASS with \( L(\mathfrak{V}) \) regular and \( L(\mathfrak{W}) = T(C_d) \). Then \( L(\mathfrak{W}) \) is regularly separable from \( C_d \).
Proof (1): Reduce to Counter Instructions

- $\Gamma_d = \{a_i, \overline{a}_i \mid 1 \leq i \leq d\}$
  - $a_i$ increase counter $i$ by 1
  - $\overline{a}_i$ decrease counter $i$ by 1
- $C_d = \{w \in \Gamma_d^* \mid \forall$ prefixes $v$ of $w, 1 \leq i \leq d: |v|_{a_i} \geq |v|_{\overline{a}_i}\}$

Lemma (Jantzen 1979)

$L \subseteq \Sigma^*$ is a VASS coverability language iff there is a rational transduction $T$ with $L = T(C_d)$.

Corollary

Let $\mathcal{L}$ and $\mathcal{W}$ be two VASS and $T$ be a rational transduction with $L(\mathcal{W}) = T(C_d)$. Then $L(\mathcal{L})$ is regularly separable from $L(\mathcal{W})$ iff $T^{-1}(L(\mathcal{L}))$ is regularly separable from $C_d$. 

\[ s \quad \text{\xrightarrow{a} (0,1)} \quad b \quad \text{\xrightarrow{1,-2}} \quad \overline{a}_2 \quad \text{\xrightarrow{a_1,a_2}} b \quad \text{\xrightarrow{a_1,a_2}} \quad b \]
Proof (1): Reduce to Counter Instructions

Γ<br>\[ d = \{ a_i \mid 1 \leq i \leq d \} \]

\[ a_i \text{ increase counter } i \text{ by 1} \]

\[ a_i \text{ decrease counter } i \text{ by 1} \]

\[ C_d = \{ w \in \Gamma^* \mid \forall \text{ prefixes } v \text{ of } w, 1 \leq i \leq d : \mid v \mid_{a_i} \geq \mid v \mid_{a_i} \} \]

Lemma (Jantzen 1979)
\[ L \subseteq \Sigma^* \text{ is a VASS coverability language iff there is a rational transduction } T \text{ with } L = T(C_d). \]

Corollary
Let \( \mathcal{V} \) and \( \mathcal{W} \) be two VASS and \( T \) be a rational transduction with \( L(\mathcal{W}) = T(C_d) \). Then \( L(\mathcal{V}) \) is regularly separable from \( L(\mathcal{W}) \) iff \( T^{-1}(L(\mathcal{V})) \) is regularly separable from \( C_d \).
Proof (2): Basic Separators

For $k \in \mathbb{N}$ let $B_k \subseteq \Gamma_d^*$ be the following language: $w \in B_k$ iff there is $1 \leq i \leq d$ with
- there is a prefix $v$ of $w$ with $|v|_{a_i} < |v|_{a_i}^-$ and
- each proper prefix $u$ of $v$ satisfies $0 \leq |u|_{a_i} - |u|_{a_i}^- \leq k$

- $B_k$ is accepted by a DFA of size $O(k^d)$.

Theorem (Czerwiński & Zetzsche @ LICS 2020)

Let $\mathcal{V}$ and $\mathcal{W}$ be two VASS with $L(\mathcal{V}) \cap L(\mathcal{W}) = \emptyset$ and let $T$ be a rational transduction with $L(\mathcal{W}) = T(C_d)$. Then $B_k$ is a regular separator of $T^{-1}(L(\mathcal{V}))$ and $C_d$ for a $k \in \mathbb{N}$. 
Proof (2): Basic Separators

For \( k \in \mathbb{N} \) let \( B_k \subseteq \Gamma_d^* \) be the following language: \( w \in B_k \) iff there is \( 1 \leq i \leq d \) with

- there is a prefix \( v \) of \( w \) with \( |v|_{a_i} < |v|_{\overline{a_i}} \) and
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\( B_k \) is accepted by a DFA of size \( O(k^d) \).

Theorem (Czerwiński & Zetzsche @ LICS 2020)

Let \( \mathcal{V} \) and \( \mathcal{W} \) be two VASS with \( L(\mathcal{V}) \cap L(\mathcal{W}) = \emptyset \) and let \( T \) be a rational transduction with \( L(\mathcal{W}) = T(C_d) \). Then \( B_k \) is a regular separator of \( T^{-1}(L(\mathcal{V})) \) and \( C_d \) for some \( k \in \mathbb{N} \).
Proof (2): Basic Separators

For \( k \in \mathbb{N} \) let \( B_k \subseteq \Gamma_d^* \) be the following language: \( w \in B_k \) iff there is \( 1 \leq i \leq d \) with
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\( B_k \) is accepted by a DFA of size \( O(k^d) \).

\[ B_k = \bigcup_{1 \leq i \leq d} L(\mathcal{B}_{k,i}) \]
Proof (3): Covering

Theorem (Rackoff 1978)

Let $\mathcal{V}$ be a VASS, $c$ be a configuration of $\mathcal{V}$, and a vector $\vec{v} \in \mathbb{N}^d$ with $c \rightarrow^*_{\mathcal{V}} (t, \vec{v}) \geq (t, \vec{0})$. Then there is $0 \leq \ell \leq (n + m)^{2^\text{poly}(d)}$ and $\vec{w} \in \mathbb{N}^d$ with $c \rightarrow^\ell_{\mathcal{V}} (t, \vec{w}) \geq (t, \vec{0})$.

Here, $n$ is the number of states in $\mathcal{V}$ and $m$ is the norm of the counter updates in $\mathcal{V}$.

Theorem

Let $\mathcal{V}$ and $\mathcal{W}$ be two VASS with $L(\mathcal{V}) \cap L(\mathcal{W}) = \emptyset$ and let $T$ be a rational transduction with $L(\mathcal{W}) = T(C_d)$. Then $B_{\text{Rackoff}(\mathcal{V} \times \mathcal{W})}$ is a regular separator of $T^{-1}(L(\mathcal{V}))$ and $C_d$.  

Finally, $T(B_{\text{Rackoff}(\mathcal{V} \times \mathcal{W})})$ is a regular separator of $L(\mathcal{V})$ and $L(\mathcal{W})$.  \qed
## Conclusion

<table>
<thead>
<tr>
<th>$d$ as input</th>
<th>NFAs</th>
<th>DFAs</th>
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<tbody>
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<td>exp.</td>
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<tr>
<td>$d = 1$</td>
<td>poly.</td>
<td>exp.</td>
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Thank you!