Rational, Recognizable, and Aperiodic Sets in the Partially Lossy Queue Monoid

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What is a Partially Lossy Queue?

- there are two types of fifo-queues:
  - Reliable Queues
    - nothing can be forgotten or injected
    - applications: software and algorithms engineering
  - Lossy Queues
    - everything can be forgotten, nothing can be injected
    - applications: verification and telematics
- natural combination of both: Partially Lossy Queues (PLQs)
  - some parts can be forgotten
  - nothing can be injected

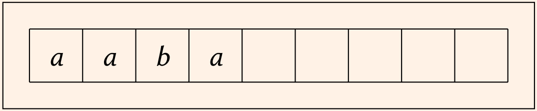
PLQs: Basics

- Let A be an alphabet ($|A| \geq 2$) and $U \subseteq A$.
  - U … unforgettable letters
  - $A \setminus U$ … forgettable letters
- two actions for each $a \in A$:
  - write letter $a \rightsquigarrow a$
  - read letter $a \rightsquigarrow \overline{a}$
- $\overline{A} := \{\overline{a} \mid a \in A\}$
- $\Sigma := A \uplus \overline{A}$
- non-controllable operation: forgetting letters from $A \setminus U$

Example

$$A = \{a, b\},\ U = \{b\}$$

$$q = aaba \qquad\qquad v = bb\overline{a}\overline{b}$$



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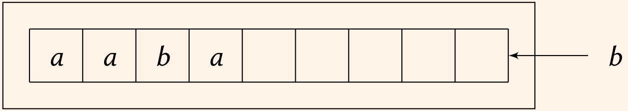

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a	a	b	a	b				

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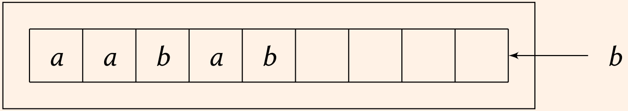

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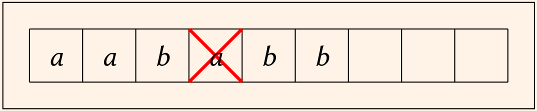

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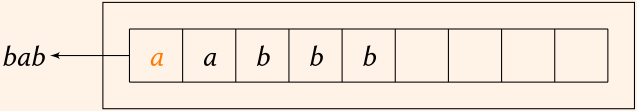

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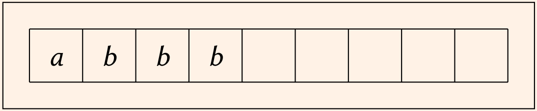

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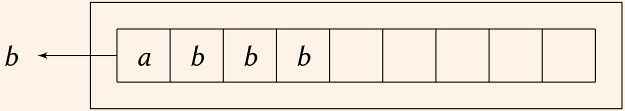

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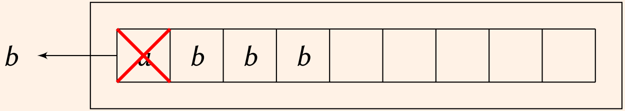

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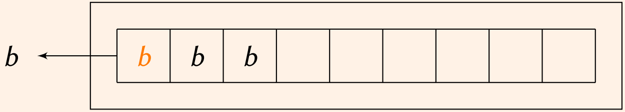

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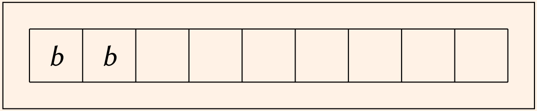

PLQ Monoids: Definition

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Definition

$u, v \in \Sigma^*$ act equally (in symbols $u \equiv v$) if, and only if,

$$\forall p, q \in A^* \colon p \xrightarrow{u} q \iff p \xrightarrow{v} q$$

Theorem (K., Kuske 2017, cf. CSR 2017)

$\equiv$ is the least congruence satisfying the following equations:

1. $a\overline{b} \equiv \overline{b}a$ *if* $a \neq b$
2. $a\overline{a}\overline{c} \equiv \overline{a}a\overline{c}$
3. $cw a\overline{a} \equiv cw\overline{a}a$ *if* $c \in U \cup \{a\}$

for any $a, b, c \in A$ *and* $w \in A^*$.

Definition

$\mathcal{Q}(A, U) := \Sigma^*/_{\equiv}$ ... the plq monoid

Rational and Recognizable Sets (1)

Definition

Let $\mathcal{M}$ be a monoid and $S \subseteq \mathcal{M}$.

- S is rational if it can be constructed from finite subsets of $\mathcal{M}$ using $\cup$, $\cdot$, and *
  - i.e., generalizes regular expressions
- S is recognizable if there is a homomorphism η into a finite monoid with $\eta^{-1}(\eta(S)) = S$.
  - i.e., generalizes acceptance of finite automata
  - closure properties: $\cup$, $\cap$, $\setminus$

Theorem (Kleene 1951)

$S \subseteq \Gamma^*$ is rational if, and only if, it is recognizable.

Rational and Recognizable Sets (2)

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Question

Is $S \subseteq \mathcal{Q}(A, U)$ rational if, and only if, it is recognizable? NO!

Proposition

- The class of rational sets is not closed under intersection.
- The class of recognizable sets is not closed under $\cdot$ and *.

- BUT: each recognizable set is rational due to [McKnight 1964]

Question

When is a rational set recognizable?

Theorem

Recognizability of rational sets is undecidable.

Definition

- $S \subseteq \mathcal{Q}(A, U)$ is q$^+$-rational if there is a rational set $R \subseteq A^*$ s.t. $S = [R \sqcup \overline{A}^*]_{\equiv}$.
- Similar: $S \subseteq \mathcal{Q}(A, U)$ is q$^-$-rational if there is a rational set $\overline{R} \subseteq \overline{A}^*$ s.t. $S = [A^* \sqcup \overline{R}]_{\equiv}$.
- $S \subseteq \mathcal{Q}(A, U)$ is q-rational if
  - S is q$^+$- or q$^-$-rational
  - $S = S_1 \cup S_2$ for some S_1, S_2 q-rational
  - $S = S_1 \cdot \mathcal{Q}(A, U) \cdot S_2$ for some S_1 q$^+$-rational, S_2 q$^-$-rational s.t. $S = [A^* \sqcup \overline{F}]_{\equiv}$ for a *finite* set $\overline{F} \subseteq \overline{A}^*$.
  - $S = \mathcal{Q}(A, U) \setminus S_1$ for some S_1 q-rational

Main Theorem

Theorem

Let $S \subseteq \mathcal{Q}(A, U)$. Then the following are equivalent:

1. *S is recognizable*
2. *S is q-rational*

Proof.

- "(1)⇒(2)": With the help of several intermediate characterizations.

Büchi's Theorem

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- Let $w = abbacba$.
- $\underline{w}$ is the following linear order:

$$a \longrightarrow b \longrightarrow b \longrightarrow a \longrightarrow c \longrightarrow b \longrightarrow a$$

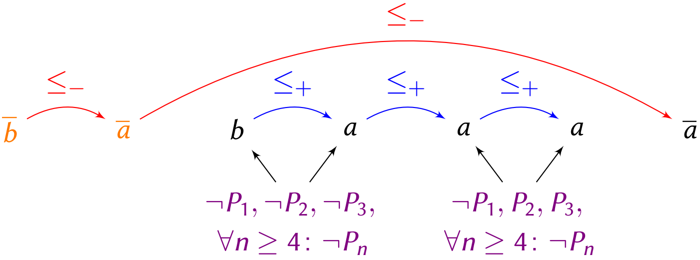
- FO ... first-order logic on these linear orders
- MSO ... FO + quantification of sets

Theorem (Büchi 1960)

$S \subseteq \Gamma^*$ is recognizable if, and only if, there is $\phi \in \mathrm{MSO}$ with $S = \{w \in \Gamma^* \mid \underline{w} \models \phi\}$.

Structures for PLQs

- Let $a, b \in A$, $b \notin U$. Consider $q = [\overline{b}\overline{a}baaa\overline{a}]_{\equiv}$.
- We model q as a structure $\widetilde{q}$ with infinitely many relations:
  - $\leq_+$, $\leq_-$, P_n for any $n \in \mathbb{N}$



- $\overline{b}\overline{a}baaa\overline{a} \equiv ba\overline{b}\overline{a}\overline{a}aa$

- Let $a, b \in A$, $b \notin U$. Consider $q = [\overline{b}\overline{a}baaa\overline{a}]_\equiv$.
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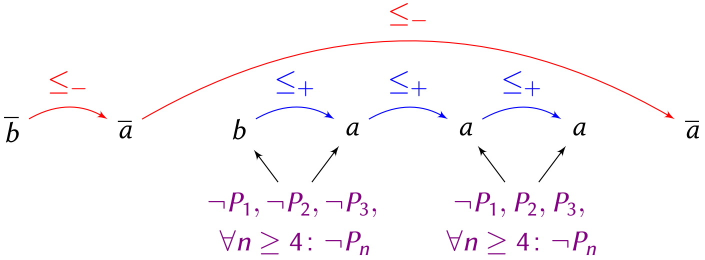

- $\mathrm{FO_q}$ ... first-order logic on these structures
- $\mathrm{MSO_q}$ ... $\mathrm{FO_q}$ + quantification of sets

Main Theorem

Theorem

Let $S \subseteq \mathcal{Q}(A, U)$. Then the following are equivalent:

1. *S is recognizable*
2. *S is q-rational*
3. *$S = \{q \in \mathcal{Q}(A, U) \mid \widetilde{q} \models \phi\}$ for some $\phi \in \mathrm{MSO_q}$*

Proof.

- "(1)⇒(2)": With the help of several intermediate characterizations.
- "(2)⇒(3)": Special product corresponds to some P_n.
- "(3)⇒(1)": Translation of $\mathrm{MSO_q}$-formulas into Büchi's MSO. □

Comparison

Data Structure	Transformation Monoid	Recognizable Sets
finite memory	finite monoid $\mathcal{F}$	$S \subseteq \mathcal{F}$
blind counter	$(\mathbb{Z}, +)$	$\bigcup_{(m,n)\in I,\, n\neq 0} m + n\mathbb{Z}$
pushdown	polycyclic monoid $\mathcal{P}$	$\emptyset, \mathcal{P}$
plq	$\mathcal{Q}(A, U)$	
reliable queue	$\mathcal{Q}(A, A)$	q-rational sets / MSO_q
lossy queue	$\mathcal{Q}(A, \emptyset)$	

Comparison

Data Structure	Transformation Monoid	Aperiodic Sets
finite memory	finite monoid $\mathcal{F}$	[...]
blind counter	$(\mathbb{Z}, +)$	$\emptyset, \mathbb{Z}$
pushdown	polycyclic monoid $\mathcal{P}$	$\emptyset, \mathcal{P}$
plq	$\mathcal{Q}(A, U)$	
reliable queue	$\mathcal{Q}(A, A)$	q-star-free sets / FO_q
lossy queue	$\mathcal{Q}(A, \emptyset)$	

Thank you!