Verifying Multi-Pushdown Automata
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Chris Köcher    Dietrich Kuske

Automata and Logics Group
Technische Universität Ilmenau

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Consider automata with one or more pushdowns.

- Model distributed systems with recursive programs.
- 2-pushdown automata are Turing-complete!
  \[\Rightarrow\] Verification problems are undecidable.
- Here: consider a special restriction to the automata.
  \[\sim\] cooperating multi-pushdown systems
A cooperating multi-pushdown system (CMPDS) $\mathcal{G}$ is a (special) subsystem of the synchronous product of 1-pushdown systems $\mathcal{G}_1, \ldots, \mathcal{G}_n$.
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$\mathcal{S}_1$: $c \mid aa \xrightarrow{a} 1 \xleftarrow{c} 2 \xrightarrow{a \mid \varepsilon} 1$

$\mathcal{S}_2$: $c \mid cb \xrightarrow{c \mid b} 1 \xleftarrow{b \mid \varepsilon} 2 \xrightarrow{b \mid \varepsilon} 2$

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$\mathcal{S}_1$:  

$$
\begin{array}{c}
\text{c | aa} \\
\text{1} \quad \text{a | } \varepsilon \\
\text{2}
\end{array}
$$

$\mathcal{S}_2$:  

$$
\begin{array}{c}
\text{c | cb} \\
\text{1} \quad \text{b | } \varepsilon \\
\text{2}
\end{array}
$$
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$$\begin{align*}
\mathcal{S}_1: & \quad c \mid aa \\
& \quad a \mid \varepsilon
\end{align*}$$

$$\begin{align*}
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& \quad b \mid \varepsilon
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\[ \begin{align*}
\mathcal{S}_1: & \quad c \mid aa \quad \xrightarrow{a} \quad 1 \quad \xrightarrow{\varepsilon} \quad 2 \\
\mathcal{S}_2: & \quad c \mid cb \quad \xrightarrow{b} \quad 1 \quad \xrightarrow{\varepsilon} \quad 2
\end{align*} \]
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\mathcal{S}_2: & \quad c \mid cb
\end{align*}

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A cooperating multi-pushdown system (CMPDS) $\mathcal{S}$ is a (special) subsystem of the synchronous product of 1-pushdown systems $\mathcal{S}_1, \ldots, \mathcal{S}_n$.
Let $\mathcal{G}$ be a CMPDS and $C$ a set of configurations.

$\text{pre}^{*}_{\mathcal{G}}(C) := \{ d \mid \exists c \in C: d \vdash^{*}_{\mathcal{G}} c \}$

**Theorem**

1. *Let $\mathcal{G}$ be a CMPDS and $C$ a recognizable set of configurations (i.e., accepted by an asynchronous automaton). Then $\text{pre}^{*}_{\mathcal{G}}(C)$ is effectively recognizable (in polynomial time).*

2. *There are a CMPDS $\mathcal{G}$ and a rational set $C$ of configurations (i.e., accepted by an NFA) such that $\text{pre}^{*}_{\mathcal{G}}(C)$ is not rational.*

Construction generalizes the one by Bouajjani, Esparza, and Maler.
Forwards Reachability

- Let $\mathcal{G}$ be a CMPDS and $C$ a set of configurations.
- $\text{post}^*_\mathcal{G}(C) := \{ d \mid \exists c \in C: c \vdash^*_\mathcal{G} d\}$

**Theorem**

1. Let $\mathcal{G}$ be a CMPDS and $C$ a rational set of configurations. Then $\text{post}^*_\mathcal{G}(C)$ is effectively rational (in polynomial time).
2. There are a CMPDS $\mathcal{G}$ and a recognizable set $C$ of configurations such that $\text{post}^*_\mathcal{G}(C)$ is not recognizable.

- Construction is inspired by the one by Finkel, Willems, and Wolper (but more involved).

Thank you!