Regular Separators for VASS Coverability Languages

43rd IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, Hyderabad

Chris Köcher   Georg Zetzsche

Max Planck Institute for Software Systems, Kaiserslautern

December 18, 2023

Motivation

- Vector Addition Systems with States (VASS)
  - NFA with finitely many non-negative counters
  - Equivalent to Petri Nets
  - Model the behavior of concurrent systems

- Why (regular) separability?
  - Safety verification consists of deciding disjointness of two languages, like event sequences
    - that are consistent with the behavior of a system component and
    - reaching an undesirable state.
  - A regular separator certifies disjointness.

Vector Addition Systems with States (1)

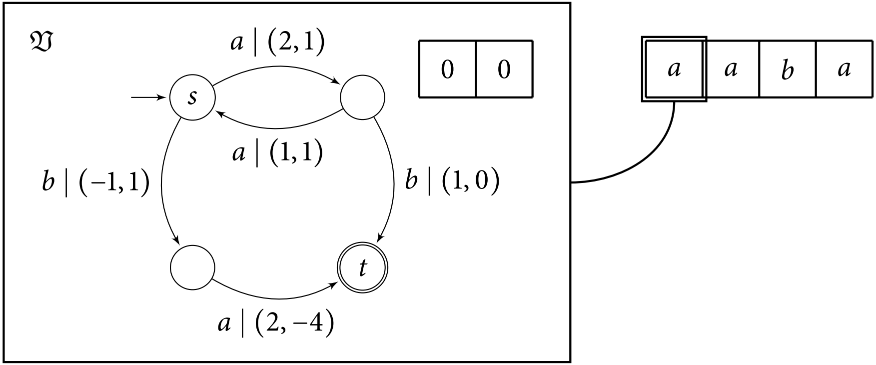
$$\mathfrak{V} = \left(Q , \Sigma , \Delta , s , t \right)$$

finite set of states

input alphabet

finite set of transitions $p \xrightarrow{a \mid \vec{v}} q$ with $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $\vec{v} \in \mathbb{Z}^d$

$s, t \in Q$ source and target state



Vector Addition Systems with States (1)

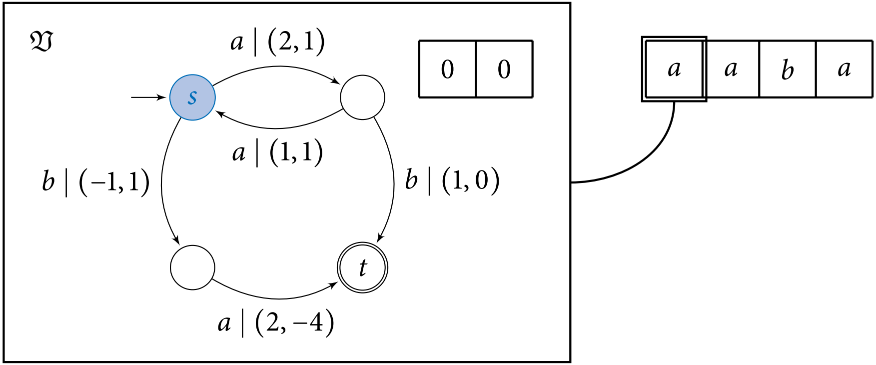
$$\mathfrak{V} = \left(Q, \Sigma, \Delta, s, t \right)$$

finite set of states

input alphabet

finite set of transitions $p \xrightarrow{a \mid \vec{v}} q$
with $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $\vec{v} \in \mathbb{Z}^d$

$s, t \in Q$ source and target state



Vector Addition Systems with States (1)

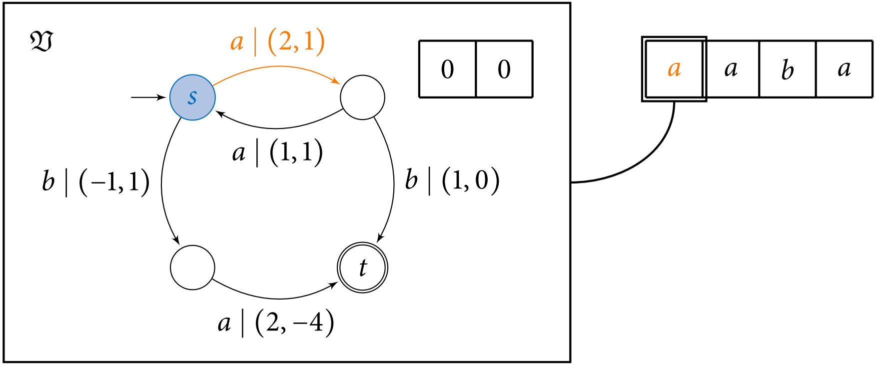
$$\mathfrak{V} = \big(Q , \Sigma , \Delta , s , t \big)$$

finite set of states

input alphabet

finite set of transitions $p \xrightarrow{a \mid \vec{v}} q$
with $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $\vec{v} \in \mathbb{Z}^d$

$s, t \in Q$ source and target state



Vector Addition Systems with States (1)

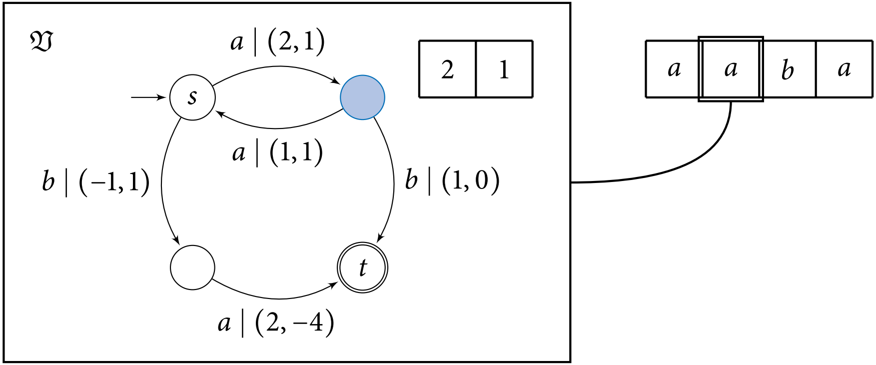
$$\mathfrak{V} = \big(Q, \Sigma, \Delta, s, t\big)$$

finite set of states

input alphabet

finite set of transitions $p \xrightarrow{a \mid \vec{v}} q$
with $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $\vec{v} \in \mathbb{Z}^d$

$s, t \in Q$ source and target state



Vector Addition Systems with States (1)

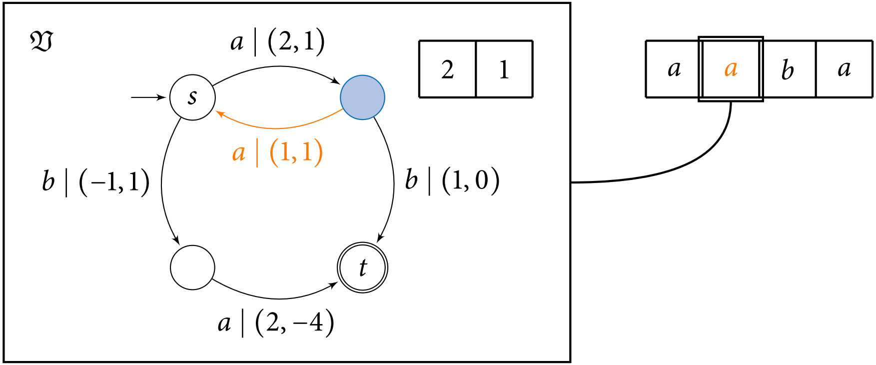
$$\mathfrak{V} = \big(Q , \Sigma , \Delta , s , t \big)$$

finite set of states

input alphabet

finite set of transitions $p \xrightarrow{a \mid \vec{v}} q$
with $p, q \in Q, a \in \Sigma \cup \{\varepsilon\}, \vec{v} \in \mathbb{Z}^d$

$s, t \in Q$ source and target state



Vector Addition Systems with States (1)

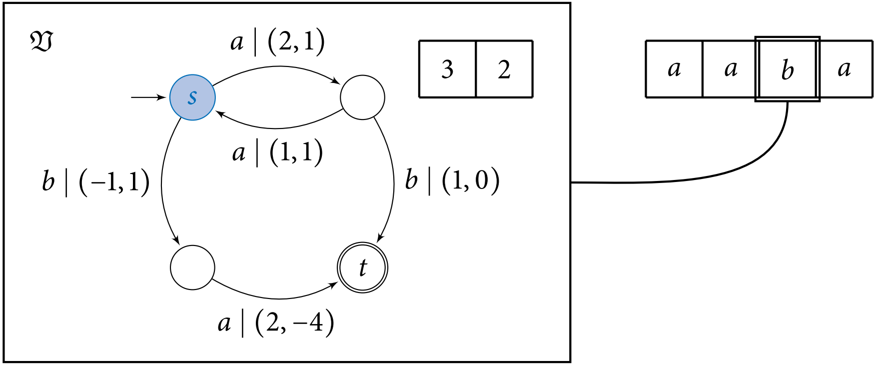
$$\mathfrak{V} = (Q, \Sigma, \Delta, s, t)$$

finite set of states

input alphabet

finite set of transitions $p \xrightarrow{a \mid \vec{v}} q$
with $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $\vec{v} \in \mathbb{Z}^d$

$s, t \in Q$ source and target state



Vector Addition Systems with States (1)

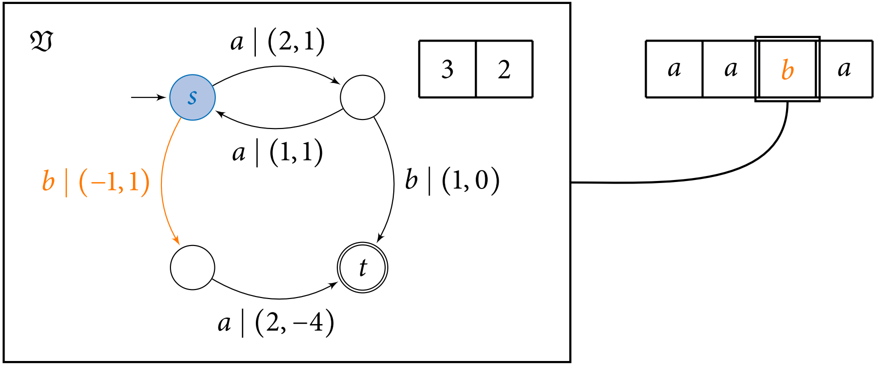
$$\mathfrak{V} = \big(Q\,,\ \Sigma\,,\ \Delta\,,\ s\,,\ t \big)$$

finite set of states

input alphabet

finite set of transitions $p \xrightarrow{a\,|\,\vec{v}} q$ with $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $\vec{v} \in \mathbb{Z}^d$

$s, t \in Q$ source and target state



Vector Addition Systems with States (1)

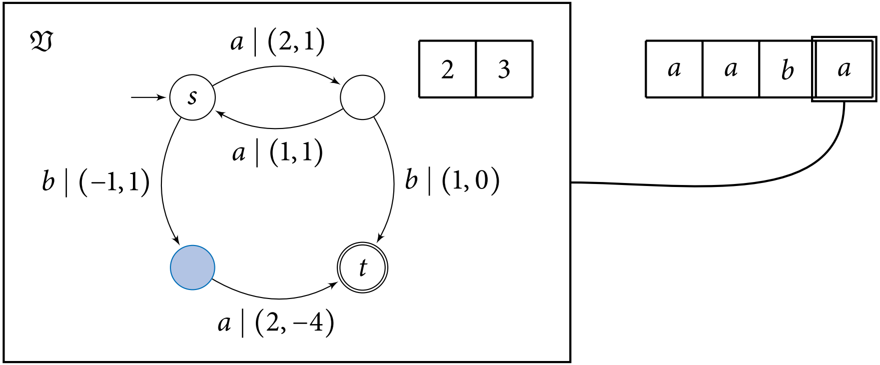
$$\mathfrak{V} = \left(Q\,,\ \Sigma\,,\ \Delta\,,\ s\,,\ t \right)$$

finite set of states

input alphabet

finite set of transitions $p \xrightarrow{a\,|\,\vec{v}} q$
with $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $\vec{v} \in \mathbb{Z}^d$

$s, t \in Q$ source and target state



Vector Addition Systems with States (1)

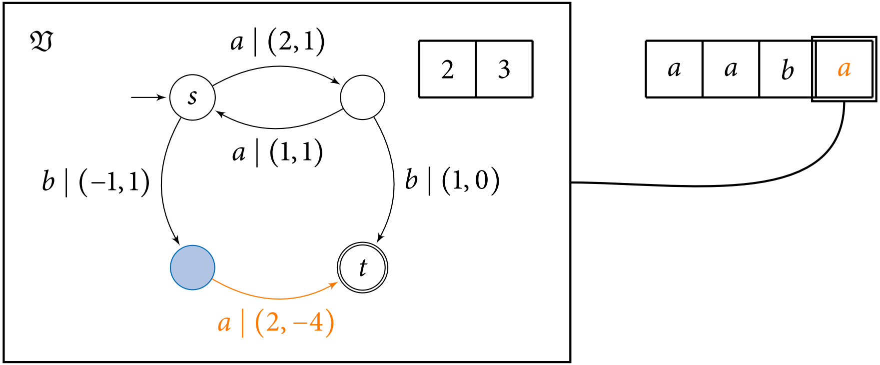
$$\mathfrak{V} = \big(Q \,,\, \Sigma \,,\, \Delta \,,\, s \,,\, t \big)$$

finite set of states

input alphabet

finite set of transitions $p \xrightarrow{a \,|\, \vec{v}} q$
with $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $\vec{v} \in \mathbb{Z}^d$

$s, t \in Q$ source and target state



Vector Addition Systems with States (1)

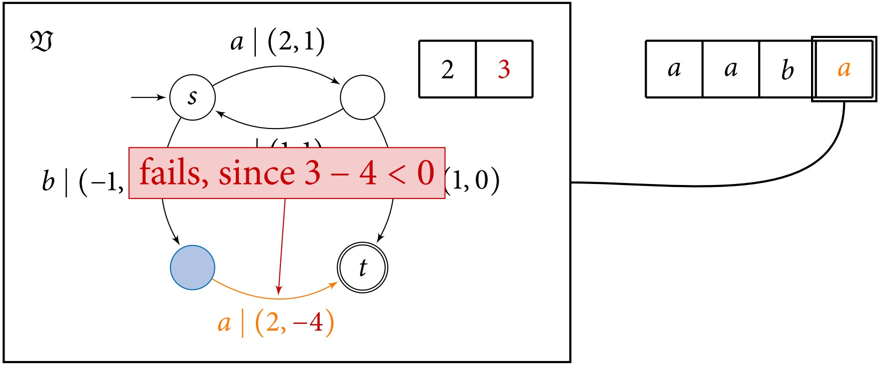
$$\mathfrak{V} = (Q, \Sigma, \Delta, s, t)$$

finite set of states

input alphabet

finite set of transitions $p \xrightarrow{a \,|\, \vec{v}} q$
with $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $\vec{v} \in \mathbb{Z}^d$

$s, t \in Q$ source and target state



Vector Addition Systems with States (2)

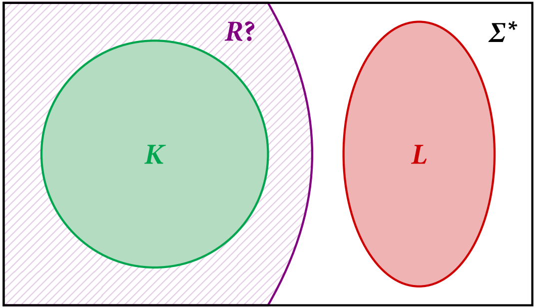
1. Reachability language:
   - $\mathrm{L}_{\mathrm{reach}}(\mathfrak{V}) = \{w \in \Sigma^* \mid (s, \vec{0}) \xrightarrow{w}_{\mathfrak{V}} (t, \vec{0})\}$

2. Coverability language:
   - $\mathrm{L}_{\mathrm{cov}}(\mathfrak{V}) = \{w \in \Sigma^* \mid \exists \vec{v} \in \mathbb{N}^d \colon (s, \vec{0}) \xrightarrow{w}_{\mathfrak{V}} (t, \vec{v}) \geq (t, \vec{0})\}$

Regular Separability (1)

Problem

- Given two languages $K, L \subseteq \Sigma^*$.
- Is there a regular language $R \subseteq \Sigma^*$ with $K \subseteq R$ and $L \cap R = \varnothing$?



- Note: Regular Separability ≠ Disjointness!

Regular Separability (2)

Theorem (Czerwiński et al. @ CONCUR 2018)

Let $\mathfrak{V}$ *and* $\mathfrak{W}$ *be two VASS. Then* $L_{\text{cov}}(\mathfrak{V})$ *and* $L_{\text{cov}}(\mathfrak{W})$ *are regular separable if, and only if,* $L_{\text{cov}}(\mathfrak{V}) \cap L_{\text{cov}}(\mathfrak{W}) = \emptyset$.

- Hence: Regular Separability for VASS coverability languages is decidable!
- Note: Decidability of Regular Separability for $L_{\text{reach}}(\mathfrak{V})$ and $L_{\text{reach}}(\mathfrak{W})$ is still open!

Question

What is the size of a regular separator of $L_{\text{cov}}(\mathfrak{V})$ and $L_{\text{cov}}(\mathfrak{W})$?

- Czerwiński et al.: doubly exp. lower bound & triply exp. upper bound

Main Theorem

Theorem

Let $\mathfrak{V}$ *and* $\mathfrak{W}$ *be two VASS with* $\leq n$ *states and updates of norm* $\leq m$*. If* $\mathrm{L}_{\mathrm{cov}}(\mathfrak{V}) \cap \mathrm{L}_{\mathrm{cov}}(\mathfrak{W}) = \varnothing$ *then there is an separating NFA with at most* $(n+m)^{2^{\mathrm{poly}(d)}}$ *many states.*

Proof (1): Reduce to Counter Instructions

- $\Gamma_d = \{\mathbf{a_i}, \overline{\mathbf{a_i}} \mid 1 \leq i \leq d\}$
  - $\mathbf{a_i}$ increase counter i by 1
  - $\overline{\mathbf{a_i}}$ decrease counter i by 1
- $C_d = \{w \in \Gamma_d^* \mid \forall$ prefixes v of $w, 1 \leq i \leq d\colon |v|_{\mathbf{a_i}} \geq |v|_{\overline{\mathbf{a_i}}}\}$

Lemma (Jantzen 1979)

$L \subseteq \Sigma^*$ is a VASS coverability language iff there is a rational transduction T with $L = T(C_d)$.

Corollary

Let $\mathfrak{V}$ and $\mathfrak{W}$ be two VASS and T be a rational transduction with $\mathrm{L}_{\mathrm{cov}}(\mathfrak{W}) = T(C_d)$. Then $\mathrm{L}_{\mathrm{cov}}(\mathfrak{V})$ is regularly separable from $\mathrm{L}_{\mathrm{cov}}(\mathfrak{W})$ iff $T^{-1}(\mathrm{L}_{\mathrm{cov}}(\mathfrak{V}))$ is regularly separable from C_d.

Proof (1): Reduce to Counter Instructions

- $\Gamma_d = \{\mathbf{a_i}, \overline{\mathbf{a_i}} \mid 1 \leq i \leq d\}$
  - $\mathbf{a_i}$ increase counter i by 1
  - $\overline{\mathbf{a_i}}$ decrease counter i by 1
- $C_d = \{w \in \Gamma_d^* \mid \forall$ prefixes v of $w, 1 \leq i \leq d\colon |v|_{\mathbf{a_i}} \geq |v|_{\overline{\mathbf{a_i}}}\}$

Lemma (Jantzen 1979)

$L \subseteq \Sigma^*$ is a VASS

vith $L = T(C_d)$.

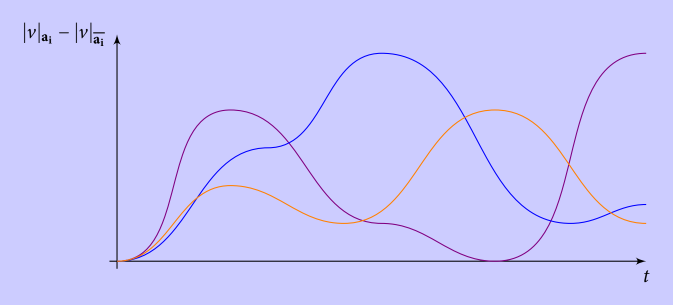
Corollary

Let $\mathfrak{V}$ and $\mathfrak{W}$ be

$= T(C_d)$. Then

$\mathrm{L_{cov}}(\mathfrak{V})$ is regula

parable from C_d.



Proof (1): Reduce to Counter Instructions

- $\Gamma_d = \{\mathbf{a_i}, \overline{\mathbf{a_i}} \mid 1 \leq i \leq d\}$
  - $\mathbf{a_i}$ increase counter i by 1
  - $\overline{\mathbf{a_i}}$ decrease counter i by 1
- $C_d = \{w \in \Gamma_d^* \mid \forall$ prefixes v of $w, 1 \leq i \leq d : |v|_{\mathbf{a_i}} \geq |v|_{\overline{\mathbf{a_i}}}\}$

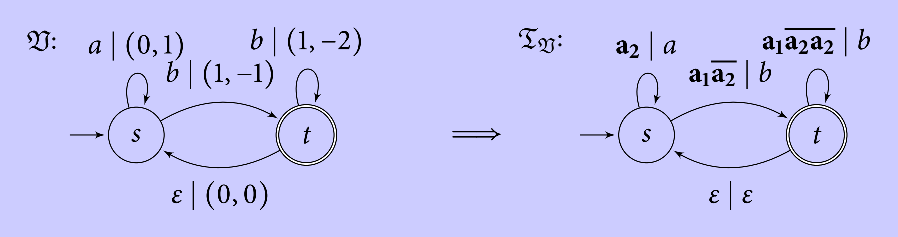
Lemma (Jantzen 1979)

$L \subseteq \Sigma^*$ is a VASS coverability language iff there is a rational transduction T with $L = T(C_d)$.

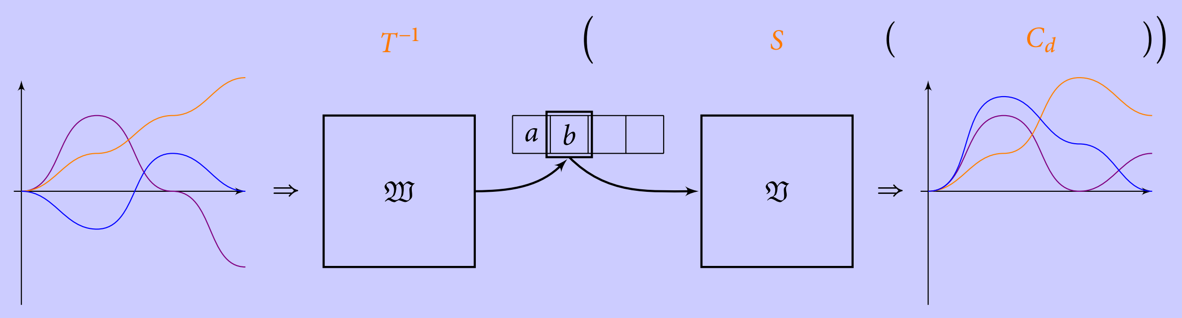
Coroll

Let $\mathfrak{V}$ a *Then*

$\mathrm{L_{cov}}(\mathfrak{V}$ *om C_d.*



Proof (1): Reduce to Counter Instructions



Corollary

Let $\mathfrak{V}$ *and* $\mathfrak{W}$ *be two VASS and* T *be a rational transduction with* $\mathrm{L}_{\mathrm{cov}}(\mathfrak{W}) = T(C_d)$*. Then* $\mathrm{L}_{\mathrm{cov}}(\mathfrak{V})$ *is regularly separable from* $\mathrm{L}_{\mathrm{cov}}(\mathfrak{W})$ *iff* $T^{-1}(\mathrm{L}_{\mathrm{cov}}(\mathfrak{V}))$ *is regularly separable from* C_d*.*

Proof (2): Basic Separators

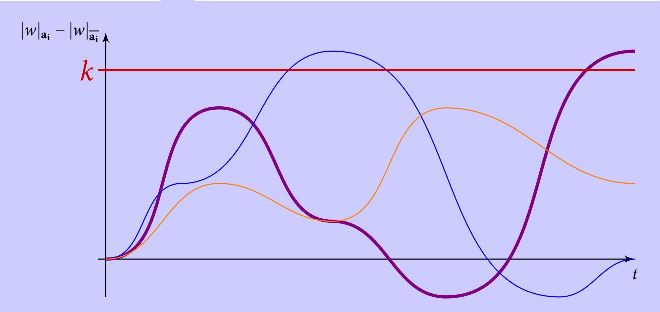
- For $k \in \mathbb{N}$ let $B_k \subseteq \Gamma_d^*$ be the following language: $w \in B_k$ iff there is $1 \le i \le d$ with
  - there is a prefix v of w with $|v|_{\mathbf{a_i}} < |v|_{\overline{\mathbf{a_i}}}$ and
  - each proper prefix u of v satisfies $0 \le |u|_{\mathbf{a_i}} - |u|_{\overline{\mathbf{a_i}}} \le k$
- B_k is accepted by a DFA of size $O(k^d)$.

Theorem (Czerwiński & Zetzsche @ LICS 2020)

Let $\mathfrak{V}$ *and* $\mathfrak{W}$ *be two VASS with* $\mathrm{L}_{\mathrm{cov}}(\mathfrak{V}) \cap \mathrm{L}_{\mathrm{cov}}(\mathfrak{W}) = \varnothing$ *and let* T *be a rational transduction with* $\mathrm{L}_{\mathrm{cov}}(\mathfrak{W}) = T(C_d)$*. Then* B_k *is a regular separator of* $T^{-1}(\mathrm{L}_{\mathrm{cov}}(\mathfrak{V}))$ *and* C_d *for a* $k \in \mathbb{N}$*.*

Proof (2): Basic Separators

- For $k \in \mathbb{N}$ let $B_k \subseteq \Gamma_d^*$ be the following language: $w \in B_k$ iff there is $1 \leq i \leq d$ with
  - there is a prefix v of w with $|v|_{\mathtt{a_i}} < |v|_{\overline{\mathtt{a_i}}}$ and
  - each proper prefix u of v satisfies $0 \leq |u|_{\mathtt{a_i}} - |u|_{\overline{\mathtt{a_i}}} \leq k$
- B_k is accepted by a DFA of size $O(k^d)$.



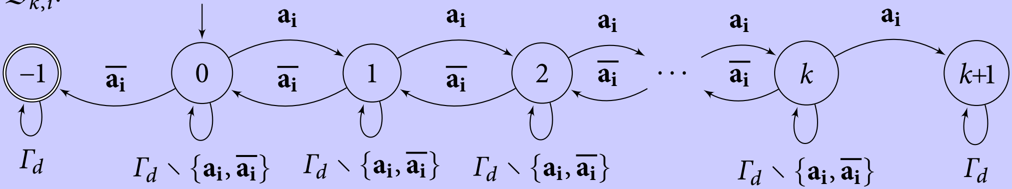
Theorem (Czerv

Let $\mathfrak{V}$ and $\mathfrak{W}$ be t … *onal transduction*
with $\mathrm{L_{cov}}(\mathfrak{W}) = T$ … *d* C_d *for a* $k \in \mathbb{N}$.

Proof (2): Basic Separators

- For $k \in \mathbb{N}$ let $B_k \subseteq \Gamma_d^*$ be the following language: $w \in B_k$ iff there is $1 \leq i \leq d$ with
  - there is a prefix v of w with $|v|_{\mathbf{a_i}} < |v|_{\overline{\mathbf{a_i}}}$ and
  - each proper prefix u of v satisfies $0 \leq |u|_{\mathbf{a_i}} - |u|_{\overline{\mathbf{a_i}}} \leq k$
- B_k is accepted by a DFA of size $O(k^d)$.

$\mathfrak{B}_{k,i}$:



$B_k = \bigcup_{1 \leq i \leq d} L(\mathfrak{B}_{k,i})$

Proof (3): Covering

Theorem (Rackoff 1978)

Let $\mathfrak{V}$ be a VASS, c be a configuration of $\mathfrak{V}$, and a vector $\vec{v} \in \mathbb{N}^d$ with $c \to_{\mathfrak{V}}^{*} (t, \vec{v}) \geq (t, \vec{0})$. Then there is $0 \leq \ell \leq \underbrace{(n+m)^{2^{\mathrm{poly}(d)}}}_{=:\,\mathrm{Rackoff}(\mathfrak{V})}$ and $\vec{w} \in \mathbb{N}^d$ with $c \to_{\mathfrak{V}}^{\ell} (t, \vec{w}) \geq (t, \vec{0})$. Here, n is the number of states in $\mathfrak{V}$ and m is the norm of the counter updates in $\mathfrak{V}$.

Theorem

Let $\mathfrak{V}$ and $\mathfrak{W}$ be two VASS with $\mathrm{L}_{\mathrm{cov}}(\mathfrak{V}) \cap \mathrm{L}_{\mathrm{cov}}(\mathfrak{W}) = \varnothing$ and let T be a rational transduction with $\mathrm{L}_{\mathrm{cov}}(\mathfrak{W}) = T(C_d)$. Then $B_{\mathrm{Rackoff}(\mathfrak{V} \times \mathfrak{W})}$ is a regular separator of $T^{-1}(\mathrm{L}_{\mathrm{cov}}(\mathfrak{V}))$ and C_d.

- Finally, $T(B_{\mathrm{Rackoff}(\mathfrak{V} \times \mathfrak{W})})$ is a regular separator of $\mathrm{L}_{\mathrm{cov}}(\mathfrak{V})$ and $\mathrm{L}_{\mathrm{cov}}(\mathfrak{W})$. ☐

Conclusion

		NFAs		DFAs	
		unary	binary	unary	binary
d as input		2-exp.		3-exp.	
d fixed	$d \geq 2$	poly.	exp.	exp.	2-exp.
	$d = 1$	poly.	exp.	exp.	exp.

Thank you!