The Cayley-Graph of the Queue Monoid: Logic and Decidability

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Motivation

Question

Is the Queue Monoid automatic?

- Queue Monoid is algebraic description of a queue’s behavior
- Automatic structures have nice properties
  - e.g. decidable FO-theory [Khoussainov, Nerode 1995]
- The Queue Monoid’s FO-theory is undecidable

Answer

No!
Motivation

Question (Huschenbett, Kuske, Zetzsche 2014)

Is the Cayley-graph of the Queue Monoid automatic?

- Queue Monoid is algebraic description of a queue’s behavior
- Automatic structures have nice properties
  - e.g. decidable FO-theory [Khoussainov, Nerode 1995]
- The Queue Monoid’s FO-theory is undecidable

Possible Approach

Prove that the FO-theory of the Queue Monoid’s Cayley-graph is undecidable.
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Is the Cayley-graph of the Queue Monoid automatic?

- Queue Monoid is algebraic description of a queue’s behavior
- Automatic structures have nice properties
  - e.g. decidable FO-theory [Khoussainov, Nerode 1995]
- The Queue Monoid’s FO-theory is undecidable

Here

The FO-theory of the Queue Monoid’s Cayley-graph is decidable.
Let $A$ be an alphabet ($|A| \geq 2$).

- two actions for each $a \in A$:
  - write letter $a$, denoted: $a$
  - read letter $a$, denoted: $\bar{a}$

- $\Sigma := \{a, \bar{a} \mid a \in A\}$

Example

$q = abaa$  
$t = b\bar{a}b\bar{b}$

\[
\begin{array}{cccc}
  a & b & a & a \\
\end{array}
\]
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$q = abaa$

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<table>
<thead>
<tr>
<th>a</th>
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Example

$q = abaa$ \hspace{1cm} $t = b\bar{a}b\bar{b}$

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```
  b  a  a  b
  
  \hline
  b
```
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The Queue Monoid

Definition

$s, t \in \Sigma^*$ act equally (in symbols $s \equiv t$) if, and only if,

$$\forall p, q \in A^*: p \xrightarrow{s} q \iff p \xrightarrow{t} q.$$  

Theorem (Huschenbett, Kuske, Zetzsche 2014)

$\equiv$ is the least congruence on $\Sigma^*$ satisfying the following equations:

1. $ab \equiv ba$ if $a \neq b$
2. $aac \equiv \overline{aac}$
3. $c\overline{a}a \equiv c\overline{aa}$

for each $a, b, c \in A$.

Definition

$Q := \Sigma^*/\equiv$ is called the queue monoid.
Let $[w] \equiv \in Q$. A **characteristic** of $[w] \equiv$ is a triple $(\lambda, a_1 \ldots a_m, \rho) \in (A^*)^3$ with

$$\bar{\lambda} \cdot a_1 \bar{a}_1 a_2 \bar{a}_2 \ldots a_m \bar{a}_m \cdot \rho \equiv w.$$ 

**Example**

Let $t = \overline{[abacabacabaaba]} \equiv$.  
- $\overline{abaca\bar{a}b\bar{b}a\bar{a}caba} \equiv \overline{abacabacabaaba}$  
- $(abac, aba, caba)$ is a characteristic of $t$
Characteristics

**Definition**

Let \([w]_\equiv \in Q\). A **characteristic** of \([w]_\equiv\) is a triple 
\((\lambda, a_1 \ldots a_m, \rho) \in (A^*)^3\) with

\[
\bar{\lambda} \cdot a_1 \bar{a_1} a_2 \bar{a_2} \ldots a_m \bar{a_m} \cdot \rho \equiv w.
\]

**Proposition (cf. Huschenbett, Kuske, Zetzsche 2014)**

*Each* \(t \in Q\) *has exactly one characteristic.*

**Definition**

Let \(t \in Q\) and \((w_L, w_M, w_R)\) be its characteristic. We denote the second component by \(\mu(t) := w_M\).
The queue monoid’s Cayley-graph is the $\Sigma$-labeled graph

$\mathcal{G} := (Q, (E_\alpha)_{\alpha \in \Sigma})$ with

$$E_\alpha = \{(t, t\alpha) \mid t \in Q\}.$$
Properties of the Cayley-Graph

Proposition

1. $\mathcal{C}$ is an acyclic graph with root $\varepsilon$.
2. $\mathcal{C}$ has bounded out-degree and unbounded in-degree.
3. $\mathcal{C}$ contains an infinite 2-dimensional grid as induced subgraph.

Corollary (cf. Seese 1991)

The MSO-theory of $\mathcal{C}$ is undecidable.
Main Theorem

Theorem

The FO-theory of \( C \) is primitive recursive.

Idea (cf. Ferrante, Rackoff 1979)

For each element from \( Q \), find another one which is equivalent and close to the root \( \varepsilon \).

- We prove this by induction on quantifier depth of an FO-formula.
Induction Step

- Let $\phi(\vec{y}) = \exists x : \psi(\vec{y}, x) \in \text{FO}$ with $m$ free variables and quantifier rank $\leq r + 1$.
- Let $\vec{s}, \vec{t} \in Q^m$ with $(\mathcal{C}, \vec{s}) \models \phi$ and $(\mathcal{C}, \vec{t}) \models \phi$; and let $s_{m+1} \in Q$ with $(\mathcal{C}, \vec{s}, s_{m+1}) \models \psi$.
- Find "small" $t_{m+1}$ with $(\mathcal{C}, \vec{t}, t_{m+1}) \models \psi$.

First case:
Induction Step

- Let $\phi(\vec{y}) = \exists x : \psi(\vec{y}, x) \in FO$ with $m$ free variables and quantifier rank $\leq r + 1$.
- Let $\bar{s}, \bar{t} \in Q^m$ with $(\mathcal{C}, \bar{s}) \models \phi$ and $(\mathcal{C}, \bar{t}) \models \phi$; and let $s_{m+1} \in Q$ with $(\mathcal{C}, \bar{s}, s_{m+1}) \models \psi$.
- Find “small” $t_{m+1}$ with $(\mathcal{C}, \bar{t}, t_{m+1}) \models \psi$.

second case:
Consider the transformations having \(abacaba\) as subsequence of write and read actions:

\[
\begin{align*}
(abacaba, \varepsilon, abacaba) & \leadsto \underline{abacabaabacaba} \\
(abacab, a, bacaba) & \leadsto \underline{abacaba\bar{a}bacaba} \\
(abac, aba, caba) & \leadsto \underline{abaca\bar{a}bba\bar{a}caba} \\
(\varepsilon, abacaba, \varepsilon) & \leadsto \underline{a\bar{a}b\bar{a}a\bar{c}\bar{a}a\bar{a}b\bar{b}\bar{a}\bar{a}}
\end{align*}
\]

Shortest path between \(abacabaabacaba\) and \(a\bar{a}b\bar{b}a\bar{a}c\bar{c}a\bar{a}b\bar{b}a\bar{a}\)?
Consider the transformations having \textit{abacaba} as subsequence of write and read actions:

\[(\text{abacaba}, \varepsilon, \text{abacaba}) \leadsto \underline{\text{abacaba}}\underline{\text{abacaba}}\]

\[(\text{abacab}, a, \text{bacaba}) \leadsto \underline{\text{abacaba}}\underline{\text{abacaba}}\]

\[(\text{abac}, aba, caba) \leadsto \underline{\text{abaca}}\underline{\text{abhaa}}\underline{\text{acaba}}\]

\[(\varepsilon, \text{abacaba}, \varepsilon) \leadsto \underline{a\bar{a}b\bar{b}a\bar{c}\bar{c}a\bar{a}}\underline{b\bar{b}a\bar{a}}\]

\begin{definition}

Let \(v, w \in A^*\).

- \(v\) is a \textbf{border} of \(w\) if it is a prefix and a suffix of \(w\).
- The \textbf{border-decomposition} \((w_0, \ldots, w_n)\) of \(w\) is the sequence of all borders of \(w\) in length-increasing order.

\end{definition}
Recall \( \vec{w} = (\varepsilon, a, aba, abacaba) \) is the border-decomposition of \( abacaba \).

Here: \( w_{i+1} = w_i x_i w_i \) for each \( 0 \leq i < 3 \) and some \( x_i \in A^* \)

**Definition**

Let \( (w_0, \ldots, w_n) \) be the border-decomposition of \( w \in A^* \) and \( r \in \mathbb{N} \). The \( r \)-skeleton \( S_r(w) \) of \( w \) is the sequence \( (s_0, \ldots, s_{n-1}) \) where \( s_i \) is the maximal prefix of length at most \( r \) of \( w_i^{-1} w \).

**Example \( (w = abacaba, r = 2) \)**

- \( abacaba \)
- \( abacaba \)
- \( abacaba \)

\( \Rightarrow S_2(w) = (ab, ba, ca) \)
Skeletons & Instantiations

- Recall $\vec{w} = (\varepsilon, a, aba, abacaba)$ is the border-decomposition of $abacaba$.
- Here: $w_{i+1} = w_ix_iw_i$ for each $0 \leq i < 3$ and some $x_i \in A^*$

**Definition**

Let $(w_0, \ldots, w_n)$ be the border-decomposition of $w \in A^*$ and $r \in \mathbb{N}$. The $r$-skeleton $S_r(w)$ of $w$ is the sequence $(s_0, \ldots, s_{n-1})$ where $s_i$ is the maximal prefix of length at most $r$ of $w_i^{-1}w$.

**Definition**

An $r$-instantiation of an $r$-skeleton $(s_0, \ldots, s_{n-1})$ is the word $v_n$ with $v_0 = \varepsilon$ and $v_{i+1} = v_is_iy_iv_i$ (for some special $y_i \in A^O(n+r)$).

**Lemma**

Let $v$ be an $r$-instantiation of an $r$-skeleton $(s_0, \ldots, s_{n-1})$. Then $|v| = O(2^{nr})$ and $S_r(v) = (s_0, \ldots, s_{n-1})$. 
Induction Step

- Let $\phi(\vec{y}) = \exists x : \psi(\vec{y}, x) \in \text{FO}$ with $m$ free variables and quantifier rank $\leq r + 1$.
- Let $\vec{s}, \vec{t} \in Q^m$ with $(\mathcal{C}, \vec{s}) \models \phi$ and $(\mathcal{C}, \vec{t}) \models \phi$; and let $s_{m+1} \in Q$ with $(\mathcal{C}, \vec{s}, s_{m+1}) \models \psi$.
- Find “small” $t_{m+1}$ with $(\mathcal{C}, \vec{t}, t_{m+1}) \models \psi$.
  - second case:
Shortening of Words

- find \( s' \in Q \) close to \( s_{m+1} \) with
  - \( \mu(s') \) has as many borders as possible
  - hence, \( S_{O(2^{r+m})}(\mu(s')) \) is as long as possible
- we can construct an automaton \( A_{s'} \) with
  \[
  L(A_{s'}) = \left\{ V \in (A^{O(2^{r+m})})^* \mid V \equiv_{r+1}^{\text{B"uchi}} S_{O(2^{r+m})}(\mu(s')) \right\}
  \]
  - \( \equiv_{r+1}^{\text{B"uchi}} \) is related to Büchi’s logic on words
- find some small word \( V \in L(A_{s'}) \) and construct an \( O(2^{r+m}) \)-instantiation \( v \) of \( V \)
- choose \( t' \in Q \) with \( \mu(t') = v \) appropriately
- recall the path from \( s' \) to \( s_{m+1} \) and go a similar path from \( t' \) to new \( t_{m+1} \)

**Problem**

There may be multiple nodes \( t_{m+1} \) with path from \( t' \) labelled with \( w \).
Let \((w_0, \ldots, w_n)\) and \((v_0, \ldots, v_{n'})\) be the border-decompositions of \(\mu(s')\) resp. \(\mu(t')\).

Recall that \(S_{\mathcal{O}(2^{r+m})}(\mu(s')) \equiv_{r+1}^\text{Büchi} S_{\mathcal{O}(2^{r+m})}(\mu(t'))\).

Let \(0 \leq k \leq n\) be maximal such that \(w_k\) is a prefix of \(\mu(s_{m+1})\).

Hence, there is \(\ell\) with
\[(S_{\mathcal{O}(2^{r+m})}(\mu(s')), k) \equiv_{r}^\text{Büchi} (S_{\mathcal{O}(2^{r+m})}(\mu(t')), \ell)\).

Find \(t_{m+1}\) such that \(v_\ell\) is maximal with \(v_\ell\) is prefix of \(\mu(t_{m+1})\).
Induction Step

- Let $\phi(\vec{y}) = \exists x : \psi(\vec{y}, x) \in \text{FO}$ with $m$ free variables and quantifier rank $\leq r + 1$.
- Let $\vec{s}, \vec{t} \in Q^m$ with $(\mathcal{C}, \vec{s}) \models \phi$ and $(\mathcal{C}, \vec{t}) \models \phi$; and let $s_{m+1} \in Q$ with $(\mathcal{C}, \vec{s}, s_{m+1}) \models \psi$.
- Find “small” $t_{m+1}$ with $(\mathcal{C}, \vec{t}, t_{m+1}) \models \psi$.
- third case:
- $s_{m+1}$ is close to $s_i$ ($1 \leq i \leq m$ minimal)

- Let $s'$ and $t'$ be constructed from $s_i$ as in 2nd case.
## Conclusion & Open Problems

### Theorem

*The FO-theory of $\mathcal{C}$ is primitive recursive.*

### Open Problems

1. Is the FO-theory of $\mathcal{C}$ decidable in elementary time?
2. Is $\mathcal{C}$ automatic?
3. Is the FO-theory of the (Partially) Lossy Queue Monoid’s Cayley-graph decidable?
   - (Partially) Lossy Queues can forget parts of their content at any time.

Thank you!