The Transformation Monoid of a Partially Lossy Queue

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What is a Partially Lossy Queue?

- we consider classical fifo-queues

Reliable Queues

- nothing can be forgotten or injected
- undecidable reachability [Brand, Zafiropulo 1983]
- can be "simulated" by queue with two distinct letters [Huschenbett, Kuske, Zetzsche 2014]

Lossy Queues

- everything can be forgotten, nothing can be injected
- decidable reachability [Abdulla, Jonsson 1994]
- cannot be "simulated" by lossy queues with less letters [Köcher 2016]

→ Partially Lossy Queues (PLQs)

Outline

1. Model the transformations on PLQs as monoid
   $\Longrightarrow$ PLQ monoid

2. Characterize which PLQ monoids embed into which others
   $\Longrightarrow$ kind of simulation of one PLQ by another

3. Characterize the trace monoids embedding into PLQ monoid

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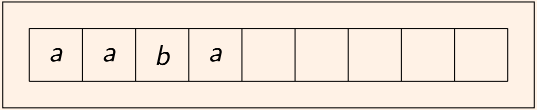
PLQs: Basics

- Let A be an alphabet ($|A| \geq 2$) and $U \subseteq A$.
  - U ... unforgettable letters
  - $A \setminus U$ ... forgettable letters
- two controllable operations for each $a \in A$:
  - write letter $a \rightsquigarrow \mathbf{a}$
  - read letter $a \rightsquigarrow \bar{\mathbf{a}}$
- $\boldsymbol{\Sigma} := \{\mathbf{a}, \bar{\mathbf{a}} \mid a \in A\}$
- non-controllable operation: forgetting letters from $A \setminus U$

Example

$$A = \{a, b\}, \; U = \{b\}$$

$q = aaba$ $\qquad$ $v = \mathbf{bb}\bar{\mathbf{a}}\bar{\mathbf{b}}$



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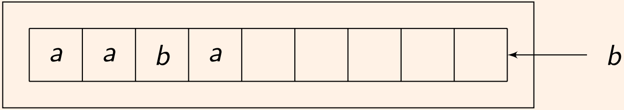

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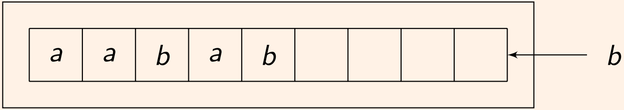

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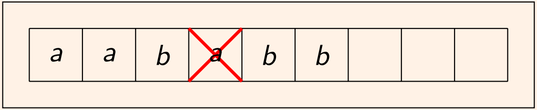

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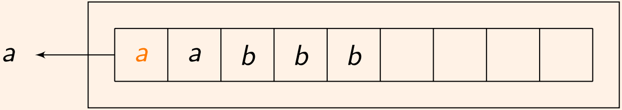

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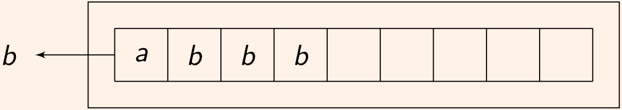

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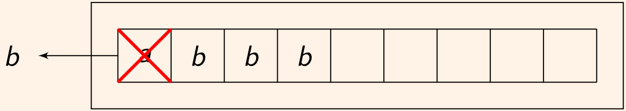

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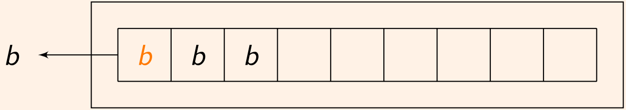

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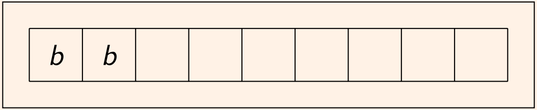

Definition

The map $\circ\colon (A^* \cup \{\bot\}) \times \mathbf{\Sigma}^* \to (A^* \cup \{\bot\})$ is defined for each $q \in A^*$, $a, b \in A$ and $\mathbf{v} \in \mathbf{\Sigma}^*$ as follows:

1. $q \circ \varepsilon = q$
2. $q \circ \mathbf{av} = qa \circ \mathbf{v}$
3. $bq \circ \overline{\mathbf{a}}\mathbf{v} = \begin{cases} q \circ \mathbf{v} & \text{if } a = b \\ q \circ \overline{\mathbf{a}}\mathbf{v} & \text{if } a \neq b, b \in A \setminus U \\ \bot & \text{otherwise} \end{cases}$
4. $\bot \circ \mathbf{v} = \bot = \varepsilon \circ \overline{\mathbf{a}}\mathbf{v}$

Example

Let $A = \{a, b\}$ and $U = \{b\}$.

$aaba \circ \mathbf{bb}\overline{\mathbf{a}}\overline{\mathbf{b}} = aabab \circ \mathbf{b}\overline{\mathbf{a}}\overline{\mathbf{b}} = aababb \circ \overline{\mathbf{a}}\overline{\mathbf{b}} = ababb \circ \overline{\mathbf{b}} = abb$

PLQ Monoids: Definition

Definition

- $\mathbf{v} \equiv \mathbf{w}$ for $\mathbf{v}, \mathbf{w} \in \mathbf{\Sigma}^*$ iff $q \circ \mathbf{v} = q \circ \mathbf{w}$ for any $q \in A^*$
- $\mathcal{Q}(A, U) := \mathbf{\Sigma}^*/_{\equiv}$ ... PLQ monoid
- $\mathcal{Q}'(f, u) := \mathcal{Q}(\{1, \ldots, f + u\}, \{1, \ldots u\})$

Theorem

The word problem of $\mathcal{Q}(A, U)$ is decidable in polynomial time. In other words: Given $\mathbf{v}, \mathbf{w} \in \mathbf{\Sigma}^*$. We can decide, whether $\mathbf{v} \equiv \mathbf{w}$ holds.

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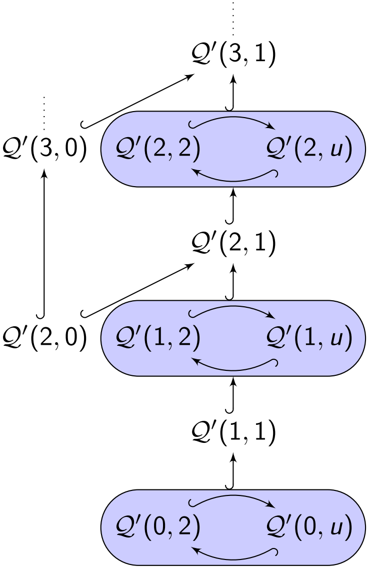

Main Theorem

Let $u + f, u' + f' \geq 2$.
Then $\mathcal{Q}'(f,u) \hookrightarrow \mathcal{Q}'(f',u')$ if, and only if, all of the following hold:

1. $f \leq f'$
2. $u' = 0 \Rightarrow u = 0$
3. $u' = 1 \Rightarrow u \leq 1$ or $f < f'$

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Trace Monoids: Definition

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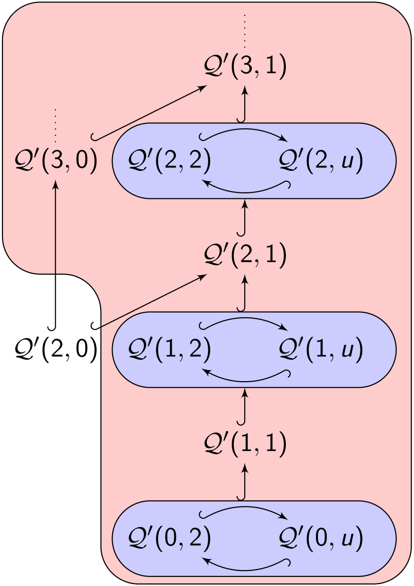
Definition

- An independence alphabet is a finite, undirected, irreflexive graph (Γ, I).
- $\equiv_I$ is the least congruence on Γ^* satisfying

$$ab \equiv_I ba \quad \text{for any } (a, b) \in I\,.$$

- $\mathbb{M}(\Gamma, I) := \Gamma^*/_{\equiv_I}$ ... trace monoid on (Γ, I).

Trace Monoids: Overview



Trace Monoids: Large Queue Alphabets

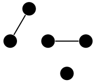
Theorem

Let $f + 2u \geq 3$ and (Γ, I) be an independence alphabet. Then the following are equivalent:

1. $\mathbb{M}(\Gamma, I) \hookrightarrow \mathcal{Q}'(f, u)$.
2. $\mathbb{M}(\Gamma, I) \hookrightarrow \mathcal{Q}'(0, 2)$.
3. $\mathbb{M}(\Gamma, I) \hookrightarrow \{a, b\}^* \times \{c, d\}^*$.
4. *One of the following conditions holds:*
   - a. *All nodes in (Γ, I) have degree ≤ 1.*
   - b. *The only non-trivial connected component of (Γ, I) is complete bipartite.*

[Kuske, Prianychnykova 2016]

a



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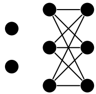
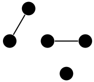


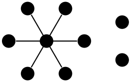
Trace Monoids: Binary Queue Alphabet

Theorem

Let (Γ, I) be an independence alphabet. Then the following are equivalent:

1. $\mathbb{M}(\Gamma, I) \hookrightarrow \mathcal{Q}'(2, 0)$.
2. *One of the following conditions holds:*
   - a. *All nodes in (Γ, I) have degree ≤ 1.*
   - b. *The only non-trivial connected component of (Γ, I) is a star graph.*

a 

b 

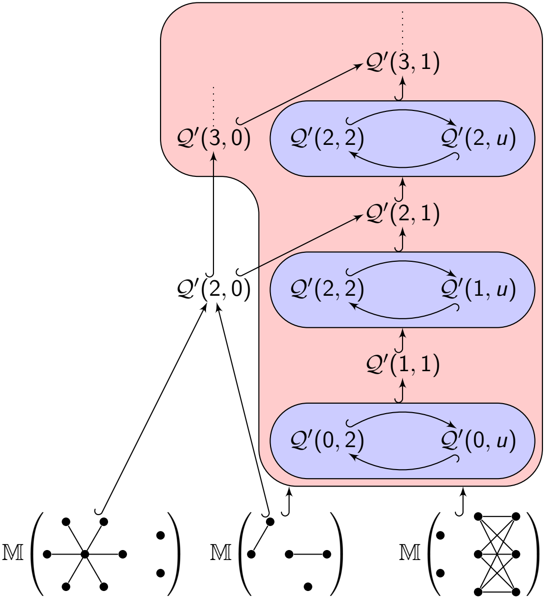
Submitted:

- Kleene-type characterization of recognizable subsets:
  - Recognizable($\mathcal{Q}(A, X)$) $\subsetneq$ Rational($\mathcal{Q}(A, X)$)
  - Recognizable($\mathcal{Q}(A, X)$) $=$ q-Rational($\mathcal{Q}(A, X)$)
- Schützenberger-type characterization of aperiodic subsets:
  - Aperiodic($\mathcal{Q}(A, X)$) $\neq$ Star-free($\mathcal{Q}(A, X)$)
  - Aperiodic($\mathcal{Q}(A, X)$) $=$ q-Star-free($\mathcal{Q}(A, X)$)

Not yet submitted:

- Algorithmic properties of Rational($\mathcal{Q}(A, X)$), e.g.,
  - Recognizability is undecidable

Summary



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