Reachability in Trace-PushDown Systems

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Abstract

We consider the reachability relation of pushdown systems whose pushdown holds a Mazurkiewicz trace instead of just a word as in classical systems. Under two natural conditions on the transition structure of such systems, we prove that the reachability relation is lc-rational, a new notion that restricts the class of rational trace relations. We also develop the theory of these lc-rational relations to the point where they allow to infer that forwards-reachability of a trace-pushdown system preserves the rationality and backwards-reachability the recognizability of sets of configurations. As a consequence it is decidable whether one recognizable set of configurations can be reached from some rational set of configurations. All our constructions are polynomial (assuming the dependence alphabet to be fixed).

These findings generalize results by Caucal on classical pushdown systems (namely the rationality of the reachability relation of such systems), complement results by Zetzsche (namely the decidability for arbitrary transition structures under severe restrictions on the dependence alphabet), and extend results from our conference papers [1] and [2].

Keywords: Reachability, Formal Verification, Pushdown Automaton, Distributed System

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1. Introduction

Much work has been done on the verification of pushdown systems that model the behavior of sequential recursive programs. As early as 1943, Post [3] claims that the reachability relation of pushdown systems (in Büchi's terminology from [4] a "canonical regular system") is decidable. In 1964, Büchi [4] constructed, from a pushdown system and a configuration, a finite automaton that accepts the set of reachable configurations as well as a finite automaton accepting the set of configurations backwards reachable from the given configuration. These constructions were later improved by Caucal [5, 6] and used to demonstrate that the reachability relation is prefix-recognizable and therefore rational. As a consequence, the set of configurations reachable forwards or backwards, respectively, from some regular set of configurations is regular, again (this result was reproved by Finkel [7] et al. using different methods). One therefore gets that the following regular reachability problem is decidable:

> Given a pushdown system and two regular sets of configurations, is it possible to reach some element of the second set from some element of the first set?

The preservation of regularity under backwards reachability was later reproved using a completely different technique by Esparza et al. [8].

Pushdown systems are a special case of valence automata: in a pushdown system, the possible pushdown contents come from a free monoid and can be accessed at the prefix, only. In a valence automaton, this free monoid is replaced by an arbitrary monoid. These automata have been considered from the purely language theoretic point of view (e.g. [9, 10, 11, 12, 13, 14, 15]) as well as in combinatorial group theory (e.g. [16, 17]). Zetzsche considered the question what monoids yield decidable reachability relations [18, 19] and decidable first-order theories of the configuration graph with reachability [20] (with D'Osualdo and Meyer). These works consider graph monoids [21] that allow an algebraic understanding of, e.g., pushdown systems, Petri nets, multi-stack automata, counter automata and many more. He describes a large class of graph monoids that result in decidable reachability relations, and another class that allows to construct valence automata with an undecidable reachability relation.

In this paper, we consider pushdown systems that hold, in their pushdown, not a word, but a Mazurkiewicz trace. Apart from systematic curiosity, these systems are interesting as they generalize cooperating multi-pushdown systems [22], a certain form of distributed pushdown systems. These models can, alternatively, be understood as valence automata over loop-free graph monoids. We are interested in the question to what extend the above regular reachability problem is decidable for these systems as well. Clearly, the reachability relation has to be decidable for a positive result which is, by Zetzsche's result, not necessarily the case. In this paper, we complement Zetzsche's work: while he was concerned with properties of the graph monoid, we describe properties of the transition structure of the pushdown automaton that guarantee decidability of the reachability relation and even of an appropriate version of the regular reachability problem; systems satisfying these properties are called *trace pushdown system*.

As in case of pushdown systems, we first ask which properties of a set of configurations are preserved under the reachability relation. In the case of pushdown systems, this is the case for regularity. In our systems, the set of configurations is a set of Mazurkiewicz traces where there are two distinct generalizations of regularity: rationality and recognizability. Our main result (Theorem 6.1) states that

(1) recognizability is preserved under backwards reachability, but not under forwards reachability and

(2) rationality is preserved under forwards reachability, but not under backwards reachability.

To obtain the preservation results for pushdown systems, Caucal first proved the rationality of the reachability relation and then employed that regularity is preserved under rational relations. This preservation does not hold for rational trace relations (Lemma 4.17). Therefore, we introduce a new notion that we call left-closed rational trace relations (short: lc-rational relations) and develop their theory as far as it is needed in our context. Namely, we show the following (see Table 1 for a comparision of the properties of rational word relations, rational trace relations, and lc-rational trace relations):

(i) A suitable generalization of Caucal's prefix-recognizable relations to the trace setting is lc-rational (Theorem 4.15).

(ii) The composition and union of lc-rational relations is lc-rational (Proposition 4.9).

(iii) The left-application of lc-rational relations preserves the rationality and the right-application preserves the recognizability of sets of traces (Theorem 4.16).

Coming back to trace-pushdown systems, we then prove that the reachability relation of a trace-pushdown system is lc-rational (cf. Table 2 for an overview of the reachability relations of pushdown systems with word and trace semantics, resp.). Using the method by Finkel et al. [7], one can transform every (classical) pushdown system into an equivalent one in which any path can be simulated by a path consisting of at most two phases: in the first one, the pushdown is shortened and, in the second, no shortening takes place. While this is not the case for trace-pushdown systems, we are able to show that a uniformly bounded number of phases suffices (the bound only depends on the trace monoid, Proposition 5.11). Using (i), we also show that the restriction of the reachability relation to single-phase paths is lc-rational (Propositions 5.4 and 5.6). Hence (ii) implies that the reachability relation of a trace-pushdown system is lc-rational (Theorem 5.1). Now the preservation results from (1) and (2) follow immediately from (iii).

As all our constructions are efficient and the non-disjointness of a recognizable and a rational set of traces is decidable in polynomial time, it follows that it

<table>
<tr><th></th><th>rational word relations</th><th>rational trace relations</th><th>lc-rational trace relations</th></tr>
<tr><td rowspan="3">examples</td><td>$(K\times L)\cdot\mathrm{Id}_{\Gamma^*}$ iff K, L regular</td><td>$(\mathcal{K}\times\mathcal{L})\cdot\mathrm{Id}_{\mathbb{M}(\mathcal{D})}$ iff $\mathcal{K}, \mathcal{L}$ rational</td><td>$(\mathcal{K}\times\mathcal{L})\cdot\mathrm{Id}_{\mathbb{M}(\mathcal{D})}$ iff $\mathcal{K}$ recognizable, $\mathcal{L}$ rational</td></tr>
<tr><td>superword</td><td>supertrace</td><td>supertrace</td></tr>
<tr><td>subword</td><td>subtrace</td><td>not: subtrace</td></tr>
<tr><td>union</td><td>yes</td><td>yes</td><td>yes</td></tr>
<tr><td>inverse</td><td>yes</td><td>yes</td><td>no</td></tr>
<tr><td>composition</td><td>yes</td><td>no</td><td>yes</td></tr>
<tr><td>concatenation</td><td>yes</td><td>yes</td><td>no</td></tr>
<tr><td>right application</td><td rowspan="2">preserve regularity</td><td rowspan="2">preserve neither rationality nor recognizability</td><td>preserve recognizability, but not rationality</td></tr>
<tr><td>left application</td><td>preserve rationality, but not recognizability</td></tr>
</table>

Table 1: Comparision of (lc-)rational word and trace relations. The first four lines describe the closure properties of the respective class of relations, the two bottom lines their preservation properties.

pushdown systems	reachability relation
with word semantics	$\bigcup_{i\in[n]}(K_i \times L_i)\cdot\mathrm{Id}_{\Gamma^*}$ for K_i, L_i regular; in particular rational
with trace semantics satisfying (P1) and (P2), i.e., tPDS	$\bigcup_{i\in[n]}\prod_{j\in[m_i]}(\mathcal{K}_{ij}\times\mathcal{L}_{ij})\cdot\mathrm{Id}_{\mathbb{M}(\mathcal{D})}$ for $\mathcal{K}_{ij}$ recognizable, $\mathcal{L}_{ij}$ rational; in particular lc-rational
violating (P1) or (P2)	undecidable

Table 2: Reachability relations. ($\prod$ denotes the composition of relations. The properties (P1) and (P2) are defined in Definition 3.2 resp. Lemma 3.6. Intuitively, (P1) says that transitions of the tPDS cannot disable the reading of a letter from the top of the pushdown. (P2) is a diamond property for the underlying automaton allowing certain commutations of operations.)

is decidable in polynomial time whether a recognizable set of traces is forwards reachable from some rational (e.g., recognizable) set of traces. In that sense, we generalize Caucal's result to the realm of trace pushdown systems.

2. Preliminaries

In this paper, 0 is a natural number and, for $n \in \mathbb{N}$, we set $[n] = \{1, 2, \dots, n\}$.

Let A be some alphabet and $w \in A^*$ a word over A. The *alphabet of* w, denoted $\mathrm{Alph}(w) \subseteq A$, is the set of letters occuring in the word w. Furthermore, $|w|_a$ denotes the number of occurrences of the letter $a \in A$ in the word w.

Trace Theory A *dependence alphabet* is a pair $\mathcal{D} = (A, D)$ where A is a finite set of *letters* and $D \subseteq A \times A$ is a reflexive and symmetric relation, the *dependence relation*. For a letter $a \in A$, we write $D(a)$ for the set $\{b \in A \mid (a, b) \in D\}$ of letters dependent from a, $D(B) = \bigcup_{b \in B} D(b)$ is the set of letters dependent from some letter in $B \subseteq A$. Note that $a \in D(a)$ since D is reflexive and $a \in D(b)$ iff $b \in D(a)$ since D is symmetric. For a word $w \in A^*$, let $D(w) = D(\mathrm{Alph}(w))$ be the set of letters dependent from some letter in the word w. The *independence relation* $I \subseteq A \times A$ is the set of pairs (a, b) of distinct letters with $(a, b) \notin D$. If $\mathrm{Alph}(u) \times \mathrm{Alph}(v) \subseteq I$ for two words $u, v \in A^*$, i.e., if every letter a from u is independent from every letter b from v, then we write $u \parallel v$.

To measure the complexity of our algorithms, we will need the following parameters of a dependence alphabet $\mathcal{D} = (A, D)$:

- The *size* $\|\mathcal{D}\|$ is the number $|A|$ of letters.
- Two letters $a, b \in A$ are *twins* if they have the same dependent letters, i.e., if $D(a) = D(b)$ (implying in particular $(a, b) \in D$ since $a \in D(a)$). We denote by $\mathrm{twns}(a) = \{b \in A \mid D(a) = D(b)\}$ the set of all twins of the letter a.

  By $\mathrm{twns}(\mathcal{D}) = \{\mathrm{twns}(a) \mid a \in A\}$ we denote the set of all equivalence classes of the relation "twin"; the *twin index* $\mathrm{TI}(\mathcal{D}) = |\mathrm{twns}(\mathcal{D})|$ is the number of these equivalence classes. Note that $\mathrm{TI}(\mathcal{D})$ equals the number of sets $D(a)$ for $a \in A$.
- The *set twin index* $\mathrm{TI}_*(\mathcal{D})$ is the number of sets $D(B)$ for $B \subseteq A$ (calling sets $B, C \subseteq A$ "set twins" if $D(B) = D(C)$, the set twin index is the number of equivalence classes of the relation "set twin").
- The *independence number* $\alpha(\mathcal{D})$ is the maximal size of a set $B \subseteq A$ of mutually independent letters (the notation $\alpha(\mathcal{D})$ is standard in graph theory).

Since independent letters cannot be twins, we get $\alpha(\mathcal{D}) \leq \mathrm{TI}(\mathcal{D})$; since any set $D(B)$ equals a union of sets $D(a)$, we have $\mathrm{TI}(\mathcal{D}) \leq \mathrm{TI}_*(\mathcal{D}) \leq 2^{\mathrm{TI}(\mathcal{D})}$.

Let $\sim \subseteq A^* \times A^*$ denote the least monoid congruence with $ab \sim ba$ for all $(a, b) \in I$. In other words, $u \sim v$ holds for two words $u, v \in A^*$ iff u can be obtained from v by successively transposing consecutive independent letters. In particular, $u \sim v$ implies $|u| = |v|$ as well as $\mathrm{Alph}(u) = \mathrm{Alph}(v)$. Furthermore, $u \parallel v$ implies $uv \sim vu$ (but the converse implication does not hold as the example $u = v = ab$ shows).

The *(Mazurkiewicz) trace monoid* induced by $\mathcal{D}$ is the quotient of the free monoid A^* wrt. the congruence $\sim$, i.e., $\mathbb{M}(\mathcal{D}) = A^*/\sim$. Its elements are equivalence classes of words denoted $[w]$; by $[w]$, we mean the equivalence class containing w, it is the *trace induced by* w.

Suppose $D = A \times A$, i.e., all letters are mutually dependent. Then $u \sim v$ iff $u = v$ holds for all words $u, v \in A^*$; hence $\mathbb{M}(\mathcal{D}) \cong A^*$ in this case. Furthermore, $\mathrm{TI}(\mathcal{D}) = \mathrm{TI}_*(\mathcal{D}) = \alpha(\mathcal{D}) = 1$.

The other extreme is $D = \{(a, a) \mid a \in A\}$ where any two distinct letters are independent. Then $u \sim v$ iff $|u|_a = |v|_a$ holds for any letter $a \in A$. Hence $\mathbb{M}(\mathcal{D}) \cong (\mathbb{N}^{|A|}, +)$ as well as $\mathrm{TI}(\mathcal{D}) = \alpha(\mathcal{D}) = |A|$ and $\mathrm{TI}_*(\mathcal{D}) = 2^A$ hold in this case.

For a set $B \subseteq A$ of letters, let $\pi_B \colon A^* \to B^*$ denote the monoid homomorphism with $\pi_B(b) = b$ for $b \in B$ and $\pi_B(a) = \varepsilon$ for $a \in A \setminus B$. We refer to this mapping as the *projection to* B. Now let $B \subseteq A$ be a set of pairwise dependent letters (i.e., $B \times B \subseteq D$). Then the very definition of $\sim$ ensures $\pi_B(u) = \pi_B(v)$ for any words u and v with $u \sim v$.

Theorem 2.1 (cf. [23, Cor. 1.4.5])**.** *Let $\mathcal{D} = (A, D)$ be a dependence alphabet and $B_1, \ldots, B_n \subseteq A$ sets of letters with $D = \bigcup_{1 \le i \le n} B_i \times B_i$. For any words $u, v \in A^*$, $u \sim v$ if, and only if, $\pi_{B_i}(u) = \pi_{B_i}(v)$ for all $1 \le i \le n$.*

It follows that the trace monoid $\mathbb{M}(\mathcal{D})$ is isomorphic to the submonoid of $\prod_{(a,b) \in D} \{a, b\}^*$ generated by all tuples $(\pi_{\{a,b\}}(c))_{(a,b) \in D}$ for $c \in A$. As a consequence, the trace monoid is cancellative, i.e., $s \cdot t \cdot u = s \cdot t' \cdot u$ implies $t = t'$ for any traces $s, t, t', u \in \mathbb{M}(\mathcal{D})$. We also get that $[a] \cdot s = [b] \cdot t$ with $(a, b) \in D$ implies $a = b$ and therefore $s = t$ (for any $a, b \in A$ and $s, t \in \mathbb{M}(\mathcal{D})$).

For a comprehensive survey of trace theory see [24].

Automata and Word Languages An *ε-NFA* or *nondeterministic finite automaton with ε-transitions* is a tuple $\mathfrak{A} = (Q, A, I, T, F)$ where Q is a finite set of *states*, A is an alphabet, $I, F \subseteq Q$ are the sets of *initial* and *final* states, respectively, and

$$T \subseteq Q \times (A \cup \{\varepsilon\}) \times Q$$

is the set of *transitions*. Its size $\|\mathfrak{A}\|$ is defined to be $|Q| + |A|$. The ε-NFA $\mathfrak{A}$ is an *NFA* if $T \subseteq Q \times A \times Q$; it is a *deterministic finite automaton* or *DFA* if, in addition, $I = \{\iota\}$ is a singleton and, for any $(p, a) \in Q \times A$, there is a unique state $q \in Q$ with $(p, a, q) \in T$.

Let $\mathfrak{A} = (Q, A, I, T, F)$ be an ε-NFA. A *path* is a sequence

$$(p_0, a_1, p_1)(p_1, a_2, p_2) \cdots (p_{n-1}, a_n, p_n)$$

of matching transitions (i.e., elements of T). Such a path is usually denoted

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \cdots \xrightarrow{a_n} p_n$$

or, if the intermediate states are of no importance,

$$p_0 \xrightarrow{a_1 a_2 \cdots a_n} p_n \,.$$

This path is *accepting* if it leads from an initial state to a final state, i.e., if $p_0 \in I$ and $p_n \in F$. It accepts the word $w = a_1 a_2 \cdots a_n$ (note that $a_i \in A \cup \{\varepsilon\}$ such that this word w can have length properly smaller than n). We denote by $L(\mathfrak{A})$ the set of words w accepted by $\mathfrak{A}$. A language $L \subseteq A^*$ is *regular* if it is accepted by some NFA, i.e., if there is some NFA $\mathfrak{A}$ with $L = L(\mathfrak{A})$.

A foundational result in the theory of finite automata states that ε-NFA, NFA, and DFA are equally expressive. Even more, ε-NFA can be transformed into equivalent NFA in polynomial time while the transformation of an NFA into an equivalent DFA requires exponential time.

A language $L \subseteq A^*$ is *rational* if it can be constructed from finite languages using the operations union, concatenation, and Kleene star. By Kleene's theorem [25], a language is regular if, and only if, it is rational.

Transducers and Word Relations For relations $R_1, R_2 \subseteq A^* \times A^*$, the *concatenation* is defined by $R_1 \cdot R_2 = \{(u_1u_2, v_1v_2) \mid (u_i, v_i) \in R_i\}$. Differently, the *composition* is the relation

$$R_1 \circ R_2 = \{(u, w) \mid \exists v \in A^* \colon (u, v) \in R_1, (v, w) \in R_2\}\,.$$

A *transducer* is a quintuple $\mathfrak{T} = (Q, A, I, T, F)$ where Q, A, I, and F are as for ε-NFAs, and

$$T \subseteq Q \times (A^* \times A^*) \times Q \text{ with } ((p, (u, v), q) \in T \;\Rightarrow\; |uv| \leq 1)$$

is a set of transitions that are labeled by pairs (a, b). Note that transitions are labeled by a pair consisting of a letter and the empty word or of two empty words. The size $\|\mathfrak{T}\|$ of the transducer $\mathfrak{T}$ is defined to be $|Q| + |A|$ which is fine since the number of transitions is polynomial in this size measure. A *path* is a sequence

$$(p_0, (a_1, b_1), p_1)(p_1, (a_2, b_2), p_2) \cdots (p_{n-1}(a_n, b_n), p_n)$$

of matching transitions. As in the case of ε-NFAs, we usually denote it by

$$p_0 \xrightarrow{(a_1,b_1)} p_1 \xrightarrow{(a_2,b_2)} p_2 \cdots \xrightarrow{(a_n,b_n)} p_n \text{ or } p_0 \xrightarrow{(a_1a_2\cdots a_n, b_1b_2\cdots b_n)} p_n$$

(note that the definition of a transducer requires $|a_ib_i| \leq 1$, hence the words $a_1a_2 \cdots a_n$ and $b_1b_2 \cdots b_n$ can have different length, the sum of these lengths is at most n). This path is *accepting* if $p_0 \in I$ and $p_n \in F$. A pair of words $(u, v) \in A^* \times A^*$ is accepted by $\mathfrak{T}$ if there is a path

$$I \ni p \xrightarrow{(u,v)} p_n \in F\,.$$

By $R(\mathfrak{T}) \subseteq A^* \times A^*$, we denote the set of pairs (u, v) that are accepted by $\mathfrak{T}$.

A word relation $R \subseteq A^* \times A^*$ is *rational* if it can be constructed from finite relations using the operations union, concatenation, and Kleene star. A foundational result on word relations ([26], cf. [27, Thm. III.6.1]) states that a word relation R is rational if, and only if, it is accepted by some transducer, i.e., there is a transducer $\mathfrak{T}$ with $R(\mathfrak{T}) = R$.

Rational and Recognizable Trace Languages Fix some dependence alphabet $\mathcal{D} = (A, D)$. For a word language $L \subseteq A^*$, we denote by $[L]$ the set of traces $[u]$ induced by words from L, i.e., $[L] = \{[u] \mid u \in L\} \subseteq \mathbb{M}(\mathcal{D})$.

Now let $\mathcal{L} \subseteq \mathbb{M}(\mathcal{D})$. The set $\mathcal{L}$ is *recognizable* if the word language $\{u \in A^* \mid [u] \in \mathcal{L}\}$ is regular; it is *rational* if there exists a regular word language $L \subseteq A^*$ with $\mathcal{L} = [L]$. It follows that every recognizable trace language is rational; the converse implication is known to fail (consider, e.g., the trace language $\{[ab]^n \mid n \in \mathbb{N}\}$ with $(a, b) \notin D$ that is rational, but not recognizable).

Since rational trace languages are images of rational word languages, we obtain that $\mathcal{L} \subseteq \mathbb{M}(\mathcal{D})$ is rational if, and only if, it can be constructed from finite trace languages using the operations union, concatenation, and Kleene star.

By the very definition, every NFA $\mathfrak{A}$ over the alphabet A represents a rational trace language $[L(\mathfrak{A})] = \{[u] \mid u \in L(\mathfrak{A})\}$. A word language $L \subseteq A^*$ is *closed* if $u \sim v \in L$ implies $u \in L$; we call an NFA *closed* if its language $L(\mathfrak{A})$ is closed. In this case, the trace language $[L(\mathfrak{A})]$ is even recognizable. Note that $\{u \in A^* \mid [u] \in \mathcal{L}\}$ is closed for any trace language $\mathcal{L}$. Hence every recognizable trace language $\mathcal{L}$ can be represented by some closed NFA $\mathfrak{A}$.

Even more, every recognizable trace language $\mathcal{L}$ is represented by some DFA that satisfies the following diamond property for any $(a, b) \in I$ and $p \in Q$ (see Fig. 1 for a vizualisation that should also explain the name "diamond property"):

(D) For each $(p, a, q), (q, b, r) \in T$, there is $q' \in Q$ with $(p, b, q'), (q', a, r) \in T$.

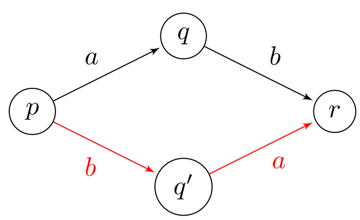


Figure 1: Visualization of the diamond property (D) of an NFA. It states that whenever we find the black transitions with $a \parallel b$, we also find a state q' with the red transitions.

Rational Trace Relations Fix some dependence alphabet $\mathcal{D} = (A, D)$. For trace relations $\mathcal{R}_1, \mathcal{R}_2 \subseteq \mathbb{M}(\mathcal{D}) \times \mathbb{M}(\mathcal{D})$, the concatenation $\mathcal{R}_1 \cdot \mathcal{R}_2$ and the composition $\mathcal{R}_1 \circ \mathcal{R}_2$ are defined as for word relations.

For a word relation $R \subseteq A^* \times A^*$, we denote by $[R]$ the set of pairs $([u],[v])$ of traces for word pairs from R, i.e., $[R] = \{([u],[v]) \mid (u,v) \in R\}$.

A trace relation $\mathcal{R} \subseteq \mathbb{M}(\mathcal{D}) \times \mathbb{M}(\mathcal{D})$ is *rational* if there exists a rational word relation $R \subseteq A^* \times A^*$ with $\mathcal{R} = \{([u],[v]) \mid (u,v) \in R\} = [R]$. Hence, rational trace relations can be represented by transducers.

Since the mapping $A^* \to \mathbb{M}(\mathcal{D})\colon u \mapsto [u]$ is a monoid homomorphism, a trace relation $\mathcal{R}$ is rational if, and only if, it can be obtained from finite relations using the operations union, concatenation, and iteration.

Left and Right Application Let X be a set, $L \subseteq X$ a subset of X, and $R \subseteq X \times X$ a binary relation on X. Then we set

$$L^R = \{y \in X \mid \exists x \in L\colon (x,y) \in R\} \text{ and } {}^R L = \{x \in X \mid \exists y \in L\colon (x,y) \in R\}\,.$$

If R is (the graph of) a function $f\colon X \to X$, then L^R is the image of L under f, i.e., $L^R = \{f(x) \mid x \in L\}$, and ${}^R L$ is the preimage of L under f, i.e., ${}^R L = \{x \in X \mid f(x) \in L\}$. Often, authors write $L\,R$ for L^R and $R\,L$ for ${}^R L$; we prefer the above notation as it stresses the different roles played by the set L and the relation R.

Example 2.2. Let $\mathcal{D} = (A, D)$ be a dependence alphabet and $u \in A^*$. Then $D(u) = \{b \in A \mid \exists a \in \mathrm{Alph}(u)\colon (a,b) \in D\} = \mathrm{Alph}(u)^D = {}^D\mathrm{Alph}(u)$—nevertheless, we will stick with the notation $D(u)$ in this case.

The mapping $2^X \to 2^X\colon L \mapsto L^R$ is the *right-application* of R while the mapping $2^X \to 2^X\colon L \mapsto {}^R L$ is the *left-application* of R. Note that for $R^{-1} = \{(y,x) \mid (x,y) \in R\}$, we have ${}^R L = L^{R^{-1}}$ and, since $(R^{-1})^{-1} = R$, also ${}^{R^{-1}} L = L^R$.

Convention In this paper, we will regularly consider subsets of and binary relations on A^* and $\mathbb{M}(\mathcal{D})$, resp. We hope to simplify understanding by using the following conventions:

- Subsets of A^* are denoted by plain capital letters K and L; binary relations on A^* are similarly denoted R, R_1, and R_2.
- Subsets of $\mathbb{M}(\mathcal{D})$ are denoted by curly letters $\mathcal{K}$ and $\mathcal{L}$; binary relations on $\mathbb{M}(\mathcal{D})$ are similarly denoted $\mathcal{R}$, $\mathcal{R}_1$, and $\mathcal{R}_2$.

3. Trace-PushDown Systems and Problem Statement

Recall that a *pushdown system* [5, 8, 7] is a pair (Q, Δ) where Q is a finite set of *states* and $\Delta \subseteq Q \times A \times A^* \times Q$ is a finite set of *transitions*. Here, $(p, a, w, q) \in \Delta$ describes that the system can move from state p to state q while replacing the letter a at the top of its pushdown by the word w. In particular, a pushdown holds, in every configuration, a word $w \in A^*$ that can be accessed at the prefix, only.

In this paper, we consider pushdowns that hold a trace $[w] \in \mathbb{M}(\mathcal{D})$ which can, as before, be accessed at the prefix, only. To this aim, we define the *trace semantics* of a pushdown system as follows. Fix a dependence alphabet $\mathcal{D} = (A, D)$ and let $\mathfrak{P} = (Q, \Delta)$ be a pushdown system. The set of configurations $\mathrm{Conf}_{\mathfrak{P}}$ of $\mathfrak{P}$ is $Q \times \mathbb{M}(\mathcal{D})$. For two configurations $(p, [u]), (q, [v]) \in \mathrm{Conf}_{\mathfrak{P}}$, we set $(p, [u]) \vdash_{\mathfrak{P}} (q, [v])$ if there is a transition $(p, a, w, q) \in \Delta$ and a word $x \in A^*$ such that $[u] = [ax]$ and $[v] = [wx]$. Note that $[u] = [ax]$ is equivalent to saying $[u] = [a] \cdot [x]$ and similarly $[v] = [wx]$ is equivalent to $[v] = [w] \cdot [x]$. Hence $(p, s) \vdash_{\mathfrak{P}} (q, t)$ for $p, q \in Q$ and $s, t \in \mathbb{M}(\mathcal{D})$ if there is a transition $(p, a, w, q) \in \Delta$ such that the trace t results from the trace s by replacing the prefix $[a]$ by $[w]$.

The reflexive and transitive closure of the one-step relation $\vdash_{\mathfrak{P}}$ is the *reachability relation* $\vdash_{\mathfrak{P}}^*$. We write $\vdash$ and $\vdash^*$ instead of $\vdash_{\mathfrak{P}}$ and $\vdash_{\mathfrak{P}}^*$ whenever the situation is clear.

Pushdown systems with trace semantics form a special case of valence automata over graph monoids. Zetzsche [19] aimed at properties of the dependence alphabet $\mathcal{D}$ that ensure decidability of the reachability relation. He obtained the following two properties that are necessary and sufficient, respectively.

Theorem 3.1 (Zetzsche [19])**.** *Let $\mathcal{D} = (A, D)$ be a dependence alphabet and $I = A^2 \setminus D$ the associated independence relation.*

- *If (A, I) contains the cycle C_4 or the path P_4 on four vertices (i.e., is not a "transitive forest" [28] or "diagonal" [29]) as an induced subgraph, then there exists a pushdown system with an undecidable reachability relation (in the trace semantics).*
- *If the independence relation I is transitive, then the reachability relation (in the trace semantics) of every pushdown system is decidable.*

Note that the dependence alphabet $\mathcal{D} = (A, D)$ with $A = \{a, b, c\}$, $(a, b) \in D$, and $(a, c), (b, c) \in I$ is not covered by any of these two conditions.

Differently from Zetzsche, we aim at properties of the transition structure of the pushdown system that ensure decidability. These properties are captured by the notion of a trace-pushdown system defined as follows.

Definition 3.2. Let $\mathcal{D} = (A, D)$ be a dependence alphabet. A *trace-pushdown system* (or *tPDS*, for short) is a pushdown system with trace semantics $\mathfrak{P} = (Q, \Delta)$ over $\mathcal{D}$ such that the following hold:

(P1) for each $(p, a, w, q) \in \Delta$ we have $D(w) \subseteq D(a)$,

(P2') for each $(p, a, v, q), (q, b, w, r) \in \Delta$ with $av \parallel bw$, there is a state $q' \in Q$ with $(p, b, w, q'), (q', a, v, r) \in \Delta$ (cf. Fig. 2).

The *size* of $\mathfrak{P} = (Q, \Delta)$ is $\|\mathfrak{P}\| := |Q| + |A| + k \cdot |\Delta|$ where $k-1$ is the maximal length of a word occurring in any transition of $\mathfrak{P}$ (i.e., $\Delta \subseteq Q \times A \times A^{<k} \times Q$).

To motivate Condition (P2'), suppose the pushdown system with trace semantics is in a configuration where it can first replace a with v and then b with

w such that av and bw are independent. Then, already before replacing a with v, the letter b was accessible at the top of the pushdown (that holds a trace). Condition (P2') ensures that already in this configuration, b can be replaced with v and then a with w.

Condition (P1) builds on top of this commutativity. Suppose the pushdown system with trace semantics is in a configuration where it has the choice between replacing the letter a and the letter b on its pushdown (in particular, $b \notin D(a)$). If it replaces a by w, popping b could get blocked by letters in w if w contains some letter that b depends on (i.e., $b \in D(w)$). Condition (P1) enforces that this cannot happen.

A more detailed motivation of the two requirements considers a pushdown system $\mathfrak{P}$ with trace semantics as a system with multiple pushdowns as follows. Let $\mathcal{D} = (A, D)$ be a dependence alphabet. Furthermore, let $C_1, \ldots, C_n \subseteq A$ be the maximal complete subgraphs of (A, D). Then two words over A are equivalent iff, for all $i \in [n]$, their projections to C_i coincide [30]. Hence a trace $[u]$ can be described by the n-tuple of projections, i.e., by a tuple $(u_1, \ldots, u_n) \in C_1^* \times C_2^* \times \cdots \times C_n^*$. We can therefore consider the single trace-pushdown with content $[u]$ as n classical pushdowns with content u_i for all $i \in [n]$.

Note that the trace $[au]$ is identified with the tuple $(v_1, \ldots, v_n)$ where $v_i = au_i$ if $a \in C_i$ and $v_i = u_i$ otherwise. In other words, reading the letter a from the trace-pushdown with contents $[au]$ corresponds to reading the letter a from all the pushdowns $i \in [n]$ with $a \in C_i$ (and similarly for writing).

Now consider the Condition (P1), i.e., consider a transition (p, a, w, q). Then $D(w) \subseteq D(a)$ is equivalent to saying $a \in C_i$ for all $i \in [n]$ such that C_i contains some letter from w. Thus, in the above view, the condition says that the transition can only write letters onto pushdown i if the letter a was read from that pushdown.

Similarly, the above view of traces also motivates Condition (P2') as follows. If $av \parallel bw$, then the transitions (p, a, v, q) and (q, b, w, r) operate on disjoint sets of pushdowns. In some sense, they do not interact. It therefore is natural to require that they can also be executed in reverse order, i.e., first b is replaced by $[w]$ and then a is replaces by $[v]$.

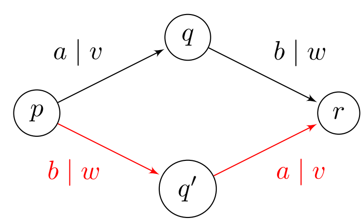


Figure 2: Visualization of the diamond property (P2') of a tPDS. It states that whenever we find the black transitions with $av \parallel bw$, we also find a state q' with the red transitions.

Example 3.3. Consider the dependence alphabet (A, D) with $A = \{a, b, c, d\}$ and $D = \{a, b\}^2 \cup \{c, d\}^2$. Let $\mathfrak{P}$ be the trace-pushdown system with $Q = \{p\}$

and the transitions $t_{a,ab} = (p, a, ab, p)$, $t_{c,cd} = (p, c, cd, p)$, $t_{a,b} = (p, a, b, p)$, $t_{c,d} = (p, c, d, p)$, $t_b = (p, b, \varepsilon, b)$, and $t_d = (p, d, \varepsilon, p)$. Let L denote the set of all those sequences of transitions that lead from the configuration $(p, [ac])$ to the configuration $(p, [\varepsilon])$. For instance, for $m, n \geq 1$, we have $t_{a,ab}^{m-1}\, t_{a,b}\, t_{c,cd}^{n-1}\, t_{c,d}\, t_b^m\, t_c^n \in L$ since

$$\begin{aligned}
(p, [a\,c]) &\vdash^* (p, [ab^{m-1}\,c]) && \text{using } m-1 \text{ times the transition } t_{a,ab}\\
&\vdash (p, [b^m\,c]) = (p, [c\,b^m]) && \text{using } t_{a,b} \text{ and } (b, c) \in I\\
&\vdash^* (p, [d^n\,b^m]) && \text{using } n-1 \text{ times } t_{c,cd} \text{ and once } t_{c,d}\\
&= (p, [b^m\,d^n]) && \text{using } (b, d) \in I\\
&\vdash^* (p, [d^n]) && \text{using } m \text{ times } t_b\\
&\vdash^* (p, [\varepsilon]) && \text{using } n \text{ times } t_d
\end{aligned}$$

More generally, L is the shuffle of the context-free languages $\{t_{a,ab}^{m-1}\, t_{a,b}\, t_b^m \mid m \geq 1\}$ and $\{t_{c,cd}^{n-1}\, t_{c,d}\, t_d^n \mid n \geq 1\}$. In particular, the language L is not contextfree.

Example 3.4. Consider the dependence alphabet $\mathcal{D} = (A, D)$ with $A = \{a, b, c\}$, $(a, c), (b, c) \in D$, and $(a, b) \notin D$. Let $\mathfrak{P}$ be the pushdown system with $Q = \{p\}$ and $\Delta = \{(p, c, ca, p), (p, c, cab, p)\}$. It can easily be checked that $\mathfrak{P}$ is a trace-pushdown system since it satisfies (P1) and (P2').

Note that $cab\,ab \sim ca^2b^2$ since $(a, b) \notin D$. Therefore, we have

$$\begin{aligned}
(p, [c]) \vdash (p, [cab]) \vdash (p, [cab\,ab]) &= (p, [ca^2b^2])\\
\vdash (p, [cab\,a^2b^2]) &= (p, [ca^3b^3]) \vdash^* (p, [ca^nb^n])
\end{aligned}$$

for all $n \geq 3$.

The configurations reachable from $(p, [c])$ together with the one-step relation between them are depicted in Fig. 3. Note that this configuration graph is isomorphic to the infinite grid. Hence its monadic second-order theory is undecidable and, consequently, there is no pushdown system with an isomorphic configuration graph. □

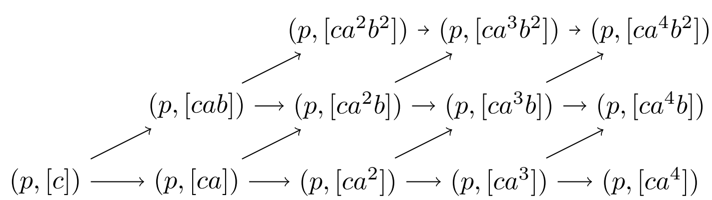


Figure 3: Configuration graph of the tPDS from Example 3.4.

Example 3.5. Consider the dependence alphabet $\mathcal{D} = (A, D)$ with alphabet $A = \{a, b, c, d\}$ and dependence relation $D = \{a, b\}^2 \cup \{c, d\}^2$. Then (A, I) is the graph C_4. Hence, by Zetzsche's Theorem 3.1, there is a pushdown system $\mathfrak{P}$

whose reachability relation (in trace semantics) is undecidable. In this example, we provide such systems $\mathfrak{P}_1$ and $\mathfrak{P}_2$ that satisfy (P1) and (P2'), respectively (but not both these conditions).

To obtain the undecidability, we use Minsky's two-counter machines, i.e., numbered sequences of commands of the form (inc_i, r) ("increment counter no. i and go to line r") and (dec_i, r, s) ("if counter no. i holds 0, go to line r, otherwise decrement it and go to line s"). By [31], there is exists a two-counter machine $\mathfrak{M}$ such that it is undecidable whether, starting from line 1 with counter values k and 0, the machine eventually reaches line 0 with both counters empty.

1. States of the PDS $\mathfrak{P}_1$ are "line numbers" of $\mathfrak{M}$, i.e., $Q = \{0, 1, \dots, n\}$. The configuration (p, k, ℓ) of the two-counter machine $\mathfrak{M}$ (where p is the line number and k and ℓ the values of the two counters) will be encoded by the configuration $(p, [a^k b\, c^\ell d])$ of the pushdown system $\mathfrak{P}_1$ with trace semantics. We will use that this configuration equals $(p, [c^\ell d\, a^k b])$. Therefore, let the set Δ of transitions be the least set satisfying the following for all $p \in [n]$:

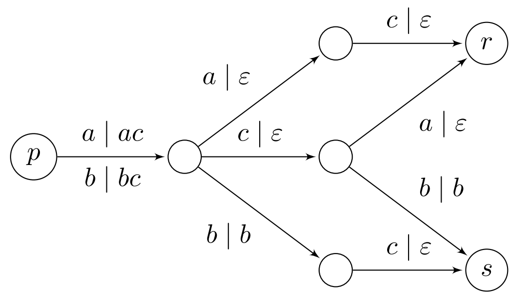
    - $(p, a, aa, r), (p, b, ab, r) \in \Delta$ if (inc_1, r) is the command in line p.
    - $(p, c, cc, r), (p, d, cd, r) \in \Delta$ if (inc_2, r) is the command in line p.
    - $(p, a, \varepsilon, r), (p, b, b, s) \in \Delta$ if (dec_1, r, s) is the command in line p.
    - $(p, c, \varepsilon, r), (p, d, d, s) \in \Delta$ if (dec_2, r, s) is the command in line p.

    Let $\mathfrak{P}_1 = (Q, \Delta)$. Since $D(a) = D(b) = \{a, b\}$ and $D(c) = D(d) = \{c, d\}$, this pushdown system satisfies (P1). We consider its configurations that are reachable from the configuration $(1, [a^k b\, d])$ for some $k \in \mathbb{N}$. They all have the form $(p, [a^\ell b\, c^m d])$ for some $\ell, m \in \mathbb{N}$. Furthermore, $(1, [a^k b\, d]) \vdash^* (p, [a^\ell b\, c^m d])$ iff the two-counter machine $\mathfrak{M}$ can reach the configuration (p, ℓ, m) from $(1, k, 0)$. In particular, $(1, [a^k b\, d]) \vdash^* (0, [b\, d])$ iff, starting from the configuration $(1, k, 0)$, the machine $\mathfrak{M}$ can reach the configuration $(0, 0, 0)$. Since this is undecidable, the reachability relation of the PDS with trace semantics $\mathfrak{P}_1$ is undecidable as well.

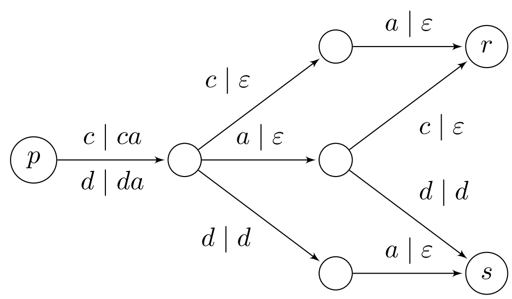
2. We now construct another pushdown system $\mathfrak{P}_2$ with undecidable reachability problem. As in the above construction, all "line numbers" $p \in \{0, 1, \dots, n\}$ are states and the two-counter machine's configuration (p, k, ℓ) is encoded by the configuration $(p, [a^k b\, c^\ell d])$ of the pushdown system with trace semantics. But depending on the command in line p, we will add additional states (see below). Then the set Δ of transitions is defined as follows:

    - $(p, a, ac, p'), (p, b, bc, p'), (p', c, a, r) \in \Delta$ if (inc_1, r) is the command in line p.
    - $(p, c, ca, p'), (p, d, da, p'), (p', a, c, r) \in \Delta$ if (inc_2, r) is the command in line p.

- If the command in line p equals (dec_1, r, s), then we have the following transitions (where the new states are not named):



- If the command in line p equals (dec_1, r, s), then we have the following transitions (where the new states are not named):



Let $\mathfrak{P}_2$ denote the result of this construction. Then, clearly, $\mathfrak{P}_2$ is a pushdown system. Note that any transition starting in a "line number" p writes a word w with $D(w) = A$. It follows that the PDS $\mathfrak{P}_2$ satisfies (P2'). We consider its configurations of the form (p, t) with $p \in \{0, 1, \dots, n\}$ that are reachable from the configuration $(1, [a^k b\, d])$ for some $k \in \mathbb{N}$. They all have the form $(p, [a^\ell b\, c^m d])$ for some $\ell, m \in \mathbb{N}$. Furthermore, $(1, [a^k b\, d]) \vdash^* (p, [a^\ell b\, c^m d])$ iff the two-counter machine $\mathfrak{M}$ can reach the configuration (p, ℓ, m) from $(1, k, 0)$. In particular, $(1, [a^k b\, d]) \vdash^* (0, [b\, d])$ iff, starting from the configuration $(1, k, 0)$, the machine $\mathfrak{M}$ can reach the configuration $(0, 0, 0)$. Since this is undecidable, the reachability relation of the PDS with trace semantics $\mathfrak{P}_2$ is undecidable as well. □

The example above indicates that the conditions (P1) and (P2') are both necessary to get the decidability of the reachability relation.

Next, we simplify the definition of trace-pushdown systems showing that, in (P2'), it suffices to require $a \parallel b$ instead of $av \parallel bw$.

Lemma 3.6. *Let $\mathcal{D} = (A, D)$ be a dependence alphabet and $\mathfrak{P} = (Q, \Delta)$ a pushdown system satisfying (P1). Then (P2') holds if, and only if,*

(P2) for each $(p, a, v, q), (q, b, w, r) \in \Delta$ with $a \parallel b$, there is a state $q' \in Q$ with $(q, b, w, q'), (q', a, v, r) \in \Delta$.

Proof. First, suppose (P2) holds. To show (P2'), let $(p, a, v, q), (q, b, w, r) \in \Delta$ with $av \parallel bw$. Then, in particular, $a \parallel b$. Hence, by (P2), there is a state q' as required by (P2').

Conversely, suppose (P2') holds and let $(p, a, v, q), (q, b, w, r) \in \Delta$ with $a \parallel b$. We want to show $av \parallel bw$. Towards a contradiction, let $(c, d) \in D$ where c is some letter from av and d some letter from bw. Then we have $c \in D(d) \subseteq D(bw) = D(b) \cup D(w) = D(b)$ since, by (P1), $D(w) \subseteq D(b)$. Consequently, $(c, b) \in D$ and therefore $b \in D(c) \subseteq D(av) = D(a) \cup D(v) = D(a)$ since, by (P1), $D(v) \subseteq D(a)$. Consequently, $(a, b) \in D$, contradicting our assumption $a \parallel b$. Thus, indeed, $av \parallel bw$. Since $\mathfrak{P}$ also satisfies (P2'), we obtain some state q' with $(p, b, w, q'), (q', a, v, r) \in \Delta$ as required. Hence, indeed, $\mathfrak{P}$ satisfies (P2). □

Let $\mathfrak{P} = (Q, \Delta)$ be a trace-pushdown system and let $C, D \subseteq \mathrm{Conf}_{\mathfrak{P}}$ be two sets of configurations. We write $C \vdash^* D$ if there are $c \in C$ and $d \in D$ with $c \vdash^* d$, i.e., if some configuration from D is reachable from some configuration from C. If $C = \{c\}$ ($D = \{d\}$, resp.) is a singleton, we also write $c \vdash^* D$ ($C \vdash^* d$, resp.). We also use similar notations for the one-step relation $\vdash$.

We consider the following decision problem: given a trace-pushdown system and two sets of configurations C and D, does $C \vdash^* D$ hold?[2] To solve this problem, it is instructive to first look at its solution for pushdown systems.

So suppose $\mathfrak{P} = (Q, \Delta)$ is a pushdown system and C, D two sets of configurations. In order to decide whether $C \vdash^* D$, it suffices to be able to decide whether, for two states p and q and two languages K and L, we have

$$\{p\} \times K \vdash^* \{q\} \times L\,. \tag{1}$$

Let $R = \{(u, v) \mid (p, u) \vdash^* (q, v)\}$ describe the modification of the pushdown content when moving from state p to state q. Then (1) holds iff $K^R \cap L \neq \emptyset$ (which is equivalent to $K \cap {}^R L \neq \emptyset$). Caucal [5] gave an algorithm that, from the transition relation Δ and two states p and q constructs finitely many regular languages U_i, V_i, W_i such that $R = \bigcup_{1 \leq i \leq n} (U_i \times V_i) \cdot \{(w, w) \mid w \in W_i\}$, i.e., he proved that R is effectively prefix-recognizable. From these regular languages, one can construct a transducer $\mathfrak{T}$ accepting the relation R. If K is regular, then an NFA for K^R can be computed from an NFA for K and the transducer $\mathfrak{T}$, i.e., K^R is effectively regular. Hence, if also L is regular, one can decide whether $K^R \cap L \neq \emptyset$, i.e., whether (1) holds. Alternatively, if K and L are regular, then $K \cap {}^R L$ is effectively regular such that one can decide the non-emptiness of this language.

Now, suppose $\mathfrak{P} = (Q, \Delta)$ is a trace-pushdown system. Then, as above, we have to decide whether

$$\{p\} \times \mathcal{K} \vdash^* \{q\} \times \mathcal{L}$$

[2] To allow the sets C and D as input of an algorithm, we will consider only special such sets.

for states p and q and trace languages $\mathcal{K}$ and $\mathcal{L}$. Let, as above, $\mathcal{R} = \{([u],[v]) \mid (p,[u]) \vdash^* (q,[v])\}$. One can, and we actually will, show that this relation is a rational trace relation (but not prefix-recognizable as in the above situation). The problem occurs in the next step: there are two generalizations (rationality and recognizability) of regularity to trace languages. But neither of them serves our purpose since, by Lemma 4.17,

- there is a rational trace language $\mathcal{L}$ and a rational trace relation $\mathcal{R}$ such that ${}^{\mathcal{R}}\mathcal{L}$ is not rational and
- there is a recognizable trace language $\mathcal{K}$ and a rational trace relation $\mathcal{R}$ such that $\mathcal{K}^{\mathcal{R}}$ is not recognizable.

The solution will be to prove that $\mathcal{R}$ is not just rational, but even "lc-rational". Since this is a new notion, the following section will introduce and study these lc-rational relations. Afterwards, we will show that the relation $\mathcal{R}$ is indeed lc-rational which will allow to complete the above program.

4. LC-Rational Trace Relations

Above, we explained that we lack a class of rational trace relations $\mathcal{R}$ whose left- and right-application preserves rationality and/or recognizablity. It turns out that the more fundamental property is the composition of relations, i.e., this section aims at a class $\mathbb{C}_{\mathbb{M}(\mathcal{D})}$ of rational trace relations that is closed under composition. Recall that a trace relation $\mathcal{R}$ is rational if, and only if, there exists a rational word relation R with $\mathcal{R} = [R]$. We will therefore first define a class $\mathbb{C}_{A^*}$ of rational word relations R that is closed under composition and satisfies

$$[R_1 \circ R_2] = [R_1] \circ [R_2] \tag{2}$$

for any relations R_1 and R_2 in $\mathbb{C}_{A^*}$ (setting $\mathbb{C}_{\mathbb{M}(\mathcal{D})} = \{[R] \mid R \in \mathbb{C}_{A^*}\}$ will then ensure that $\mathbb{C}_{\mathbb{M}(\mathcal{D})}$ is closed under composition, cf. Def. 4.7).

But first, we show that Eq. (2) does not hold for arbitrary rational word relations; the example also demonstrates that the composition of two rational trace relations need not be rational.

Example 4.1. Suppose there are $a, b, c, d \in A$ with $(a, b) \in D$ and $c \parallel d$. Consider the rational word relations

$$R_1 = \{(abab, cdcd)\} \cdot \big\{(ab, cd)\}^* = \big\{\big((ab)^n, (cd)^n\big) \;\big|\; n \geq 2\big\} \text{ and}$$
$$R_2 = \{(c, a)\}^* \{(d, b)\}^* = \{(c^m d^n, a^m b^n) \mid m, n \geq 0\}.$$

Since $c \parallel d$, we get $(cd)^n \sim c^n d^n$. Consequently

$$\mathcal{R}_1 := [R_1] = \{([ab]^n, [c^n d^n]) \mid n \geq 2\} \text{ and}$$
$$\mathcal{R}_2 := [R_2] = \{([c^m d^n], [a^m b^n]) \mid m, n \geq 0\}.$$

Hence $\mathcal{R} := \mathcal{R}_1 \circ \mathcal{R}_2 = \{([ab]^n, [a^n b^n]) \mid n \geq 2\}$. Thus, we have $[R_1 \circ R_2] = [\emptyset] = \emptyset \neq \mathcal{R} = \mathcal{R}_1 \circ \mathcal{R}_2 = [R_1] \circ [R_2]$, i.e., Eq. (2) does not hold for all rational word relations.

To show that the trace relation $\mathcal{R} = \mathcal{R}_1 \circ \mathcal{R}_2$ is not rational, let $R \subseteq A^* \times A^*$ be a word relation with $[R] = \mathcal{R}$. Since $(a, b) \in D$, any equivalence class $[v]$ is a singleton for $v \in \{a, b\}^*$. Hence $[R] = \mathcal{R}$ implies $R = \{((ab)^n, a^n b^n) \mid n \geq 2\}$ and therefore $(\{ab\}^*)^R = \{a^n b^n \mid n \geq 2\}$. Since the language $\{ab\}^*$ is regular and $\{a^n b^n \mid n \geq 2\}$ is not regular, the relation R cannot be rational [27, Cor. III.4.2]. Hence the composition $\mathcal{R} = \mathcal{R}_1 \circ \mathcal{R}_2$ of two rational trace relations is not necessarily rational.

4.1. LC-Rational Word Relations

It is well-known (cf. [27]) that left- and right-application of rational word relations to regular languages yield regular languages, that the composition of rational word relations is rational, and that the inverse of a rational word relation is rational. The following theorem summarizes the algorithmic aspects of these facts.

Theorem 4.2.

(R1) From an NFA $\mathfrak{A}$ and a transducer $\mathfrak{T}$, one can compute in polynomial time NFAs $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that $L(\mathfrak{A}_1) = L(\mathfrak{A})^{R(\mathfrak{T})}$ and $L(\mathfrak{A}_2) = {}^{R(\mathfrak{T})}L(\mathfrak{A})$ (cf. [27, Cor. III.4.2]).

(R2) From transducers $\mathfrak{T}_i$ for $1 \leq i \leq n$, one can compute a transducer $\mathfrak{T}$ with

$$\begin{aligned} R(\mathfrak{T}) &= R(\mathfrak{T}_1) \circ R(\mathfrak{T}_2) \circ \cdots \circ R(\mathfrak{T}_n) \\ &= \left\{ (u, v) \in A^* \times A^* \;\middle|\; \begin{array}{ll} \exists u_i \in A^* \colon & u = u_0, u_n = v, \text{ and} \\ & (u_{i-1}, u_i) \in R(\mathfrak{T}_i) \end{array} \right\}. \end{aligned}$$

This computation can be carried out in time $t^{O(n)}$ where $t = \sum_{1 \leq i \leq n} \|\mathfrak{T}_i\|$ is the total size of the transducers $\mathfrak{T}_i$ (cf. [27, Thm. III.4.4]).

(R3) From a transducer $\mathfrak{T}$, one can compute in polynomial time a transducer $\mathfrak{T}'$ such that $R(\mathfrak{T}') = \{(v, u) \mid (u, v) \in R(\mathfrak{T})\}$.

Proof. We only sketch the construction of the automata and transducers claimed to exist.

(R1) Here, we construct an ε-NFA $\mathfrak{A}_1$ (the construction of the ε-NFA $\mathfrak{A}_2$ is analogous). The idea is to run the NFA $\mathfrak{A}$ and the transducer $\mathfrak{T}$ in parallel; the NFA checks whether the input is accepted and the transducer, at the same time, produces its output. So let $\mathfrak{A} = (Q, A, I, T, F)$ and $\mathfrak{T} = (Q', A, I', T', F')$. Then $\mathfrak{A}_1 = (Q \times Q', A, I \times I', T_1, F \times F')$ with $((p, p'), b, (q, q')) \in T_1$ iff

- there exists $a \in A$ with $(p, a, q) \in T$ and $(p', (a, b), q') \in T'$ or
- $p = q$ and $(p', (\varepsilon, b), q') \in T'$.

Hence we have $(p,p') \xrightarrow{v} (q,q')$ in $\mathfrak{A}_1$ iff there is a word $u \in A^*$ with $p \xrightarrow{u} q$ in $\mathfrak{A}$ and $p' \xrightarrow{(u,v)} q'$ in $\mathfrak{T}$.

(R2) Let $\mathfrak{T}_i = (Q_i, A, I_i, T_i, F_i)$ for $1 \le i \le n$ be transducers. We construct a transducer $\mathfrak{T} = (Q, A, I, T, F)$ with $Q = \prod_{1\le i\le n} Q_i$, $I = \prod_{1\le i\le n} I_i$, and $F = \prod_{1\le i\le n} F_i$. To define the set of transitions, let $\overline{p} = (p_i)_{1\le i\le n}$ and $\overline{q} = (q_i)_{1\le i\le n}$ be two states from Q and let $a, b \in A \cup \{\varepsilon\}$ with $|ab| \le 1$. A straightforward product construction would set $(\overline{p}, (a,b), \overline{q}) \in T$ iff there are words $u_0, \dots, u_n \in A^*$ such that $a = u_0$, $p_i \xrightarrow{(u_{i-1},u_i)} q_i$ in $\mathfrak{T}_i$ for all $i \in [n]$, and $u_n = b$. But then the shortest word u_{n-1} witnessing this transition can be of exponential length implying a doubly exponential running time for the search of such words. While this construction runs the transducers in parallel in an unrestricted manner, the following construction allows only "interleaving" runs where at most two of the transducers move. More precisely, set $(\overline{p}, (a,b), \overline{q}) \in T$ if, and only if, one of the following hold:

- $a = b = \varepsilon$ and there exist $1 \le i < n$ and $c \in A$ with $(p_i, (\varepsilon, c), q_i) \in T_i$ and $(p_{i+1}, (c, \varepsilon), q_{i+1}) \in T_{i+1}$ and $p_j = q_j$ for $j \notin \{i, i+1\}$
- $a = b = \varepsilon$ and there exists $1 \le i \le n$ with $(p_i, (\varepsilon,\varepsilon), q_i) \in T_i$ and $p_j = q_j$ for $j \ne i$
- $a = \varepsilon$, $(p_n, (\varepsilon, b), q_n) \in T_n$ and $p_j = q_j$ for $j < n$
- $b = \varepsilon$, $(p_1, (a, \varepsilon), q_1) \in T_1$ and $p_j = q_j$ for $j > 1$

By induction on the length of the words u and v, one then gets $\overline{p} \xrightarrow{(u,v)} \overline{q}$ in $\mathfrak{T}$ if, and only if, there are words u_i with $u = u_0$, $p_i \xrightarrow{(u_{i-1},u_i)} q_i$ in $\mathfrak{T}_i$ for all $1 \le i \le n$, and $u_n = v$ implying that $\mathfrak{T}$ has the desired semantics. The size of $\mathfrak{T}$ can be estimated by

$$\|\mathfrak{T}\| = \prod_{1\le i\le n} |Q_i| + |A| \le \prod_{1\le i\le n} (|Q_i| + |A|) = \prod_{1\le i\le n} \|\mathfrak{T}_i\| \le t^n$$

and it is easily seen that $\mathfrak{T}$ can be constructed in time $t^{O(n)}$.

(R3) Here, we simply switch input and output: a transition $(p, (a,b), q)$ is replaced by the transition $(p, (b,a), q)$ which can be done in polynomial time. □

Now suppose $\mathcal{R} \subseteq \mathbb{M}(\mathcal{D})^2$ is a rational trace relation, i.e., $\mathcal{R} = [R(\mathfrak{T})]$ for some transducer $\mathfrak{T}$. Then, by (R3) above, also $\mathcal{R}^{-1} = [R(\mathfrak{T})^{-1}]$ is a rational trace relation. Thus, we have an analogue of (R3) in the trace monoid, but similar analogues of (R1) and (R2) fail. Regarding (R1), there are two possible formulations in the trace monoid, but Lemma 4.17 will demonstrate that the application of rational relations does neither preserve the rationality nor the recognizability of a trace language. Regarding (R2), Example 4.1 above demonstrates that the class of rational trace relations is not closed under composition.

Note that Eq. (2) fails in Example 4.1 since there, $R_1 \circ R_2 = \emptyset$ and $\mathcal{R}_1 \circ \mathcal{R}_2 \neq \emptyset$. The reason is that there are words $t, u, u', v' \in A^*$ such that $(t, u) \in R_1$, $(u', v') \in R_2$, and $u \sim u'$ distinct such that $([t], [v']) \in \mathcal{R}_1 \circ \mathcal{R}_2$, but $(t, v') \notin R_1 \circ R_2$. The following definition circumvents this problem.

Definition 4.3. A relation $R \subseteq A^* \times A^*$ is *left-closed* if $\sim \circ R \subseteq R \circ \sim$, i.e.,

$$\left(\exists u' \in A^* \colon u \sim u' \ R \ v'\right) \implies \left(\exists v \in A^* \colon u \ R \ v \sim v'\right)$$

holds for all $u, v' \in A^*$. The relation R is *lc-rational* if it is left-closed and rational. We call a transducer $\mathfrak{T}$ *left-closed* or an *lc-transducer* if the relation $R(\mathfrak{T})$ is left-closed.

Note that the relation R is left-closed if, for all $u \sim u'$, the closures of the languages $\{v \mid (u, v) \in R\}$ and $\{v \mid (u', v) \in R\}$ coincide, i.e., iff $[\{u\}^R] = [\{u'\}^R]$. Furthermore note that the definition of a left-closed transducer is based on the relation accepted by the transducer.

A very simple example for an lc-rational relation is the identity relation $\mathrm{Id}_{A^*} = \{(u, u) \mid u \in A^*\}$: if $u \sim u' \ \mathrm{Id}_{A^*} \ v'$, then $u' = v'$. Setting $v = u$, we obtain $u \ \mathrm{Id}_{A^*} \ v = u \sim u' = v'$. Since this relation is clearly rational, it is indeed lc-rational. Other examples are $A^* \times \{\varepsilon\}$ and $\{\varepsilon\} \times A^*$.

Example 4.4. A word $u \in A^*$ is a *subword* of $v \in A^*$ if $u = u_1 u_2 \cdots u_n$ and $v = u_1 v_1 u_2 v_2 \cdots u_n v_n$ for some $n \in \mathbb{N}$ and $u_1, u_2, \ldots, u_1, v_1, v_2, \ldots, v_n \in A^*$. In this case, we write $u \preceq v$. The subword-relation is rational since $\preceq = \{(a, a), (\varepsilon, a) \mid a \in A\}^*$.

Suppose $(a, c) \in I$ and $(a, b), (b, c) \in D$. Then $ca \sim ac \preceq abc$, but there is no superword of ca that is equivalent to abc. Hence the subword-relation is not left-closed. But the inverse relation $\succeq$ (the superword relation) is left-closed, see Appendix.

We next show that the class of lc-rational word relations has the desired properties: it is closed under composition and the homomorphism [.] commutes with composition.

Proposition 4.5. *Let $R_1, R_2 \subseteq A^* \times A^*$.*

(i) If R_2 is left-closed, then Eq. (2) holds, i.e., $[R_1 \circ R_2] = [R_1] \circ [R_2]$.

(ii) If R_1 and R_2 are lc-rational, then $R_1 \circ R_2$ is lc-rational. More precisely, from left-closed transducers $\mathfrak{T}_i$ for $1 \leq i \leq n$, one can compute a left-closed transducer $\mathfrak{T}$ with

$$[R(\mathfrak{T})] = [R(\mathfrak{T}_1)] \circ [R(\mathfrak{T}_2)] \circ \cdots [R(\mathfrak{T}_n)]$$

in time $t^{O(n)}$ where t is the total size of the transducers $\mathfrak{T}_i$.

Proof. To demonstrate the first claim, let R_2 be left-closed. For the inclusion $[R_1 \circ R_2] \subseteq [R_1] \circ [R_2]$, let $(u, w) \in R_1 \circ R_2$. Then there exists $v \in A^*$ with $u\ R_1\ v\ R_2\ w$ and therefore $[u]\ [R_1]\ [v]\ [R_2]\ [w]$ implying $([u], [w]) \in [R_1] \circ [R_2]$.

For the converse inclusion, let $(x, z) \in [R_1] \circ [R_2]$. There is some trace y with $x\ [R_1]\ y\ [R_2]\ z$. Hence there are words u, v, v', w with

- $x = [u]$, $y = [v]$, and $(u, v) \in R_1$ and
- $y = [v']$, $z = [w]$, and $(v', w) \in R_2$.

Hence we have $u\ R_1\ v \sim v'\ R_2\ w$. Since R_2 is left-closed, there exists a word $w' \in A^*$ such that $u\ R_1\ v\ R_2\ w' \sim w$. Hence we have $(x, z) = ([u], [w]) = ([u], [w']) \in [R_1 \circ R_2]$. This finishes the verification of the first claim.

Now, assume both relations R_1 and R_2 to be left-closed such that $\sim \circ R_i \subseteq R_i \circ \sim$ holds for all $i \in [2]$. Consequently, we get $\sim \circ R_1 \circ R_2 \subseteq R_1 \circ \sim \circ R_2 \subseteq R_1 \circ R_2 \circ \sim$. Hence, indeed, $R_1 \circ R_2$ is left-closed such that the second claim follows using Theorem 4.2(R2) and statement (i). □

The following proposition characterizes the lc-rational word relations of the form $K \times L$ for languages $K, L \subseteq A^*$. This characterization should also explain the name "left-closed".

Proposition 4.6. *Let $K, L \subseteq A^*$ be nonempty.*

(i) Then $K \times L$ is rational if, and only if, K and L both are regular. More precisely, we have the following.

- *From a transducer $\mathfrak{T}$, one can compute in polynomial time NFAs $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with $L(\mathfrak{A}_1) = \{u \mid \exists v\colon (u, v) \in R(\mathfrak{T})\}$ and $L(\mathfrak{A}_2) = \{v \mid \exists u\colon (u, v) \in R(\mathfrak{T})\}$.*
- *Conversely, from two NFAs $\mathfrak{A}_1$ and $\mathfrak{A}_2$, one can construct in polynomial time a transducer $\mathfrak{T}$ with $R(\mathfrak{T}) = L(\mathfrak{A}_1) \times L(\mathfrak{A}_2)$.*

(ii) $K \times L$ is left-closed if, and only if, K is closed. More precisely, we have the following.

- *If $R \subseteq A^* \times A^*$ is left-closed, then $\{u \in A^* \mid \exists v\colon (u, v) \in R\}$ is closed.*
- *If K is closed, then $K \times L$ is left-closed.*

Consequently, $K \times L$ is lc-rational if, and only if, K and L both are regular and K is closed.

Proof. Fix an NFA $\mathfrak{A}$ accepting A^*. Now let $\mathfrak{T} = (Q, A, I, T, F)$ be a transducer. Then, by Theorem 4.2(R1), one can construct in polynomial time NFAs $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with $L(\mathfrak{A}_1) = {}^{R(\mathfrak{T})}L(\mathfrak{A}) = {}^{R(\mathfrak{T})}(A^*) = \{u \mid \exists v\colon (u, v) \in R(\mathfrak{T})\}$ and, similarly, $L(\mathfrak{A}_2) = L(\mathfrak{A})^{R(\mathfrak{T})} = (A^*)^{R(\mathfrak{T})} = \{v \mid \exists u\colon (u, v) \in R(\mathfrak{T})\}$.

Regarding the second claim, let $\mathfrak{A}_i = (Q_i, A, I_i, T_i, F_i)$ be NFAs. Construct transducers $\mathfrak{T}_i = (Q_i, A, I_i, T_i', F_i)$ by setting $(p, (a, b), q) \in T_1'$ iff $(p, a, q) \in T_1$ and $b = \varepsilon$ as well as $(p, (a, b), q) \in T_2'$ iff $(p, b, q) \in T_2$ and $a = \varepsilon$. Then

$R(\mathfrak{T}_1) = L(\mathfrak{A}_1) \times \{\varepsilon\}$ and $R(\mathfrak{T}_2) = \{\varepsilon\} \times L(\mathfrak{A}_2)$. Furthermore, the transducers $\mathfrak{T}_i$ can be constructed in polynomial time. The second claim now follows from Theorem 4.2(R2) since $L(\mathfrak{A}_1) \times L(\mathfrak{A}_2) = R(\mathfrak{T}_1) \circ R(\mathfrak{T}_2)$.

Suppose $R \subseteq A^* \times A^*$ to be left-closed and let $K = \{u \mid \exists v\colon (u, v) \in R\}$. To show that K is closed, let $u \sim u' \in K$. By the definition of K, there exists $v' \in A^*$ with $(u', v') \in R$, i.e., $u \sim u' \; R \; v'$. Since R is assumed to be left-closed, there exists $v \in A^*$ with $u \; R \; v \sim v'$. This implies in particular $u \in K$. Hence, K is closed.

Conversely, suppose K to be closed and let $u \sim u' \; (K \times L) \; v'$. Then $u \sim u' \in K$ implying $u \in K$ such that (with $v = v'$) we get $u \; (K \times L) \; v \sim v'$, i.e., $K \times L$ is left-closed. □

4.2. LC-Rational Trace Relations

Recall that a trace relation $\mathcal{R} \subseteq \mathbb{M}(\mathcal{D}) \times \mathbb{M}(\mathcal{D})$ is rational if there exists a rational word relation $R \subseteq A^* \times A^*$ with $[R] = \mathcal{R}$. Similarly, we now lift the concept of lc-rational relations from words to traces.

Definition 4.7. A trace relation $\mathcal{R} \subseteq \mathbb{M}(\mathcal{D}) \times \mathbb{M}(\mathcal{D})$ is *lc-rational* if there exists some lc-rational word relation $R \subseteq A^* \times A^*$ with $\mathcal{R} = [R]$.

Simple examples are $\mathbb{M}(\mathcal{D}) \times \{[\varepsilon]\}$ and $\{[\varepsilon]\} \times \mathbb{M}(\mathcal{D})$ since $A^* \times \{\varepsilon\}$ and $\{\varepsilon\} \times A^*$ are lc-rational word relations. Also the identity relation $\mathrm{Id}_{\mathbb{M}(\mathcal{D})} = \{(x, x) \mid x \in \mathbb{M}(\mathcal{D})\}$ is lc-rational since Id_{A^*} is an lc-rational word relation.

Example 4.8. Another, more involved example, is the supertrace-relation [32]: $x \in \mathbb{M}(\mathcal{D})$ is a *supertrace* of $y \in \mathbb{M}(\mathcal{D})$ if $x = x_1 y_1 \cdots x_n y_n x_{n+1}$ and $y = y_1 y_2 \cdots y_n$ for some $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_{n+1}, y_1, y_2, \ldots, y_n \in \mathbb{M}(\mathcal{D})$. In this case, we write $x \sqsupseteq y$. It is easily checked that $x \sqsupseteq y$ if, and only if, there are words u and v such that $x = [u]$, $y = [v]$, and $u \succeq v$, i.e., $\sqsupseteq \; = [\succeq]$. Since the superword-relation $\succeq$ is lc-rational by Example 4.4, we obtain that the supertrace-relation $\sqsupseteq$ is lc-rational.

By Example 4.1, the composition of rational trace relations is, in general, not rational. The following proposition demonstrates that the composition is rational provided the second relation is lc-rational (differently, if the first relation is lc-rational, the composition is not necessarily rational as Example 4.1 demonstrates as well). If both relations are lc-rational, then so is the composition.

Proposition 4.9. *Let $\mathcal{R}_1, \mathcal{R}_2 \subseteq \mathbb{M}(\mathcal{D}) \times \mathbb{M}(\mathcal{D})$ be rational trace relations.*

(i) If $\mathcal{R}_2$ is lc-rational, then $\mathcal{R}_1 \circ \mathcal{R}_2$ is rational. More precisely, from a transducer $\mathfrak{T}_1$ and a left-closed transducer $\mathfrak{T}_2$, one can compute in polynomial time a transducer $\mathfrak{T}$ such that $[R(\mathfrak{T})] = [R(\mathfrak{T}_1)] \circ [R(\mathfrak{T}_2)]$.

(ii) If $\mathcal{R}_1$ and $\mathcal{R}_2$ both are lc-rational, then $\mathcal{R}_1 \circ \mathcal{R}_2$ is even lc-rational. More precisely, from left-closed transducers $\mathfrak{T}_i$ for $1 \leq i \leq n$, one can compute a left-closed transducer $\mathfrak{T}$ with $[R(\mathfrak{T})] = [R(\mathfrak{T}_1)] \circ [R(\mathfrak{T}_2)] \circ \cdots \circ [R(\mathfrak{T}_n)]$ in time $t^{O(n)}$ where t is the total size of the transducers $\mathfrak{T}_i$.

Proof. Let $\mathfrak{T}_1$ be a transducer and $\mathfrak{T}_2$ a left-closed transducer. By Theorem 4.2(R2), we can compute in polynomial time a transducer $\mathfrak{T}$ with $R(\mathfrak{T}) = R(\mathfrak{T}_1) \circ R(\mathfrak{T}_2)$. Since the relation $R(\mathfrak{T}_2)$ is left-closed, Prop. 4.5(i) implies

$$[R(\mathfrak{T})] = \big[R(\mathfrak{T}_1) \circ R(\mathfrak{T}_2)\big] = [R(\mathfrak{T}_1)] \circ [R(\mathfrak{T}_2)]\,.$$

Next consider n left-closed transducers $\mathfrak{T}_i$. By Theorem 4.2(R2), one can compute a transducer $\mathfrak{T}$ in the given time bound that accepts the composition of the relations $R(\mathfrak{T}_i)$. By Prop. 4.5(ii), this composition is lc-rational (implying that $\mathfrak{T}$ is a left-closed transducer) and, by Prop. 4.5(i), satisfies

$$[R(\mathfrak{T})] = \big[R(\mathfrak{T}_1) \circ R(\mathfrak{T}_2) \circ \cdots \circ R(\mathfrak{T}_n)\big] = [R(\mathfrak{T}_1)] \circ [R(\mathfrak{T}_2)] \circ \cdots \circ [R(\mathfrak{T}_n)]\,.$$

□

Next, we want to characterize the lc-rational relations among the direct products $\mathcal{K} \times \mathcal{L}$ of sets of traces $\mathcal{K}$ and $\mathcal{L}$ (we have done so for word relations in Prop. 4.6).

Proposition 4.10. *Let $\mathcal{K}, \mathcal{L} \subseteq \mathbb{M}(\mathcal{D})$ be nonempty.*

(i) $\mathcal{K} \times \mathcal{L}$ is rational if, and only if, $\mathcal{K}$ and $\mathcal{L}$ both are rational.

(ii) $\mathcal{K} \times \mathcal{L}$ is lc-rational if, and only if, $\mathcal{K}$ is recognizable and $\mathcal{L}$ is rational.

More precisely, we have the following.

(a) From a transducer $\mathfrak{T}$, one can compute in polynomial time NFAs $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with $[L(\mathfrak{A}_1)] = \{[u] \mid \exists v\colon ([u],[v]) \in [R(\mathfrak{T})]\}$ and $[L(\mathfrak{A}_2)] = \{[v] \mid \exists u\colon ([u],[v]) \in [R(\mathfrak{T})]\}$. If $\mathfrak{T}$ is a left-closed transducer, then $\mathfrak{A}_1$ is a closed NFA (implying that $[L(\mathfrak{A}_1)]$ is recognizable).

(b) Conversely, from two NFAs $\mathfrak{A}_1$ and $\mathfrak{A}_2$, one can construct in polynomial time a transducer $\mathfrak{T}$ with $[R(\mathfrak{T})] = [L(\mathfrak{A}_1)] \times [L(\mathfrak{A}_2)]$. If $\mathfrak{A}_1$ is closed, then $\mathfrak{T}$ is left-closed.

Proof. Let $\mathfrak{T}$ be a transducer and let $\mathcal{K} = \{[u] \mid \exists [v]\colon ([u],[v]) \in [R(\mathfrak{T})]\}$. By Prop. 4.6(i), we can construct in polynomial time an NFA $\mathfrak{A}_1$ that accepts $K := \{u \mid \exists v\colon (u,v) \in R(\mathfrak{T})\}$. To show $[L(\mathfrak{A}_1)] = \mathcal{K}$, let $u \in A^*$ be some word. Then we have the following

$$\begin{aligned}
[u] \in \mathcal{K} &\iff \exists v \in A^*\colon ([u],[v]) \in [R(\mathfrak{T})] \\
&\iff \exists u', v'\colon u \sim u' \; R(\mathfrak{T}) \; v' \\
&\iff \exists u'\colon u \sim u' \in K \\
&\iff [u] \in [K] = [L(\mathfrak{A}_1)]\,.
\end{aligned}$$

Hence, indeed, $[L(\mathfrak{A}_1)] = \{[u] \mid \exists [v]\colon ([u],[v]) \in [R(\mathfrak{T})]\}$. This finishes the proof of the first part of claim (a) regarding $\mathfrak{A}_1$, the NFA $\mathfrak{A}_2$ can be obtained symmetrically.

Now let $\mathfrak{T}$ be a left-closed transducer. Then $R(\mathfrak{T})$ is a left-closed relation. Hence, by Prop. 4.6(ii), the language $\{u \mid \exists v\colon (u,v) \in R(\mathfrak{T})\}$ is closed. But this language equals $L(\mathfrak{A}_1)$, i.e., $\mathfrak{A}_1$ is a closed NFA. This finishes the proof of claim (a).

To also prove (b), let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be NFAs. Then, by Prop. 4.6(i), one can construct in polynomial time a transducer $\mathfrak{T}$ with $R(\mathfrak{T}) = L(\mathfrak{A}_1) \times L(\mathfrak{A}_2)$. By the very definition of $[R]$ for a word relation R, we get

$$[L(\mathfrak{A}_1)] \times [L(\mathfrak{A}_2)] = \big[L(\mathfrak{A}_1) \times L(\mathfrak{A}_2)\big] = [R(\mathfrak{T})]\,.$$

Now suppose $\mathfrak{A}_1$ to be closed. To show that the transducer $\mathfrak{T}$ is left-closed, we have to prove that the relation $R(\mathfrak{T})$ is left-closed. But this is the case by Prop. 4.6(ii) since $R(\mathfrak{T}) = L(\mathfrak{A}_1) \times L(\mathfrak{A}_2)$ and $L(\mathfrak{A}_1)$ is closed. □

Proposition 4.9(i) ensures that the composition of a rational and an lc-rational trace relation is rational, again. This holds in particular if the first relation is the inverse of an lc-rational relation. We now demonstrate that all rational trace relations arise in this way (provided there are at least two dependent letters).

Proposition 4.11. *Suppose there are $a, b \in A$ distinct with $(a,b) \in D$. Let $\mathcal{R} \subseteq \mathbb{M}(\mathcal{D})^2$ be a rational trace relation. There exist lc-rational trace relations $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $\mathcal{R} = \mathcal{R}_1^{-1} \circ \mathcal{R}_2$.*

Proof. There exists a rational relation $R \subseteq A^* \times A^*$ with $\mathcal{R} = [R]$. By Nivat's theorem [33] (cf. [27, Theorem III.3.2]), there exist an alphabet B, homomorphisms $f, g\colon B^* \to A^*$, and a regular language $K \subseteq B^*$ such that

$$R = \{(f(u), g(u)) \mid u \in K\}\,.$$

Suppose $B = \{c_1, c_2, \ldots, c_n\}$. Let $h\colon B^* \to A^*$ be the injective homomorphism defined by $h(c_i) = a^i b$.

Now consider the relations

$$R_1 = \{(h(u), f(u)) \mid u \in K\} \text{ and } R_2 = \{(h(u), g(u)) \mid u \in K\}\,.$$

We first show that these relations are lc-rational (by symmetry, we only consider the relation R_1). From Nivat's theorem, we obtain that R_1 is rational. To show that it is left-closed, let $v, v', w' \in A^*$ with $v \sim v'\ R_1\ w'$. Since $R_1 \subseteq \{a,b\}^* \times A^*$, we obtain $v' \in \{a,b\}^*$. Since $(a,b) \in D$, this implies $v = v'$. Hence, setting $w := w'$, we obtain $v\ R_1\ w \sim w'$. Hence, indeed, the relations R_1 and R_2 are lc-rational.

Next, we show $R = R_1^{-1} \circ R_2$. For the inclusion "$\subseteq$", let $(v,w) \in R$. Then there exists $u \in K$ with $v = f(u)$ and $w = g(u)$. Hence we obtain

$$v = f(u)\ R_1^{-1}\ h(u)\ R_2\ g(u) = w$$

and therefore $(v,w) \in R_1^{-1} \circ R_2$. For the converse inclusion, suppose $(v,w) \in R_1^{-1} \circ R_2$. Then there exists some word x with $v\ R_1^{-1}\ x\ R_2\ w$. By the definition of the relations R_1 and R_2, there are words $u_1, u_2 \in K \subseteq B^*$ such that

$$v = f(u_1)\,, x = h(u_1) \text{ and } x = h(u_2)\,, w = g(u_2)\,.$$

Since the homomorphism h is injective, we get $u_1 = u_2$ and therefore

$$(v, w) = (f(u_1), g(u_2)) = (f(u_1), g(u_1)) \in R\,.$$

Thus, indeed, $R = R_1^{-1} \circ R_2$.

Finally, let $\mathcal{R}_1 = [R_1]$ and $\mathcal{R}_2 = [R_2]$. Note that R_1^{-1} is rational and satisfies $[R_1^{-1}] = \mathcal{R}_1^{-1}$. From Prop. 4.5(i), we obtain that

$$[R_1^{-1} \circ R_2] = [R_1^{-1}] \circ [R_2] = \mathcal{R}_1^{-1} \circ \mathcal{R}_2$$

since R_2 is lc-rational. Hence, we obtain

$$\mathcal{R} = [R] = [R_1^{-1} \circ R_2] = \mathcal{R}_1^{-1} \circ \mathcal{R}_2\,. \qquad \square$$

4.3. Concatenations of lc-rational trace relations

The (componentwise) concatenation of two rational trace relations is rational again: if $\mathcal{R}_1, \mathcal{R}_2 \subseteq \mathbb{M}(\mathcal{D})^2$ are rational, then there are transducers $\mathfrak{T}_1$ and $\mathfrak{T}_2$ with $\mathcal{R}_i = [R(\mathfrak{T}_i)]$. From these transducers, one can construct a transducer $\mathfrak{T}$ with $R(\mathfrak{T}) = R(\mathfrak{T}_1) \cdot R(\mathfrak{T}_2)$. It follows that $\mathcal{R}_1 \cdot \mathcal{R}_2 = [R(\mathfrak{T}_1)] \cdot [R(\mathfrak{T}_2)] = [R(\mathfrak{T}_1) \cdot R(\mathfrak{T}_2)] = [R(\mathfrak{T})]$ is rational. The following lemma demonstrates that this does not hold for lc-rational relations.

Lemma 4.12. *There exist lc-rational relations $\mathcal{R}$ and $\mathcal{R}'$ such that $\mathcal{R} \cdot \mathcal{R}'$ is not lc-rational.*

Proof. Consider the rational trace relations $\mathcal{R}_1$ and $\mathcal{R}_2$ from Example 4.1. Note that $\mathcal{R}_2$ is the concatenation of the lc-rational trace relations $\mathcal{R} = \{([c], [a])\}^*$ and $\mathcal{R}' = \{([d], [b])\}^*$ and recall that $\mathcal{R}_1 \circ \mathcal{R}_2$ is not rational. Hence, by Prop. 4.9(i), $\mathcal{R}_2 = \mathcal{R} \cdot \mathcal{R}'$ cannot be lc-rational. $\square$

We now come to two special cases of relations $\mathcal{R}_1$ that ensure the lc-rationality of $\mathcal{R}_1 \cdot \mathcal{R}_2$:

Lemma 4.13. *Let $\mathcal{K} \subseteq \mathbb{M}(\mathcal{D})$ be recognizable. Then the relation*

$$\mathcal{R} = (\mathcal{K} \times \{[\varepsilon]\}) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})} = \{(xy, y) \mid x \in \mathcal{K}, y \in \mathbb{M}(\mathcal{D})\}$$

is lc-rational.

More precisely, from a dependence alphabet $\mathcal{D} = (A, D)$ and a closed NFA $\mathfrak{A} = (Q, A, I, T, F)$, one can compute a left-closed transducer $\mathfrak{T}$ with $[R(\mathfrak{T})] = ([L(\mathfrak{A})] \times \{[\varepsilon]\}) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})}$; this computation can be carried out in time polynomial in the size $\|\mathfrak{A}\|$ of $\mathfrak{A}$ and the set twin index $\mathrm{TI}_*(\mathcal{D})$ of $\mathcal{D}$.

Proof. Let R denote the set of pairs of words

$$(u_1 v_1 u_2 v_2 \cdots u_n v_n, v_1 v_2 \cdots v_n)$$

with $n \in \mathbb{N}$ and $u_1, u_2, \ldots, u_n, v_1, \ldots, v_n \in A^*$ such that

(i) $u_1 u_2 \cdots u_n \in L(\mathfrak{A})$,

(ii) $v_1 v_2 \cdots v_i \parallel u_{i+1}$ for all $i \in [n-1]$.

We construct a transducer $\mathfrak{T}$ with $R(\mathfrak{T}) = R$, prove $[R] = ([L(\mathfrak{A})] \times \{[\varepsilon]\}) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})}$, and show that R is left-closed (implying that $\mathfrak{T}$ is a left-closed transducer).

States of the transducer $\mathfrak{T}$ are triples $(q, D(B), a)$ of a state $q \in Q$, a set of letters $B \subseteq A$, and $a \in A \cup \{\varepsilon\}$. A state $(q, D(B), a)$ is initial if $q \in I$, $D(B) = \emptyset$ (i.e., $B = \emptyset$), and $a = \varepsilon$. It is accepting if $q \in F$ and $a \in \varepsilon$. The transducer has two types of transitions (with $a \in A$):

- There are transitions from $(p, D(B), \varepsilon)$ to $(p, D(B), a)$ and then to $(q, D(C), \varepsilon)$ labeled (a, ε) and (ε, a), respectively iff $p = q$ and $D(C) = D(B) \cup D(a)$
- There is a transition from $(p, D(B), \varepsilon)$ to $(q, D(C), \varepsilon)$ labeled (a, ε) iff $(p, a, q) \in T$ is a transition of the automaton $\mathfrak{A}$, $a \notin D(B)$, and $D(B) = D(C)$.

Clearly, this transducer can be computed in time polynomial in $\|\mathfrak{A}\| \cdot \mathrm{TI}_*(\mathcal{D})$.

Let $p \in I$, $q \in Q$, $B \subseteq A$, and $v, w \in A^*$. Then the transducer $\mathfrak{T}$ has a path labeled (w, v) from $(p, \emptyset, \varepsilon)$ to $(q, D(B), \varepsilon)$ iff the following hold:

- $D(B) = D(\mathrm{Alph}(v))$ is the set of letters dependent from some letter of v.
- w results from v by injecting some letters (using transitions of the second type (a, ε)) that are independent from all letters of v read so far.
- The sequence u of injected letters leads from p to q in the automaton $\mathfrak{A}$.

Consequently, (w, v) labels a path from some initial to some accepting state iff $(w, v) \in R$. Hence, indeed, $R = R(\mathfrak{T})$ is rational.

Next, we verify $[R] = ([L(\mathfrak{A})] \times \{[\varepsilon]\}) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})}$. First, suppose $(w, v) = (u_1 v_1 \cdots u_n v_n, v_1 \cdots v_n) \in R$ with the properties from above. From (ii), we obtain $u_1 v_1 \cdots u_n v_n \sim u_1 u_2 \cdots u_n \, v_1 v_2 \cdots v_n$ and therefore $([w], [v]) = ([u_1 \cdots u_n] \cdot [v], [v])$ which belongs to $\mathcal{R}$ since $u_1 \cdots u_n \in K$ by (i). Thus, $[R] \subseteq \mathcal{R}$. Conversely, let $([uv], [v]) \in \mathcal{R}$, i.e., $u \in K$ and $v \in A^*$. With $n = 1$, $u_1 = u$, and $v_1 = v$, we get $(uv, v) = (u_1 v_1, v_1) \in R$ and therefore $\mathcal{R} \subseteq [R]$. Thus, indeed, $[R] = \mathcal{R}$.

It remains to be shown that R is left-closed. So let $n \in \mathbb{N}$ be a number and $u_1, \ldots, u_n, v_1, \ldots, v_n \in A^*$ words satisfying (i) and (ii) from above and let $w \in A^*$ such that

$$w \sim u_1 v_1 u_2 v_2 \cdots u_n v_n \; R \; v_1 \cdots v_n \,.$$

With $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$, (ii) implies

$$w \sim u_1 v_1 u_2 v_2 \cdots u_n v_n \sim u_1 \cdots u_n \, v_1 \cdots v_n = uv \,.$$

Application of Levi's Lemma for traces [24, p. 74] to the equivalence $w \sim uv$ yields $m \in \mathbb{N}$ and words $u'_1, \ldots, u'_m$ and $v'_1, \ldots, v'_m$ such that

(1) $w = u'_1 v'_1 \, u'_2 v'_2 \cdots u'_m v'_m$,

(2) $u \sim u'_1 u'_2 \cdots u'_m =: u'$,

(3) $v \sim v'_1 v'_2 \cdots v'_m =: v'$, and

(4) $v'_1 v'_2 \cdots v'_i \parallel u'_{i+1}$ for all $i \in [m-1]$.

Note that $u' \sim u \in L(\mathfrak{A})$ implies $u' \in L(\mathfrak{A})$ since the NFA $\mathfrak{A}$ is closed. Hence we get $w \; R \; v' \sim v$. Thus, indeed, the relation R is left-closed. □

Lemma 4.14. *Let $\mathcal{L} \subseteq \mathbb{M}(\mathcal{D})$ be rational. Then the relation*

$$\mathcal{R} = (\{[\varepsilon] \times \mathcal{L}\}) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})} = \{(y, xy) \mid x \in \mathcal{L}, y \in \mathbb{M}(\mathcal{D})\}$$

is lc-rational.

More precisely, from a dependence alphabet $\mathcal{D} = (A, D)$ and an NFA $\mathfrak{B} = (Q, A, I, T, F)$, one can compute a left-closed transducer $\mathfrak{T}$ with $[R(\mathfrak{T})] = (\{[\varepsilon]\} \times [L(\mathfrak{B})]\}) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})}$; this computation can be carried out in time polynomial in the size $\|\mathfrak{B}\|$ of the NFA $\mathfrak{B}$.

Proof. From $\mathfrak{B}$, one can compute in polynomial time a transducer $\mathfrak{T}_1$ with $R(\mathfrak{T}_1) = \{\varepsilon\} \times L(\mathfrak{B})$ by Prop. 4.6(i). Let $\mathfrak{T}_2$ be a transducer for Id_{A^*}. From the transducers $\mathfrak{T}_1$ and $\mathfrak{T}_2$, one can compute a transducer $\mathfrak{T}$ with $R(\mathfrak{T}) = (\{\varepsilon\} \times L(\mathfrak{B})) \cdot \mathrm{Id}_{A^*}$ in polynomial time (just add $(\varepsilon, \varepsilon)$-transitions from any accepting state of $\mathfrak{T}_1$ to any initial state of $\mathfrak{T}_2$).

We first show that $R(\mathfrak{T})$ is even left-closed. So let $u \in L(\mathfrak{B})$ and $v \sim v'$ be arbitrary words such that $v \sim v' \; R(\mathfrak{T}) \; uv'$. Then we have $v \; R(\mathfrak{T}) \; uv \sim uv'$. Hence, indeed, $R(\mathfrak{T})$ is left-closed implying that $\mathfrak{T}$ is left-closed.

Next we show $(\{[\varepsilon]\} \times [L(\mathfrak{B})]) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})} = [R(\mathfrak{T})]$. For the inclusion "$\supseteq$", let $(v, uv) \in R(\mathfrak{T})$, i.e., $u \in L(\mathfrak{B})$ and $v \in A^*$. Then $([v], [uv]) = ([\varepsilon], [u]) \cdot ([v], [v])$ is contained in the left-hand side, i.e, we showed the inclusion "$\supseteq$". Conversely, let $(y, xy) = ([\varepsilon], x) \cdot (y, y)$ belong to the left-hand side, i.e., $x \in [L(\mathfrak{B})]$ and $y \in \mathbb{M}(\mathcal{D})$. From $x \in [L(\mathfrak{B})]$, we obtain a word $u \in L(\mathfrak{B})$ with $[u] = x$. Further, there is a word $v \in A^*$ with $[v] = y$. It follows that $(v, uv) \in R(\mathfrak{T})$ and therefore $(y, xy) = ([v], [uv]) \in [R(\mathfrak{T})]$. □

Now the following sufficient condition for the lc-rationality of $\mathcal{R}_1 \cdot \mathcal{R}_2$ follows:

Theorem 4.15. *Let $\mathcal{K} \subseteq \mathbb{M}(\mathcal{D})$ be recognizable, $\mathcal{L} \subseteq \mathbb{M}(\mathcal{D})$ rational, and $\mathcal{R} \subseteq \mathbb{M}(\mathcal{D})^2$ lc-rational. Then $(\mathcal{K} \times \mathcal{L}) \cdot \mathcal{R}$ is lc-rational.*

More precisely, from a dependence alphabet $\mathcal{D} = (A, D)$, a closed NFA $\mathfrak{A}$, an NFA $\mathfrak{B}$, and a left-closed transducer $\mathfrak{T}$, one can compute a left-closed transducer $\mathfrak{T}'$ with $[R(\mathfrak{T}')] = ([L(\mathfrak{A})] \times [L(\mathfrak{B})]) \cdot [R(\mathfrak{T})]$; this computation can be carried out in time polynomial in $\|\mathfrak{A}\| + \|\mathfrak{B}\| + \|\mathfrak{T}\| + \mathrm{TI}_*(\mathcal{D})$, i.e., in the size of the NFAs, the transducer, and the set twin index $\mathrm{TI}_*(\mathcal{D})$ of the dependence alphabet $\mathcal{D}$.

Proof. In the following, let $\mathcal{K} = [L(\mathfrak{A})]$, $\mathcal{L} = [L(\mathfrak{B})]$, and $\mathcal{R} = [R(\mathfrak{T})]$.

By Lemma 4.13, one can compute in time polynomial in the size of the closed NFA $\mathfrak{A}$ and the set twin index of $\mathcal{D}$ a left-closed transducer $\mathfrak{T}_1$ for the lc-rational relation $\mathcal{R}_1 = (\mathcal{K} \times \{[\varepsilon]\}) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})}$. Lemma 4.14 allows to construct in time polynomial in the size of the NFA $\mathfrak{B}$ a left-closed transducer $\mathfrak{T}_2$ for the lc-rational relation $\mathcal{R}_2 = (\{[\varepsilon]\} \times \mathcal{L}) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})}$. From Prop. 4.9(ii), we obtain a left-closed transducer $\mathfrak{T}'$ for the relation $\mathcal{R}_1 \circ \mathcal{R} \circ \mathcal{R}_2$ in time polynomial in the size of the transducers $\mathfrak{T}_1$, $\mathfrak{T}$, and $\mathfrak{T}_2$. Hence, in summary, the construction of $\mathfrak{T}'$ can be carried out in the given time bound.

It remains to be shown that the relation $[R(\mathfrak{T}')]$ equals $(\mathcal{K} \times \mathcal{L}) \cdot \mathcal{R}$, i.e., that $\mathcal{R}_1 \circ \mathcal{R} \circ \mathcal{R}_2 = (\mathcal{K} \times \mathcal{L}) \cdot \mathcal{R}$ holds.

Note that $(x, z) \in \mathcal{R}_1 \circ \mathcal{R} \circ \mathcal{R}_2$ iff there are $y_1, y_2 \in \mathbb{M}(\mathcal{D})$ with $(x, y_1) \in \mathcal{R}_1$, $(y_1, y_2) \in \mathcal{R}$, and $(y_2, z) \in \mathcal{R}_2$. But $(x, y_1) \in \mathcal{R}_1$ is equivalent to the existence of $k \in \mathcal{K}$ with $x = k \cdot y_1$. Similarly, $(y_2, z) \in \mathcal{R}_2$ iff there is $\ell \in \mathcal{L}$ with $z = \ell \cdot y_2$. In summary, we have $(x, z) \in \mathcal{R}_1 \circ \mathcal{R} \circ \mathcal{R}_2$ iff there exist $k \in \mathcal{K}$, $(y_1, y_2) \in \mathcal{R}$, and $\ell \in \mathcal{L}$ with $(x, z) = (k\, y_1, \ell\, y_2)$. But this holds iff $(x, z) \in (\mathcal{K} \times \mathcal{L}) \cdot \mathcal{R}$. □

4.4. Preservation of language properties

Recall that rational word relations preserve the regularity of languages under left- and right-application. Since rationality and recognizability are different notions in the trace monoid, this leads to two possible generalisations; later, Lemma 4.17 will show that none of them holds for rational trace relations (but, by Theorem 4.18, the right-application of a rational trace relation transforms a recognizable trace language into a rational one).

Now restrict attention to lc-rational trace relations $\mathcal{R}$. Since $\mathcal{R}^{-1}$ need not be lc-rational, we now get four possible preservation results: we could consider rationality or recognizability as well as left- or right-application. The following theorem shows that two of them hold, Lemma 4.17 proves that the other two fail.

Theorem 4.16. *Let $\mathcal{R} \subseteq \mathbb{M}(\mathcal{D}) \times \mathbb{M}(\mathcal{D})$ be an lc-rational trace relation.*

(i) If $\mathcal{K} \subseteq \mathbb{M}(\mathcal{D})$ is recognizable, then also ${}^{\mathcal{R}}\mathcal{K}$ is recognizable.

More precisely, from a dependence alphabet $\mathcal{D}$, a left-closed transducer $\mathfrak{T}$, and a closed NFA $\mathfrak{A}$, one can compute in polynomial time a closed NFA $\mathfrak{B}$ with $[L(\mathfrak{B})] = {}^{[R(\mathfrak{T})]}[L(\mathfrak{A})]$.

(ii) If $\mathcal{L} \subseteq \mathbb{M}(\mathcal{D})$ is rational, then also $\mathcal{L}^{\mathcal{R}}$ is rational.

More precisely, from a dependence alphabet $\mathcal{D}$, a left-closed transducer $\mathfrak{T}$, and an NFA $\mathfrak{A}$, one can compute in polynomial time an NFA $\mathfrak{B}$ with $[L(\mathfrak{B})] = [L(\mathfrak{A})]^{[R(\mathfrak{T})]}$.

Proof. First, let $\mathfrak{T}$ be a left-closed transducer and $\mathfrak{A}$ a closed NFA. By Prop. 4.10(b), we can construct in polynomial time a left-closed transducer $\mathfrak{T}_1$ with $[R(\mathfrak{T}_1)] = [L(\mathfrak{A})] \times \{[\varepsilon]\}$. In a second step, using Prop. 4.9(ii), we can construct in polynomial time a left-closed transducer $\mathfrak{T}_2$ with $[R(\mathfrak{T}_2)] = [R(\mathfrak{T})] \circ [R(\mathfrak{T}_1)]$. Finally, by Prop. 4.10(a), we can construct in polynomial time a closed NFA $\mathfrak{B}$ with $[L(\mathfrak{B})] = \{[v] \mid \exists u\colon ([u], [v]) \in [R(\mathfrak{T}_2)]\}$.

This finishes the proof of the first claim since

$$[R(\mathfrak{T}_2)] = [R(\mathfrak{T})] \circ [R(\mathfrak{T}_1)] = [R(\mathfrak{T})] \circ \big([L(\mathfrak{A})] \times \{[\varepsilon]\}\big) = {}^{[R(\mathfrak{T})]}[L(\mathfrak{A})] \times \{[\varepsilon]\}$$

and therefore

$$[L(\mathfrak{B})] = {}^{[R(\mathfrak{T})]}[L(\mathfrak{A})]\,.$$

The proof of the second claim is analogous. □

We now come to the announced non-preservation results; they hold for lc-rational trace relations and therefore, in particular, for the larger class of rational trace relations.

Lemma 4.17. *Suppose there are $a, b, c, d \in A$ with $(a, b) \in D$ and $c \parallel d$.*

There exist an lc-rational relation $\mathcal{R} \subseteq \mathbb{M}(\mathcal{D})^2$, a rational set $\mathcal{K} \subseteq \mathbb{M}(\mathcal{D})$, and a recognizable set $\mathcal{L} \subseteq \mathbb{M}(\mathcal{D})$ such that ${}^{\mathcal{R}}\mathcal{K}$ is not rational and $\mathcal{L}^{\mathcal{R}}$ is not recognizable.

(We refer, again, to Theorem 4.18 expressing that the left- or right-application of a rational trace relation transforms a recognizable trace language into a rational one.)

Proof. Let $R = \{(a, c), (b, d)\}^*$. Then R is rational and, since $(a, b) \in D$, even lc-rational. We consider the lc-rational trace relation $\mathcal{R} = [R]$. Since $c \parallel d$, we obtain $([u], [v]) \in \mathcal{R}$ if, and only if, $u \in \{a, b\}^*$, $v \in \{c, d\}^*$, $|u|_a = |v|_c$, and $|u|_b = |v|_d$.

Consider the regular language $K = \{cd\}^*$ and let $\mathcal{K}$ denote the rational set $[K]$. Since $c \parallel d$, we get $[v] \in \mathcal{K}$ iff $v \in \{c, d\}^*$ and $|v|_c = |v|_d$. It follows that $[u] \in {}^{\mathcal{R}}\mathcal{K}$ iff $u \in \{a, b\}^*$ and $|u|_a = |u|_b$. Let $H \subseteq A^*$ denote the set of words $u \in \{a, b\}^*$ with $|u|_a = |u|_b$. Since $(a, b) \in D$, this language H is the only language with $[H] = {}^{\mathcal{R}}\mathcal{K}$. Since H is not regular, it follows that ${}^{\mathcal{R}}\mathcal{K}$ is not rational which proves the first claim.

Next, let $L = (ab)^*$ and $\mathcal{L} = [L]$. Then $[u] \in \mathcal{L}$ iff $u \in L$ since $(a, b) \in D$. Hence $\{u \in A^* \mid [u] \in \mathcal{L}\}$ is the regular language L, implying that $\mathcal{L}$ is recognizable. Note that $\mathcal{L}^{\mathcal{R}}$ is the set of traces $[v]$ with $v \in \{c, d\}^*$ and $|v|_c = |v|_d$ (i.e., it equals $\mathcal{K}$). Hence the language $\{v \in A^* \mid [v] \in \mathcal{L}^{\mathcal{R}}\}$, i.e., $\mathcal{L}^{\mathcal{R}}$ is not recognizable. □

The above lemma implies, in particular, that the right-application of rational trace relations does neither preserve the rationality nor the recognizability of a trace language. Proposition 4.11 allows to prove the weaker result that the right-application of a rational relation to a recognizable set yields a rational set.

Theorem 4.18. *Let $\mathcal{R} \subseteq \mathbb{M}(\mathcal{D}) \times \mathbb{M}(\mathcal{D})$ be rational and $\mathcal{L} \subseteq \mathbb{M}(\mathcal{D})$ recognizable. Then $\mathcal{L}^{\mathcal{R}}$ is rational.*

Proof. By Prop. 4.11, there are lc-rational relations $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $\mathcal{R} = \mathcal{R}_1^{-1} \circ \mathcal{R}_2$. Hence $\mathcal{L}^{\mathcal{R}} = \mathcal{L}^{\mathcal{R}_1^{-1} \circ \mathcal{R}_2} = (\mathcal{L}^{\mathcal{R}_1^{-1}})^{\mathcal{R}_2} = ({}^{\mathcal{R}_1}\mathcal{L})^{\mathcal{R}_2}$. From Theorem 4.16, we know that $\mathcal{K} := {}^{\mathcal{R}_1}\mathcal{L}$ is recognizable and therefore in particular rational. Hence, again using Theorem 4.16, $\mathcal{K}^{\mathcal{R}_2}$ is rational. □

Note that the above lemma cannot be improved: by Lemma 4.17, there exist an lc-rational relation $\mathcal{R}$ and a recognizable set $\mathcal{L}$ such that $\mathcal{L}^{\mathcal{R}}$ is not recognizable. Furthermore, there is also a rational set $\mathcal{L}$ such that ${}^{\mathcal{R}}\mathcal{L}$ is not rational. Note that ${}^{\mathcal{R}}\mathcal{L} = \mathcal{L}^{\mathcal{R}^{-1}}$ and that $\mathcal{R}^{-1}$ is rational. Hence, indeed, the above lemma is optimal.

5. The Reachability Relation of tPDS is LC-Rational

In this section we consider the reachability relation of trace-pushdown systems. Concretely, we will show that, by application of the results from the previous section, this relation is lc-rational. Recall that $\vdash_{\mathfrak{P}}^*$ denotes the reachability relation of the tPDS $\mathfrak{P} = (Q, \Delta)$, i.e., the set of pairs (c, d) of configurations such that d is reachable from c.

Next, we want to introduce the notions of rationality and lc-rationality for the reachability relation. Note that we cannot derive them from the classical definitions presented in Section 2 since the set of configurations $\mathrm{Conf}_{\mathfrak{P}}$ is not a monoid. However, when fixing any pair of states $p, q \in Q$ and projecting to the pushdown, we obtain the trace relation

$$\mathrm{Reach}_{p,q}(\mathfrak{P}) := \{([u],[v]) \in \mathbb{M}(\mathcal{D})^2 \mid (p,[u]) \vdash_{\mathfrak{P}}^* (q,[v])\}\,.$$

We say that the reachability relation of $\mathfrak{P}$ is *(lc-)rational* if, and only if, for each pair $p, q \in Q$ of states, the trace relation $\mathrm{Reach}_{p,q}(\mathfrak{P})$ is (lc-)rational.

The main theorem of this section shows that this holds for any trace-pushdown system:

Theorem 5.1. *Let $\mathcal{D} = (A, D)$ be a dependence alphabet and $\mathfrak{P} = (Q, \Delta)$ a trace-pushdown system. Then the reachability relation $\vdash_{\mathfrak{P}}^*$ is lc-rational.*

More precisely, from $\mathcal{D}$, $\mathfrak{P}$, and $p, q \in Q$, one can construct in time polynomial in $\|\mathfrak{P}\|^{O(\mathrm{TI}(\mathcal{D}))}$ a left-closed transducer $\mathfrak{T}$ such that $[R(\mathfrak{T})] = \mathrm{Reach}_{p,q}(\mathfrak{P})$.

Suppose $D = A \times A$. Then $\mathrm{TI}(\mathcal{D}) = 1$ and the trace-pushdown system $\mathfrak{P}$ is actually a classical pushdown system. Furthermore, in this case, the transducer $\mathfrak{T}$ can be constructed in time polynomial in the size of the pushdown system $\mathfrak{P}$—thus, the above theorem generalizes the classical results from [5, 7].

The proof of this theorem (that can be found on page 47) is inspired by the work by Finkel et al. [7].[3] To explain its idea and particularities, we first start with a classical pushdown system $\mathfrak{P} = (Q, \Delta)$ (i.e., with a trace-pushdown system over the trace monoid A^*). Suppose there are states p, q, and r and transitions (p, a, bv, q) and (q, b, ε, r) such that $(p, ax) \vdash_{\mathfrak{P}} (q, bvx) \vdash_{\mathfrak{P}} (r, vx)$ holds for any word x. If we add the transition (q, a, v, r) to Δ that allows to go from (p, ax) to (r, vx) in one step, the reachability relation does not change.

[3] The result originates from [5] where Caucal demonstrates that the reachability relation of a pushdown system is effectively prefix-recognizable, i.e., a finite union of sets of the form $\{(uw, vw) \mid u \in U, v \in V, w \in A^*\}$ for regular languages U and V, but our construction generalizes the one from [7].

We keep adding such "shortcuts" and call the resulting set of transitions $\Delta^{(\infty)}$. Note that this set of transitions is finite since any added "shortcut" writes a word that is shorter than the word bv from the transition (p,a,bv,q) above. Further, the reachability relations of $\mathfrak{P}$ and of $\mathfrak{P}^{(\infty)}=(Q,\Delta^{(\infty)})$ are the same. We next split $\Delta^{(\infty)}$ into the set of transitions shortening the pushdown and the set of transitions that do not shorten the pushdown:

$$\begin{aligned} \Delta_\varepsilon^{(\infty)} &= \{(p,a,\varepsilon,q)\in\Delta^{(\infty)}\} & \mathfrak{P}_\varepsilon^{(\infty)} &= (Q,\Delta_\varepsilon^{(\infty)}) \\ \Delta_+^{(\infty)} &= \{(p,a,v,q)\in\Delta^{(\infty)}\colon |v|\geq 1\} & \mathfrak{P}_+^{(\infty)} &= (Q,\Delta_+^{(\infty)})\,. \end{aligned}$$

The crucial point of the arguments by Finkel et al. is the following: for any two configurations (p,u) and (r,w), we then get $(p,u)\vdash_{\mathfrak{P}}^{*}(r,w)$ if, and only if, $(p,u)\vdash_{\mathfrak{P}^{(\infty)}}^{*}(r,w)$ if, and only if, there exists a configuration (q,v) such that $(p,u)\vdash_{\mathfrak{P}_\varepsilon^{(\infty)}}^{*}(q,v)\vdash_{\mathfrak{P}_+^{(\infty)}}^{*}(r,w)$. In other words, any run of the original system $\mathfrak{P}$ can be simulated by a run of the system with shortcuts $\mathfrak{P}^{(\infty)}$ that first shortens the pushdown (using transitions from $\Delta_\varepsilon^{(\infty)}$) and then writes onto the pushdown (using transitions from $\Delta_+^{(\infty)}$).

It follows that, for any set of configurations C, we have

$$\mathrm{Reach}_{p,r}(\mathfrak{P})=\bigcup_{q\in Q}\mathrm{Reach}_{p,q}(\mathfrak{P}_\varepsilon^{(\infty)})\circ\mathrm{Reach}_{q,r}(\mathfrak{P}_+^{(\infty)})\,.$$

Due to the very restricted type of transitions in the two subsystems of $\mathfrak{P}^{(\infty)}$, it follows that $\mathrm{Reach}_{p,q}(\mathfrak{P}_\varepsilon^{(\infty)})$ and $\mathrm{Reach}_{q,r}(\mathfrak{P}_+^{(\infty)})$ are rational word relations. Consequently, by Theorem 4.2(R2) the reachability relation $\vdash_{\mathfrak{P}}^{*}$ is rational for pushdown systems $\mathfrak{P}$.

The crucial point of the above construction is that any run of the system $\mathfrak{P}^{(\infty)}$ can be brought into some "simple form" by using shortcuts. Here, "simple form" means that it consists of two phases: the pushdown decreases properly in every step of the first phase and does not decrease in any step of the second phase.

Our strategy in the proof of Theorem 5.1 will extend the above idea:

1. First, Propositions 5.4 and 5.6 demonstrate that Theorem 5.1 holds for "homogeneous" systems that formalize and strengthen the two types of systems $\mathfrak{P}_\varepsilon^{(\infty)}$ and $\mathfrak{P}_+^{(\infty)}$ from above:

   Definition 5.2. Let $\mathfrak{P}=(Q,\Delta)$ be a trace-pushdown system. It is *homogeneous* if one of the following hold:

   (1) $\Delta\subseteq Q\times A\times\{\varepsilon\}\times Q$, i.e., all transitions $(p,a,w,q)\in\Delta$ satisfy $w=\varepsilon$, or

   (2) $\Delta\subseteq Q\times\mathrm{twns}(a)\times A^+\times Q$ for a letter $a\in A$, i.e., all transitions $(p,b,w,q)\in\Delta$ satisfy $D(a)=D(b)$ and $w\neq\varepsilon$.

Note that the set of transitions Δ of any trace-pushdown system $\mathfrak{P}$ can be split into the set of transitions Δ_ε as above, and the sets $\Delta_{\mathrm{twns}(b)} = \{(p,a,w,q) \in \Delta \mid D(a) = D(b) \text{ and } w \neq \varepsilon\}$. Hence, the number of these subsystems (that was 2 above) is linear in the twin index $\mathrm{TI}(\mathcal{D})$ of the dependence alphabet $\mathcal{D}$.

2. Secondly, Theorem 5.12 demonstrates Theorem 5.1 for "saturated" systems, i.e., systems where no new "shortcuts" can be added:

   Definition 5.3. Let $\mathfrak{P} = (Q, \Delta)$ be a trace-pushdown system. It is *saturated* if $(p, a, ubv, q), (q, b, \varepsilon, r) \in \Delta$ with $u \parallel b$ implies $(p, a, uv, r) \in \Delta$.

   The central argument in this proof is that any run can be transformed into an equivalent one that consists of a bounded number of phases. As explained above, this bound is 2 for pushdown systems, Example 5.7 will show that this small bound does not suffice for trace-pushdown systems over trace monoids other than A^*. But we show that this number of phases is linear in the twin index $\mathrm{TI}(\mathcal{D})$ of $\mathcal{D}$. This number of phases is the reason why our procedure is exponential in $\mathrm{TI}(\mathcal{D})$.

3. Finally, Prop. 5.18 proves Theorem 5.1 in full generality by showing that any system can be saturated by adding shortcuts.

We will do the aforementioned steps in the following three subsections.

5.1. Reachability in homogeneous systems

We first consider the reachability relation for systems that either shorten their pushdown in each transition or that do not shorten it in any transition, but only replace letters a from the pushdown with $D(a) = D(b)$ for some fixed letter b. Accordingly, we prove two propositions.

The first result considers systems that shorten their pushdown in every step.

Proposition 5.4. *Let $\mathcal{D} = (A, D)$ be a dependence alphabet, $\mathfrak{P} = (Q, \Delta)$ a trace-pushdown system with $\Delta \subseteq Q \times A \times \{\varepsilon\} \times Q$, and $p, q \in Q$ two states. Then the relation $\mathrm{Reach}_{p,q}(\mathfrak{P})$ is lc-rational.*

Even more, from $\mathcal{D}$, $\mathfrak{P}$, and $p, q \in Q$, one can compute a left-closed transducer $\mathfrak{T}$ with $[R(\mathfrak{T})] = \mathrm{Reach}_{p,q}(\mathfrak{P})$; this computation can be carried out in time polynomial in $\|\mathfrak{P}\| + \mathrm{TI}_*(\mathcal{D})$.

Proof. We first transform the tPDS $\mathfrak{P}$ into the NFA $\mathfrak{A} = (Q, A, \{p\}, T, \{q\})$ setting

$$(p_1, a, p_2) \in T \iff (p_1, a, \varepsilon, p_2) \in \Delta\,.$$

Note that the tPDS $\mathfrak{P}$ only reads letters from the pushdown and never writes anything onto the pushdown. Essentially, $\mathfrak{A}$ is the tPDS $\mathfrak{P}$, but we read letters from the input instead of the stack.

Let $\mathcal{K} = [L(\mathfrak{A})]$. We prove that $\mathcal{K}$ is recognizable and that

$$\mathrm{Reach}_{p,q}(\mathfrak{P}) = (\mathcal{K} \times \{[\varepsilon]\}) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})}\,.$$

To prove the recognizability of $\mathcal{K}$, it suffices to prove

$$u \in L(\mathfrak{A}) \iff [u] \in \mathcal{K} \tag{3}$$

for all $u \in A^*$ since $L(\mathfrak{A})$ is regular (note that this implies in particular that the NFA $\mathfrak{A}$ is closed). The implication "$\Rightarrow$" is immediate by the definition of the trace language $\mathcal{K}$. For the implication "$\Leftarrow$", let $[u] \in \mathcal{K}$. By the definition of $\mathcal{K}$, there exists $v \in L(\mathfrak{A})$ with $[u] = [v]$, i.e., $u \sim v$. Hence u can be obtained from v by transposing consecutive independent letters. Since the tPDS $\mathfrak{P}$ satisfies the diamond property (P2), the NFA $\mathfrak{A}$ satisfies the diamond property (D). Hence, the v-labeled path in $\mathfrak{A}$ from p to q can be transformed into a u-labeled path from p to q. But this means that the word u is accepted by the NFA $\mathfrak{A}$, i.e., $u \in L(\mathfrak{A})$. Thus, indeed, the trace language $\mathcal{K}$ is recognizable.

We now prove the above characterization of the relation $\mathrm{Reach}_{p,q}(\mathfrak{P})$. So let $u, v \in A^*$. Then $([u], [v]) \in \mathrm{Reach}_{p,q}(\mathfrak{P})$ iff $(p, [u]) \vdash^* (q, [v])$. But this is equivalent to the existence of $n \geq 0$, transitions $(p_{i-1}, a_i, \varepsilon, p_i) \in \Delta$, and words x_i for $1 \leq i \leq n$ such that

1. $(p, [u]) = (p_0, [a_1 x_1])$,
2. $x_i \sim a_{i+1} x_{i+1}$ for all $1 \leq i < n$, and
3. $(p_n, [x_n]) = (q, [v])$.

Now suppose these transitions and words exist. The construction of the NFA $\mathfrak{A}$ yields $(p_{i-1}, a_i, p_i) \in T$ for all $1 \leq i \leq n$. Hence

$$p = p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \cdots \xrightarrow{a_n} p_n = q$$

is a path in $\mathfrak{A}$ implying $a_1 \cdots a_n \in L(\mathfrak{A})$ and therefore $[a_1 \cdots a_n] \in \mathcal{K}$. Recall that we also have $[x_n] = [v]$ and, by induction, $[x_i] = [a_{i+1} a_{i+2} \cdots a_n v]$, in particular $u \sim a_1 x_1 \sim a_1 a_2 \cdots a_n v$. Consequently,

$$([u], [v]) = ([a_1 \cdots a_n], [\varepsilon]) \cdot ([v], [v]) \in (\mathcal{K} \times \{[\varepsilon]\}) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})}$$

which proves the first inclusion.

Conversely, suppose $[u] \in \mathcal{K}$ and $v \in A^*$, i.e., $([u], [\varepsilon]) \cdot ([v], [v]) \in (\mathcal{K} \times \{[\varepsilon]\}) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})}$. Since $[u] \in \mathcal{K}$, we obtain from Equation (3) a u-labeled accepting path, say

$$p = p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \cdots \xrightarrow{a_n} p_n = q$$

is such a path. The construction of the NFA $\mathfrak{A}$ yields $(p_{i-1}, a_i, \varepsilon, p_i) \in \Delta$ for all $1 \leq i \leq n$. With $x_i = a_i a_{i+1} \cdots a_n\, v$, there are also words such that the above three properties hold, implying $([u], \varepsilon) \cdot ([v], [v]) \in \mathrm{Reach}_{p,q}(\mathfrak{P})$ and therefore the converse inclusion.

Since the closed NFA $\mathfrak{A}$ can be computed in time polynomial in the size of $\mathfrak{P}$, the claim follows from Lemma 4.13. □

In the above proof, we constructed a recognizable trace language $\mathcal{K}$, proved that the trace relation $\text{Reach}_{p,q}(\mathfrak{P})$ equals $(\mathcal{K} \times \{[\varepsilon]\}) \cdot \text{Id}_{\mathbb{M}(\mathcal{D})}$ (this part of the proof used that $\mathfrak{P}$ can only shorten its pushdown) and used that such relations are lc-rational.

We next consider systems where no transition shortens the pushdown. Here, we make the additional assumption that all transitions replace letters b with $D(a) = D(b)$ for some fixed letter a. In this situation, we analogously to the above proof construct a rational trace language $\mathcal{L}$, prove that the trace relation $\text{Reach}_{p,q}(\mathfrak{P})$ equals $(\{[\varepsilon]\} \times \mathcal{L}) \cdot \text{Id}_{\mathbb{M}(\mathcal{D})}$, and use that such relations are lc-rational.

Recall that $\text{twns}(a)$ is the set of twins of the letter a, hence we restrict to systems where all transitions replace twins of a.

Lemma 5.5. *Let $\mathcal{D} = (A, D)$ be a dependence alphabet, $\mathfrak{P} = (Q, \Delta)$ a trace-pushdown system with $\Delta \subseteq Q \times \text{twns}(a) \times A^+ \times Q$ for some $a \in A$, and $p, q \in Q$ be two states. There exists a regular language $H_a \subseteq A^*$ such that, for any $v \in A^*$,*

$$\{[w] \mid (p, [av]) \vdash_{\mathfrak{P}}^* (q, [w])\} = [H_a] \cdot \{[v]\}\,.$$

More precisely, from $\mathfrak{P}$, $a \in A$, and $p, q \in Q$, one can construct in time polynomial in the size of $\mathfrak{P}$ an NFA $\mathfrak{A}_a$ with

$$\{[w] \mid (p, [av]) \vdash_{\mathfrak{P}}^* (q, [w])\} = [L(\mathfrak{A}_a)] \cdot \{[v]\}$$

for all $v \in A^*$.

Proof. Let $\mathfrak{A} = (Q_{\mathfrak{A}}, A, I, \delta, F)$ be the following ε-NFA that simulates a path from $(p, [av])$ to $(q, [w])$ backwards: we start with the only initial state (q, ε) and all pairs $(r, c) \in Q \times A$ as further states (recall that Q is the set of states of the trace-pushdown system $\mathfrak{P}$). The idea is that the second component stores the top letter of the pushdown that is replaced in the next step of the path from $(p, [av])$ to $(q, [w])$ (if such a step exists). To start, we add c-labeled transitions from (q, ε) to (q, c) for any letter $c \in A$. Then, for any transition $(r, c, udv, s) \in \Delta$ with $d \in A$ and $u, v \in A^*$ such that $u \parallel d$, we add a uv-labeled path from (s, d) to (r, c). The set of initial states is $I = \{(q, \varepsilon)\}$ and the set of final states is $F = \{(p, a)\}$.

We set $H_a = L(\mathfrak{A})$.

Now let $v \in A^*$ be arbitrary and set $\mathcal{L}_v = \{[w] \mid (p, [av]) \vdash_{\mathfrak{P}}^* (q, [w])\}$. Hence, it remains to be shown that $\mathcal{L}_v = [H_a] \cdot \{[v]\}$.

First, we verify the inclusion "$\supseteq$". Therefore, let $x \in [H_a] \cdot \{[v]\}$ be arbitary. Then there is $u \in H_a$ with $x = [u] \cdot [v] = [uv]$. Since u is accepted by the ε-NFA $\mathfrak{A}$, we find a letter $a_0 \in A$, pairs $(a_j, w_j) \in \text{twns}(a) \times A^*$ for $j \in [n]$, and tPDS-states $q_j \in Q$ for $j \in \{0, 1, \ldots, n\}$ such that $u = a_0 w_1 w_2 \cdots w_n$ as well as

- $q = q_0$,
- there is a w_{j+1}-labeled path from (q_j, a_j) to (q_{j+1}, a_{j+1}) that does not contain any inner state from $Q \times \text{twns}(a)$, and
- $(q_n, a_n) = (p, a)$.

The construction of the ε-NFA $\mathfrak{A}$ implies that for all $0 \leq j < n$ there are transitions $(q_{j+1}, a_{j+1}, u_{j+1}a_jv_{j+1}, q_j) \in \Delta$ with $u_{j+1} \parallel a_j$ and $w_{j+1} = u_{j+1}v_{j+1}$. Hence, we have

$$\begin{aligned}
(p,[av]) = (q_n,[a_nv]) &\vdash_{\mathfrak{P}} (q_{n-1},[u_na_{n-1}v_nv]) \\
&\qquad = (q_{n-1},[a_{n-1}u_nv_nv]) = (q_{n-1},[a_{n-1}w_nv]) \\
&\vdash_{\mathfrak{P}} (q_{n-2},[u_{n-1}a_{n-2}v_{n-1}w_nv]) \\
&\qquad = (q_{n-2},[a_{n-2}u_{n-1}v_{n-1}w_nv]) = (q_{n-2},[a_{n-2}w_{n-1}w_nv]) \\
&\ \ \vdots \\
&\vdash_{\mathfrak{P}} (q_0,[a_0w_1w_2\cdots w_nv]) = (q,[uv])\,.
\end{aligned}$$

Consequently, $x = [u] \cdot [v] = [uv] \in \mathcal{L}_v$. Since $x \in [H_a] \cdot \{[v]\}$ was chosen arbitrary, we have $\mathcal{L}_v \supseteq [H_a] \cdot \{[v]\}$.

For the converse inclusion, let $w \in A^*$ with $[w] \in \mathcal{L}_v$, i.e., $(p,[av]) \vdash_{\mathfrak{P}}^* (q,[w])$. In order to prove $[w] \in [H_a] \cdot \{[v]\}$, we will construct $u \in H_a$ such that $w \sim uv$.

From $(p,[av]) \vdash_{\mathfrak{P}}^* (q,[w])$, we get a natural number $n \geq 0$, states $q_j \in Q$, and words $x_j \in A^*$ for all $j \in \{0,1,\ldots,n\}$ such that

$$\begin{aligned}
(p,[av]) = (q_n,[x_n]) &\vdash_{\mathfrak{P}} (q_{n-1},[x_{n-1}]) \\
&\vdash_{\mathfrak{P}} (q_{n-2},[x_{n-2}]) \\
&\ \ \vdots \\
&\vdash_{\mathfrak{P}} (q_1,[x_1]) \\
&\vdash_{\mathfrak{P}} (q_0,[x_0]) = (q,[w])\,.
\end{aligned}$$

Consequently, for any j with $n \geq j > 0$, there is a transition $(q_j, a_j, u'_j, q_{j-1}) \in \Delta$ and a word $y_j \in A^*$ with $x_j \sim a_jy_j$ and $x_{j-1} \sim u'_jy_j$ (i.e., the trace-pushdown system replaces the letter a_j with the trace $[u'_j]$). The requirement on $\mathfrak{P}$ implies $a_j \in \operatorname{twns}(a)$ and $u'_j \neq \varepsilon$, hence there are $b_{j-1} \in A$ and $u_j \in A^*$ with $u'_j = b_{j-1}u_j$. Note that $b_{j-1} \in D(b_{j-1}) \subseteq D(u'_j) \subseteq D(a_j)$ since $\mathfrak{P}$ is a trace-pushdown system.

Now let $n \geq j > 1$. Then we obtain

$$b_{j-1}u_jy_j = u'_jy_j \sim x_{j-1} \sim a_{j-1}y_{j-1}\,.$$

From $b_{j-1} \in D(a_j) = D(a) = D(a_{j-1})$, we infer $b_{j-1} = a_{j-1}$ as well as $u_jy_j \sim y_{j-1}$.

Consequently, we have

$$\begin{aligned}
x_0 \sim u'_1y_1 = b_0u_1y_1 &\sim b_0u_1u_2y_2 \\
&\ \ \vdots \\
&\sim b_0u_1u_2\cdots u_ny_n
\end{aligned}$$

Next recall $av \sim x_n \sim a_n y_n$ and $a \in D(a) = D(a_n)$. Hence $a = a_n$ and $v \sim y_n$. Thus, we get

$$w \sim x_0 \sim b_0 u_1 u_2 \cdots u_n y_n \sim b_0 u_1 u_2 \cdots u_n v = uv$$

with $u := b_0 u_1 u_2 \cdots u_n$.

Finally consider the following path in the ε-NFA $\mathfrak{A}$:

$$(q, \varepsilon) = (q_0, \varepsilon) \xrightarrow{b_0}_{\mathfrak{A}} (q_0, b_0) \xrightarrow{u_1}_{\mathfrak{A}} (q_1, a_1) \cdots \xrightarrow{u_n}_{\mathfrak{A}} (q_n, a_n) = (q, a)\,.$$

It witnesses that the word u is accepted by the ε-NFA $\mathfrak{A}$, i.e., that $u \in H_a$.

Thus, indeed, we found a word $u \in H_a$ such that $w \sim uv$ and therefore $[w] \in [H_a] \cdot \{[v]\}$.

Since $w \in A^*$ with $[w] \in \mathcal{L}_v$ was chosen arbitrary, we also proved the inclusion $\mathcal{L}_v \subseteq [H_a] \cdot \{[v]\}$.

Finally note that the ε-NFA $\mathfrak{A}$ can be constructed in time polynomial in the size of the trace-pushdown system $\mathfrak{P}$ and that $\mathfrak{A}$ can be transformed into an equivalent NFA $\mathfrak{A}_a$ in time polynomial in the size of $\mathfrak{A}$. □

Proposition 5.6. *Let $\mathcal{D} = (A, P)$ be a dependence alphabet, $\mathfrak{P} = (Q, \Delta)$ a trace-pushdown system with $\Delta \subseteq Q \times \mathrm{twns}(b) \times A^+ \times Q$ for some $b \in A$, and $p, q \in Q$ two states. Then the trace relation $\mathrm{Reach}_{p,q}(\mathfrak{P})$ is lc-rational.*

More precisely, from $\mathcal{D}$, $\mathfrak{P}$ and $p, q \in Q$, one can compute a left-closed transducer $\mathfrak{T}$ such that $[R(\mathfrak{T})] = \mathrm{Reach}_{p,q}(\mathfrak{P})$; this computation can be carried out in time polynomial in $\|\mathfrak{P}\| + \mathrm{TI}_*(\mathcal{D})$.

Proof. For a letter $a \in \mathrm{twns}(b)$, let H_a denote the regular set from Lemma 5.5. Furthermore, set $\mathcal{I}_{p,q} = \mathrm{Id}_{\mathbb{M}(\mathcal{D})}$ if $p = q$ and $\mathcal{I}_{p,q} = \emptyset$ if $p \neq q$ (note that $\mathcal{I}_{p,q}$ is efficiently lc-rational).

Now consider two words $v', w \in A^*$. Since all transitions from Δ read some letter from $\mathrm{twns}(b)$, we obtain $(p, [v']) \vdash^*_{\mathfrak{P}} (q, [w])$ iff

- $p = q$ and $v' \sim w$ (i.e., $([v'], [w]) \in \mathcal{I}_{p,q}$) or
- there exist $a \in \mathrm{twns}(b)$ and $v \in A^*$ with $v' \sim av$ and $(p, [av]) \vdash^*_{\mathfrak{P}} (q, [w])$.

By Lemma 5.5, $(p, [av]) \vdash^*_{\mathfrak{P}} (q, [w])$ holds iff there is $u \in H_a$ with $w \sim uv$. Thus, $(p, [v']) \vdash^*_{\mathfrak{P}} (q, [w])$ iff

$$([v'], [w]) \in \mathcal{I}_{p,q} \cup \bigcup_{a \in \mathrm{twns}(b)} \big(\{[a]\} \times [H_a]\big) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})}\,.$$

In other words, this relation equals $\mathrm{Reach}_{p,q}(\mathfrak{P})$. Since $\{[a]\}$ is recognizable and $[H_a]$ rational, the relation $\mathrm{Reach}_{p,q}(\mathfrak{P})$ is indeed lc-rational by Theorem 4.15.

More precisely, note that NFAs for $\{[a]\}$ and H_a can be computed in polynomial time. Hence, by Theorem 4.15, a transducer for some left-closed relation R_a with $[R_a] = (\{[a]\} \times [H_a]) \cdot \mathrm{Id}_{\mathbb{M}(\mathcal{D})}$ can be computed in time polynomial in the size of the tPDS and the set twin index of $\mathcal{D}$. The disjoint union of these transducers (for $a \in \mathrm{twns}(b)$) together with some transducer for $\mathcal{I}_{p,q}$ is a left-closed transducer $\mathfrak{T}$ with $[R(\mathfrak{T})] = \mathrm{Reach}_{p,q}(\mathfrak{P})$ and can be computed in the available time since $|A| \leq \|\mathfrak{P}\|$. □

5.2. Reachability in saturated systems

We saw that any run in a saturated pushdown system over the trace monoid A^* can be simulated by a run consisting of two phases: first, the pushdown shortens and then, it increases. In this section, we want to prove that a similar property holds for saturated trace-pushdown systems, i.e., that every run can be simulated by a run consisting of a uniformly bounded number (that only depends linearly on $\mathrm{TI}(\mathcal{D}) \leq |A|$) of phases. By Prop. 4.9, 5.4, and 5.6, it then follows immediately that the reachability relation of a saturated trace-pushdown system is lc-rational. The following example shows that, differently from the pushdown case, we cannot bound the number of phases to two.

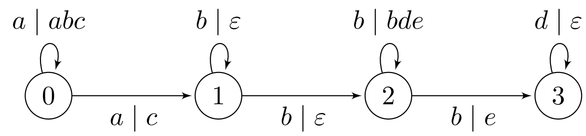


Figure 4: The trace-pushdown system from Example 5.7.

Example 5.7. We consider the dependence alphabet $\mathcal{D} = (A, D)$ with $A = \{a, b, c, d, e\}$ and the dependence relation D given by the following table:

x	a	b	c	d	e
$D(x)$	A	$\{a, b, d, e\}$	$\{a, c\}$	$\{a, b, d\}$	$\{a, b, e\}$

Further, let $\mathfrak{P} = (Q, \Delta)$ be the trace-pushdown system from Fig. 4. One can check that $\mathfrak{P}$ is saturated. The following is the only run from the configuration $(0, [a])$ to the configuration $(3, [e^4c^4])$:

$$\begin{aligned}(0, [a]) &\vdash^3 (0, [abcbcbc]) \vdash (1, [cbcbcbc]) = (1, [b^3c^4])\\ &\vdash^2 (2, [bc^4])\\ &\vdash^3 (2, [bdededec^4]) \vdash (3, [edededec^4]) = (3, [d^3e^4c^4])\\ &\vdash^3 (3, [e^4c^4])\end{aligned}$$

Note that this run splits into four phases (that correspond to the four lines above); it increases the pushdown in the first and third and decreases it in the second and fourth.

So far, we used the term “phase” without defining it formally. To be precise, a “phase” is a run of a maximal homogeneous subsystem of $\mathfrak{P}$. These maximal homogeneous subsystems $\mathfrak{P}_\varepsilon$ and $\mathfrak{P}_T$ for $T \in \mathrm{twns}(\mathcal{D}) \subseteq 2^A$ are defined as follows:

$$\Delta_\varepsilon = \Delta \cap (Q \times A \times \{\varepsilon\} \times Q) \qquad \mathfrak{P}_\varepsilon = (Q, \Delta_\varepsilon)$$
$$\Delta_T = \Delta \cap (Q \times T \times A^+ \times Q) \qquad \mathfrak{P}_T = (Q, \Delta_T)$$

The set Δ_T contains all transitions that replace some letter $a \in T$ by some non-empty word. Note that the definition of a homogeneous subsystem of $\mathfrak{P}$ requires that $T = \mathrm{twns}(a)$ holds for some letter $a \in A$. We will later use that the reachability relation in such subsystems is efficiently lc-rational. We should also note that there is only a linear number of such maximal homogeneous subsystems $\mathfrak{P}_T$ since there are only $\mathrm{TI}(\mathcal{D}) \leq |A|$ many sets $T \in \mathrm{twns}(\mathcal{D})$.

Since $\mathrm{twns}(\mathcal{D})$ is a partitioning of A into the equivalence classes of the relation "twin", the set of transitions Δ is the disjoint union of its subsets Δ_ε and Δ_T for $T \in \mathrm{twns}(\mathcal{D})$. Therefore, any run of $\mathfrak{P}$ splits into maximal subruns of these subsystems and these subruns are precisely what we called "phase". In particular, any run of the system $\mathfrak{P}_\varepsilon$ or $\mathfrak{P}_T$ is a single phase of a run of the complete system $\mathfrak{P}$.

We write $\vdash_\varepsilon$ for the one-step relation $\vdash_{\mathfrak{P}_\varepsilon}$ of the system $\mathfrak{P}_\varepsilon$; $\vdash_T$ is to be understood similarly.

For two binary relations $\models_1$ and $\models_2$ on a set C, we write $\models_1\models_2$ as shorthand for the composition $\models_1 \circ \models_2$.

Definition 5.8. Let $\mathfrak{P}$ be a trace-pushdown system.

- For $T \in \mathrm{twns}(\mathcal{D})$, we set $\Vdash_T = \vdash_\varepsilon^* \vdash_T^+ \subseteq \mathrm{Conf}_{\mathfrak{P}} \times \mathrm{Conf}_{\mathfrak{P}}$.
- For $\overline{T} = (T_1, T_2, \ldots, T_n)$ with $T_i \in \mathrm{twns}(\mathcal{D})$, we set $\Vdash_{\overline{T}} = \Vdash_{T_1} \Vdash_{T_2} \cdots \Vdash_{T_n}$.

In other words, a pair of configurations (c_1, c_2) belongs to $\Vdash_T$ if the system $\mathfrak{P}$ has a run from c_1 to c_2 that first shortens the pushdown and then, in the second phase, uses transitions from Δ_T, only (that replace letters with $\mathrm{twns}(a) = T$ by non-empty words). It should be noted that the first (deleting) phase is allowed to be empty while the second (writing) phase is required to be non-empty. More generally, the pair (c_1, c_2) belongs to $\Vdash_{\overline{T}}$ if there is a run r from c_1 to c_2 that can be split into $r_1, r_2, \ldots, r_{2n}$ where r_{2i-1} shortens the pushdown and r_{2i} uses transitions from Δ_{T_i}, only (for all $1 \leq i \leq n$); again, the odd subruns can be empty while the even ones are required to be of length at least one.

Clearly, the binary relation $\vdash^*$ is the union of all relations $\Vdash_{\overline{T}} \vdash_\varepsilon^*$ for $\overline{T}$ a sequence of sets from $\mathrm{twns}(\mathcal{D})$ of arbitrary length. Our next aim is to show that we only need to consider sequences $\overline{T}$ of bounded length. To this aim, we will prove that any run witnessing $c \Vdash_{\overline{T}} d$ for some "long" sequence $\overline{T}$ implies the existence of some "short" sequence $\overline{U}$ such that $c \Vdash_{\overline{U}} d$.

To see that the following lemma achieves this for certain runs, suppose $u_0 \neq \varepsilon$. Then using the condition (4) from the lemma we infer in particular $(p_0, [a_0 x_0]) \Vdash_{(\mathrm{twns}(a_0), T_1, \ldots, T_n, T_{n+1})} d_3$. In the very specific situation of the lemma, we then get $(p_0, [a_0 x_0]) \Vdash_{(T_1, \ldots, T_n, T_{n+1})} d_3$, i.e., the sequence $(\mathrm{twns}(a_0), T_1, \ldots, T_{n+1})$ can be shortened to $(T_1, \ldots, T_{n+1})$.

Lemma 5.9. *Let $\mathfrak{P} = (Q, \Delta)$ be a saturated trace-pushdown system. Let $T_1, \ldots, T_{n+1} \in \mathrm{twns}(\mathcal{D})$, $(p_0, a_0, u_0, p_1) \in \Delta$, $d_1, d_2, d_3 \in \mathrm{Conf}_{\mathfrak{P}}$, and $x_0 \in A^*$ such that*

- *$D(a_0) \subseteq D(T_{n+1})$ and*
- *the set $D(T_{n+1})$ is incomparable wrt. inclusion with all the sets $D(T_i)$ for all $1 \leq i \leq n$.*

Then

$$(p_0, [a_0 x_0]) \vdash (p_1, [u_0 x_0]) \Vdash_{(T_1, T_2, \ldots, T_n)} d_1 \vdash_\varepsilon^* d_2 \vdash_{T_{n+1}} d_3 \tag{4}$$

implies

$$(p_0, [a_0 x_0]) \Vdash_{(T_1, \ldots, T_n, T_{n+1})} d_3 \,. \tag{5}$$

Proof. The lemma is shown by induction on the length of the word u_0.

If $u_0 = \varepsilon$, we have

$$(p_0, [a_0 x_0]) \vdash_\varepsilon (p_1, [u_0 x_0]) \vdash_\varepsilon^* \vdash_{T_1}^+ \Vdash_{(T_2, \ldots, T_n)} d_1 \vdash_\varepsilon^* d_2 \vdash_{T_{n+1}} d_3$$

and therefore (5).

Now suppose $u_0 = b u_0'$ with $b \in A$ and $u_0' \in A^*$. By (4), there are $k \in \mathbb{N}$ and configurations $c_j = (q_j, [w_j])$ for all $0 \leq j \leq k$ such that

- $c_0 = (p_0, [a_0 x_0])$ and $c_1 = (p_1, [u_0 x_0])$,
- $c_j \vdash c_{j+1}$ for all $1 \leq j < k$, and
- $c_k = d_3$.

Let $j \in \{1, 2, \ldots, k-1\}$. Then $(q_j, [w_j]) = c_j \vdash c_{j+1} = (q_{j+1}, [w_{j+1}])$ implies the existence of a word $x_j \in A^*$ and a transition $(q_j, a_j, u_j, q_{j+1}) \in \Delta$ such that

$$[w_j] = [a_j x_j] \text{ and } [w_{j+1]} = [u_j x_j]$$

(note that this also holds for $j = 0$ by the assumption in the lemma).

Again by (4), the sequence of pairs

$$\big((a_j, u_j)\big)_{1 \leq j < k}$$

can be chosen from

$$\left(\prod_{1 \leq m \leq n} \big(A \times \{\varepsilon\}\big)^* \big(T_m \times A^+\big)^+ \right) \big(A \times \{\varepsilon\}\big)^* \big(T_{n+1} \times A^+\big) \,.$$

In particular, we have $a_{k-1} \in T_{n+1} = \mathrm{twns}(a_{k-1})$, i.e., $D(a_{k-1}) = D(T_{n+1})$.

Note that

$$\begin{aligned} b &\in D(u_0) && \text{(since } u_0 \in bA^*\text{)} \\ &\subseteq D(a_0) && \text{(since } (p_0, a_0, u_0, p_1) \in \Delta\text{)} \\ &\subseteq D(T_{n+1}) = D(a_{k-1}) \,. \end{aligned}$$

In other words, there is some $\ell \in \{1, 2, \ldots, k-1\}$ with $b \in D(a_\ell)$, we choose ℓ minimal with this property implying $a_i \parallel b$ for all $1 \leq i < \ell$.

Claim. $a_\ell = b$.

PROOF OF THE CLAIM. By induction, we construct words x_i' with $x_i \sim bx_i'$ for all $1 \leq i < \ell$. For $i = 1$, we have

$$a_1 x_1 \sim u_0 x_0 = bu_0' x_0 \,.$$

Since $a_1 \parallel b$, this implies the existence of some word x_1' with $x_1 \sim bx_1'$. Now let $1 < i < \ell - 1$. Then

$$\begin{aligned} a_i x_i &\sim u_{i-1} x_{i-1} \sim u_{i-1} b x_{i-1}' \\ &\sim b u_{i-1} x_{i-1}' \qquad\qquad (\text{since } b \notin D(a_{i-1}) \supseteq D(u_{i-1}))\,. \end{aligned}$$

Since $a_i \parallel b$, this implies the existence of some word x_i' with $x_i \sim bx_i'$. Thus, in particular,

$$a_\ell x_\ell \sim u_{\ell-1} x_{\ell-1} \sim u_{\ell-1} b x_{\ell-1}' \sim b u_{\ell-1} x_{\ell-1}'$$

where, as before, the last equivalence follows from $b \notin D(a_{\ell-1}) \supseteq D(u_{\ell-1})$. Recall that $b \in D(a_\ell)$. Hence $a_\ell x_\ell \sim b u_{\ell-1} x_{\ell-1}'$ implies $a_\ell = b$. Q.E.D.

Now, we distinguish two cases, namely whether the word u_ℓ is empty or not.

First, suppose $u_\ell = \varepsilon$.

The diamond property of $\mathfrak{P}$ and the independence of $b = a_\ell$ from all letters $a_1, a_2, \ldots, a_{\ell-1}$ implies that we can reorder these transitions. More precisely, there are states $p_1, \ldots, p_\ell \in Q$ with $p_\ell = q_{\ell+1}$ such that we obtain the following new run from c_0 to c_k:

- first apply the transition (q_0, a_0, bu_0', q_1),
- then do the a_ℓ-transition $(q_1, a_\ell, u_\ell, p_1) = (q_1, b, \varepsilon, p_1)$,
- then follow the a_j-transitions (p_j, a_j, u_j, p_{j+1}) for $j = 1, 2, \ldots, \ell - 1$,
- and finally follow the original path $c_{\ell+2}, c_{\ell+3}, \ldots, c_k$.

So far, we reordered the path from c_0 to c_k in such a way that the second transition (starting in the configuration c_1) is $(q_1, b, \varepsilon, p_1)$.

Note that this reordered path has all the properties of the original path $c_0, c_1, c_2, \ldots, c_k$, so we abuse notation and call the configurations in this new path $c_0, c_1, c_2, \ldots, c_k$. The crucial effect is that then $\ell = 1$ and $u_1 = \varepsilon$. In other words, we have transitions

$$(q_0, a_0, bu_0', q_1) \text{ and } (q_1, b, \varepsilon, q_2)$$

in the saturated system $\mathfrak{P}$. Saturation implies

$$(q_0, a_0, u_0', q_2) \in \Delta\,.$$

Hence, also the sequence of configurations $c_0, c_2, c_3, \ldots, c_k$ (omitting c_1) is a path in $\mathfrak{P}$ that leads from c_0 to c_k. Note that, in the first transition in this path, a_0 is replaced by the word u_0' which is properly shorter than u_0. Hence, by the induction hypothesis, we get (5) which settles the case $u_\ell = \varepsilon$.

Now let $u_\ell \neq \varepsilon$. Then there is $i \in \{1, 2, \ldots, n+1\}$ such that $a_\ell \in T_i$, i.e., $D(a_\ell) = D(T_i)$. If $i \leq n$, we get

$$\begin{aligned} D(T_i) = D(a_\ell) = D(b) \qquad & (\text{since } a_\ell = b) \\ \subseteq D(a_0) \qquad & (\text{since } (p_0, a_0, bu_0', p_1) = (p_0, a_0, u_0, p_1) \in \Delta) \\ \subseteq D(T_{n+1}) \qquad & \end{aligned}$$

which contradicts the incomparability of $D(T_i)$ and $D(T_{n+1})$ for all $i \in [n]$. Hence $i = n+1$, i.e., $D(a_\ell) = D(T_{n+1})$. Note that, among all the transitions (q_j, a_j, u_j, q_{j+1}) for $j \in \{1, 2, \ldots, k-1\}$, only the last one satisfies $D(a_j) = D(T_{n+1})$ and $u_j \neq \varepsilon$. Hence, $\ell = k-1$. Thus, we showed $i = n+1$ and $\ell = k-1$.

Note that

$$D(b) \subseteq D(a_0) \subseteq D(T_{n+1}) = D(a_\ell) = D(b)$$

implies $D(b) = D(a_0)$. Since a_ℓ is independent from all the letters from $\{a_1, a_2, \ldots, a_{\ell-1}\} = \{a_1, a_2, \ldots, a_{k-2}\}$, the same applies to a_0. Hence, the diamond property of $\mathfrak{P}$ and the independence of a_0 from all letters $a_1, a_2, \ldots, a_{k-2}$ imply that we can reorder these transitions. More precisely, there are states $p_0, \ldots, p_{k-2} \in Q$ with $p_0 = q_0$ such that we have the following run from c_0 to c_k:

- first apply the a_j-transitions (p_{j-1}, a_j, u_j, p_j) for $j = 1, 2, \ldots, k-2$,
- then the b-transition $(p_{k-2}, a_0, u_0, q_{k-1})$,
- and then the a_{k-1}-transition $(q_{k-1}, a_{k-1}, u_{k-1}, q_k)$.

Since $D(a_0) = D(b) = D(a_\ell) = D(T_{n+1})$, the resulting run satisfies (5). □

As explained before the previous lemma, in certain situations, we can shorten the sequence $\overline{T}$. To apply the lemma, we require in particular that, if the run starts with a transition of $\mathfrak{P}_{\mathrm{twns}(a_0)}$, then this very first phase has length one. Similarly, the very last phase (a run of $\mathfrak{P}_{T_{n+1}}$) has length one. The following lemma replaces these requirements by "non-empty phase" and therefore generalizes the above lemma.

Lemma 5.10. *Let $\mathfrak{P} = (Q, \Delta)$ be a saturated trace-pushdown system. Let $T_0, T_1, \ldots, T_{n+1} \in \mathrm{twns}(\mathcal{D})$ such that*

- *$D(T_0) \subseteq D(T_{n+1})$ and*
- *the sets $D(T_i)$ and $D(T_{n+1})$ are incomparable wrt. inclusion for all $1 \leq i \leq n$.*

Then $\Vdash_{(T_0, T_1, \ldots, T_{n+1})} \subseteq \Vdash_{(T_1, \ldots, T_{n+1})}$.

Proof. Let $c, d \in \mathrm{Conf}_{\mathfrak{P}}$ be two configurations with $c \Vdash_{(T_0,T_1,\ldots,T_{n+1})} d$. Then there exist $k \geq 1$ such that

$$c \vdash_{\varepsilon}^{*} \vdash_{T_0}^{k} \Vdash_{(T_1,\ldots,T_n)} \vdash_{\varepsilon}^{*} \vdash_{T_{n+1}}^{+} d\,. \tag{6}$$

Let $k \geq 0$ be minimal such that (6) holds. If $k > 0$, there exist configurations $c', d' \in \mathrm{Conf}_{\mathfrak{P}}$ such that

$$c \vdash_{\varepsilon}^{*} \vdash_{T_0}^{k-1} c' \vdash_{T_0} \Vdash_{(T_1,\ldots,T_n)} \vdash_{\varepsilon}^{*} \vdash_{T_{n+1}} d' \vdash_{T_{n+1}}^{*} d$$

From Lemma 5.9, we obtain

$$c' \Vdash_{(T_1,\ldots,T_n)} \vdash_{\varepsilon}^{*} \vdash_{T_{n+1}}^{+} d'$$

and therefore

$$c \vdash_{\varepsilon}^{*} \vdash_{T_0}^{k-1} \Vdash_{(T_1,\ldots,T_n)} \vdash_{\varepsilon}^{*} \vdash_{T_{n+1}}^{+} d\,,$$

contradicting our choice of k. Hence $k = 0$ such that $c \Vdash_{(T_1,\ldots,T_{n+1})} d$ follows from (6). □

We will now show that, indeed, the reachability relation of a saturated system is efficiently lc-rational. The central argument will be that the above lemma allows to bound the number of phases uniformly.

Proposition 5.11. *Let $\mathcal{D} = (A, D)$ be a dependence alphabet, $\mathfrak{P} = (Q, \Delta)$ a saturated trace-pushdown system, and $c, d \in \mathrm{Conf}_{\mathfrak{P}}$ two configurations of $\mathfrak{P}$ with $c \vdash_{\mathfrak{P}}^{*} d$. Then there exist $m \leq \mathrm{TI}(\mathcal{D})$ and $\overline{T} = (T_1, \ldots, T_m)$ with $T_i \in \mathrm{twns}(\mathcal{D})$ such that $c \Vdash_{\overline{T}} \vdash_{\varepsilon}^{*} d$.*

Proof. Since $c \vdash^{*} d$, there is some sequence of sets $\overline{U} = (U_1, U_1, \ldots, U_m)$ with $U_i \in \mathrm{twns}(\mathcal{D})$ such that

$$c \Vdash_{\overline{U}} \vdash_{\varepsilon}^{*} d\,.$$

Let $\overline{U}$ be such a sequence of minimal length and suppose $m > \mathrm{TI}(\mathcal{D})$. Then there are natural numbers i and n such that

$$1 \leq i < i + n + 1 \leq m \text{ and } D(U_i) \subseteq D(U_{i+n+1})\,. \tag{7}$$

Let $n \geq 0$ be minimal such that this holds for some i.

Then there are configurations $c', d' \in \mathrm{Conf}_{\mathfrak{P}}$ such that

$$c \Vdash_{(U_1,\ldots,U_{i-1})} c' \Vdash_{(U_i,\ldots,U_{i+n+1})} d' \Vdash_{(U_{i+n+2},\ldots,U_m)} \vdash_{\varepsilon}^{*} d$$

where we understand $\Vdash_{()}$ as identity relation. For notational simplicity, we set $T_j = U_{i+j}$ for all $0 \leq j \leq n+1$ such that

$$c' \Vdash_{(T_0,T_1,\ldots,T_{n+1})} d' \text{ and } D(T_0) \subseteq D(T_{n+1})\,.$$

Let $j \in \{1, 2, \ldots, n\}$. Then $D(T_j) \nsubseteq D(T_{n+1})$ since otherwise we would have chosen $n - j < n$ for n. Similarly, $D(T_{n+1}) \nsubseteq D(T_j)$ since otherwise $D(T_0) \subseteq$

$D(T_{n+1}) \subseteq D(T_j)$ and we would have chosen $j - i < n$ for n. Hence we showed that the sets $D(T_j)$ and $D(T_{n+1})$ are incomparable for all $j \in \{1, 2, \ldots, n\}$. From Lemma 5.10, we obtain

$$c' \Vdash_{(T_1,\ldots,T_{n+1})} d' .$$

But this implies

$$c \Vdash_{(U_1,\ldots,U_{i-1},U_{i+1},\ldots,U_m)} \vdash_\varepsilon^* d$$

contradicting our choice of m.

Hence, indeed, $m \leq \mathrm{TI}(\mathcal{D})$. □

Theorem 5.12. *Let $\mathcal{D} = (A, D)$ be a dependence alphabet, $\mathfrak{P} = (Q, \Delta)$ a saturated trace-pushdown system over $\mathcal{D}$, and $p, q \in Q$. Then the reachability relation $\mathrm{Reach}_{p,q}(\mathfrak{P})$ is lc-rational.*

More precisely, from $\mathcal{D}$, $\mathfrak{P}$, p, and q, one can compute a left-closed transducer $\mathfrak{T}$ with $\mathrm{Reach}_{p,q}(\mathfrak{P}) = [R(\mathfrak{T})]$; this computation can be carried out in time $\|\mathfrak{P}\|^{O(\mathrm{TI}(\mathcal{D}))}$.

Proof. In this proof, let $n = 2 \cdot \mathrm{TI}(\mathcal{D}) + 1$.

For states $r_1, r_2 \in Q$, let

$$\mathcal{R}_{r_1,r_2}(\mathfrak{P}) = \mathrm{Reach}_{r_1,r_2}(\mathfrak{P}_\varepsilon) \cup \bigcup_{T \in \mathrm{twns}(\mathcal{D})} \mathrm{Reach}_{r_1,r_2}(\mathfrak{P}_T)$$

denote the union of the reachability relations of the homogeneous subsystems $\mathfrak{P}_\varepsilon$ and $\mathfrak{P}_T$ of $\mathfrak{P}$. Recall that $\Delta_T \neq \emptyset$ implies the existence of some $a \in A$ with $\mathrm{twns}(a) = T$, hence

$$\mathcal{R}_{r_1,r_2}(\mathfrak{P}) = \mathrm{Reach}_{r_1,r_2}(\mathfrak{P}_\varepsilon) \cup \bigcup_{b \in A} \mathrm{Reach}_{r_1,r_2}(\mathfrak{P}_{\mathrm{twns}(b)}) .$$

This relation is lc-rational according to Prop. 5.4 and 5.6. Now Proposition 5.11 implies

$$\mathrm{Reach}_{p,q}(\mathfrak{P}) = \bigcup_{\substack{q_0,q_1,\ldots,q_n \in Q,\\ q_0=p,q_n=q}} \prod_{0 \leq i < n} \mathcal{R}_{q_i,q_{i+1}}(\mathfrak{P})$$

for any $p, q \in Q$ (here, $\prod$ represents the iterated application of composition $\circ$ of relations). Since $\mathcal{R}_{q_i,q_{i+1}}(\mathfrak{P})$ is lc-rational for any pair $0 \leq i < n$ and since the class of lc-rational trace relations is closed under composition (Prop. 4.9) and union, the trace relation $\mathrm{Reach}_{p,q}(\mathfrak{P})$ is also lc-rational.

It remains to verify the claimed upper bound.

First, fix some sequence $\overline{q} = (q_0, \ldots, q_n)$ of states of $\mathfrak{P}$. Let $0 \leq i < n$. By Prop. 5.4 and 5.6, we can construct, for all $X \in \{\varepsilon\} \cup \mathrm{twns}(\mathcal{D})$, a left-closed transducer $\mathfrak{T}_{q_i,q_{i+1},X}$ with $[R(\mathfrak{T}_{q_i,q_{i+1},X})] = \mathrm{Reach}_{q_i,q_{i+1}}(\mathfrak{P}_X)$ in time polynomial in $\|\mathfrak{P}\| + \mathrm{TI}_*(\mathcal{D})$. Let the left-closed transducer $\mathfrak{T}_{q_i,q_{i+1}}$ be the disjoint union of all the transducers $\mathfrak{T}_{q_i,q_{i+1},X}$ such that $[R(\mathfrak{T}_{q_i,q_{i+1}})] = \mathcal{R}_{q_i,q_{i+1}}$. Since the number of possible values for X is linear in $\mathrm{TI}(\mathcal{D})$, the left-closed transducer $\mathfrak{T}_{q_i,q_{i+1}}$ can be constructed in time $\mathrm{poly}(\|\mathfrak{P}\| + \mathrm{TI}_*(\mathcal{D})) + O(\mathrm{TI}(\mathcal{D}))$.

Let $t_{\overline{q}}$ denote the total size of all these transducers $\mathfrak{T}_{q_i,q_{i+1}}$ as well as the tPDS $\mathfrak{P}$ (since the number n of these transducers is linear in $\mathrm{TI}(\mathcal{D})$, the number $t_{\overline{q}}$ is bounded by $O(\mathrm{TI}(\mathcal{D})) \cdot [\mathrm{poly}(\|\mathfrak{P}\| + \mathrm{TI}_*(\mathcal{D})) + O(\mathrm{TI}(\mathcal{D}))]$ and is therefore polynomial in $\|\mathfrak{P}\| + \mathrm{TI}_*(\mathcal{D}) + \mathrm{TI}(\mathcal{D})$). By Prop. 4.9, a left-closed transducer $\mathfrak{T}_{\overline{q}}$ with $[R(\mathfrak{T}_{\overline{q}})] = \prod_{0\le i\le n}[R(\mathfrak{T}_{q_i,q_{i+1}})]$ can be computed in time $O(t_{\overline{q}}^n)$.

There is a number t with $t_{\overline{q}} \le t$ for all $\overline{q} \in Q^{n+1}$ that is polynomial in $\|\mathfrak{P}\| + \mathrm{TI}_*(\mathcal{D}) + \mathrm{TI}(\mathcal{D})$. Since there are $|Q|^{n+1} \le t^{n+1}$ many such sequences $\overline{q}$ of states, a left-closed transducer $\mathfrak{T}$ for the union of all these relations can be obtained in time $t^{O(n)} = t^{O(\mathrm{TI}(\mathcal{D}))}$. Recall that t is polynomial in $\|\mathfrak{P}\| + \mathrm{TI}_*(\mathcal{D}) + \mathrm{TI}(\mathcal{D})$. The number of sets $D(B)$ for $B \subseteq A$ is at most $2^{\mathrm{TI}(\mathcal{D})}$ since any set $D(B)$ is the union of the sets $D(b)$ for $b \in B$. Hence $\mathrm{TI}_*(\mathcal{D}) \le 2^{\mathrm{TI}(\mathcal{D})}$ implying that t is polynomial in $\|\mathfrak{P}\|^{O(\mathrm{TI}(\mathcal{D}))}$. But this implies that the construction of $\mathfrak{T}$ can, indeed, be carried out in time $\|\mathfrak{P}\|^{O(\mathrm{TI}(\mathcal{D}))}$. □

5.3. Saturating a system

So far, we showed that the reachability relation is lc-rational, provided the system is saturated. To get the result in full generality, it remains to transform an arbitrary system into an equivalent saturated one. For a classical pushdown system (i.e., a trace-pushdown system over the trace monoid A^*), the idea is very simple: if there are transitions (p, a, bw, q) and (q, b, ε, r), then adding the transition (p, a, w, r) does not change the behavior and transforms the system closer to a saturated one. In the trace-pushdown setting, the technicalities are a bit more involved: suppose we have the transitions (p, a, cbw, q) and (q, b, ε, r) with $b \parallel c$. Then $[cbw] = [bcw]$, i.e., after doing the first transition (that writes the trace $[cbw] = [bcw]$ onto the pushdown), the second transition (eliminating b) can be executed immediately. Therefore, also in this situation, we add the transition (p, a, cw, r) to get closer to a saturated system.

We also want to keep the system "small". In particular, we do not want to add transitions (p, a, u, q) and (p, a, v, q) with $u \sim v$ as they are redundant. To achieve this technically, we use lexicographic normal forms defined as follows.

Definition 5.13. Let $\mathcal{D} = (A, D)$ be a dependence alphabet and $\le$ a linear order on A. Further, let $u \in A^*$. Then $\mathrm{lnf}(u)$ is the (wrt. $\le$) lexicographically minimal word $v \in A^*$ with $u \sim v$; it is called the *lexicographic normal form* of u.

Note that, since only finitely many words v satisfy $u \sim v$, the lexicographic normal form exists for any word u. Further, it can easily be computed from u in polynomial time (cf. [23, Section 1.5]).

Formally, we construct the pushdown systems $\mathfrak{P}^{(k)} = (Q, \Delta^{(k)})$ for any $k \in \mathbb{N}$ as follows:

- we set $\Delta^{(0)} := \{(p, a, \mathrm{lnf}(w), q) \mid (p, a, w, q) \in \Delta\}$.
- To obtain $\Delta^{(k+1)}$, we add to the set $\Delta^{(k)}$ all transitions $(p, a, \mathrm{lnf}(uv), r)$ for which there are a letter $b \in A$ and a state $q \in Q$ such that (p, a, ubv, q), $(q, b, \varepsilon, r) \in \Delta^{(k)}$ and $u \parallel b$.

Let $\Delta^{(\infty)} = \bigcup_{k\geq 0} \Delta^{(k)}$ be the "limit" of the increasing sequence of sets $\Delta^{(k)}$. Since Δ is finite, there exists a natural number N with $\Delta \subseteq Q \times A \times A^{\leq N} \times Q$, i.e., all words written by the trace-pushdown system $\mathfrak{P}$ are of length at most N. The construction ensures that this also holds for all pushdown systems $\mathfrak{P}^{(k)} = (Q, \Delta^{(k)})$. Hence the limit set $\Delta^{(\infty)}$ is finite, i.e., the pair $\mathfrak{P}^{(\infty)} = (Q, \Delta^{(\infty)})$ is a pushdown system. Since the sequence of sets $\Delta^{(k)}$ is increasing, there is some natural number $\ell \leq \left|Q \times A \times A^{\leq N} \times Q\right|$ with $\Delta^{(\infty)} = \Delta^{(\ell)}$ (later, we will see that a much smaller number ℓ suffices).

Example 5.14. In Fig. 5 we depict our construction of the pushdown systems $\mathfrak{P}^{(k)}$ for $k \in \{0, 1, 2\}$ starting with the tPDS $\mathfrak{P} = \mathfrak{P}^{(0)}$ on the left of Fig. 5. There, $(a, b) \in D$ and $(a, c), (b, c) \notin D$. It can be verified that $\mathfrak{P}^{(2)} = \mathfrak{P}^{(3)}$. Further, all the pushdown systems $\mathfrak{P}^{(k)}$ satisfy conditions (P1) and (P2).

Below, we prove these two properties for all trace-pushdown systems $\mathfrak{P}$.

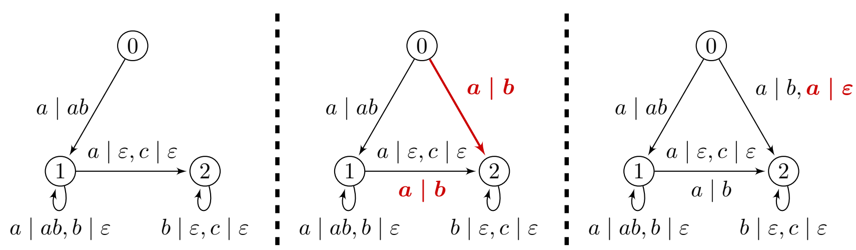


Figure 5: The trace-pushdown system $\mathfrak{P} = \mathfrak{P}^{(0)}$, $\mathfrak{P}^{(1)}$, and $\mathfrak{P}^{(2)} = \mathfrak{P}^{(\infty)}$ (from left to right). New transitions are marked in bold and red.

By definition, the pair $\mathfrak{P}^{(k)} = (Q, \Delta^{(k)})$ is a pushdown system. The following lemma states that this pushdown system is even a trace-pushdown system.

Lemma 5.15. *The pair $\mathfrak{P}^{(\infty)}$ is a trace-pushdown system.*

Proof. Since $\mathfrak{P}^{(\infty)}$ equals one of the systems $\mathfrak{P}^{(k)}$, it suffices to prove this statement by induction on k for all systems $\mathfrak{P}^{(k)}$. The base case $k = 0$ is obvious since $\mathfrak{P}$ is a trace-pushdown system.

Now let $k \geq 0$. We have to verify that $\mathfrak{P}^{(k+1)}$ satisfies the two conditions (P1) and (P2) from Definition 3.2 and Lemma 3.6, respectively.

To show (P1), let $(p, a, w, r) \in \Delta^{(k+1)}$. If this transition belongs to $\Delta^{(k)}$, we obtain $D(w) \subseteq D(a)$ by the induction hypothesis. So suppose, in the other case, $(p, a, w, r) \in \Delta^{(k+1)} \setminus \Delta^{(k)}$. Then, by definition of $\Delta^{(k+1)}$, there are words $u, v \in A^*$ with $w = \mathrm{lnf}(uv)$, a letter $b \in A$ with $u \parallel b$, and a state $q \in Q$ such that $(p, a, ubv, q), (q, b, \varepsilon, r) \in \Delta^{(k)}$. Then we obtain

$$D(w) = D(uv) \subseteq D(ubv) \subseteq D(a)$$

where the last inclusion follows from the induction hypothesis and $(p, a, ubv, q) \in \Delta^{(k)}$. Hence $\mathfrak{P}^{(k+1)}$ satisfies property (P1).

We next show (P2). So let $(p, a, w, q), (q, c, x, r) \in \Delta^{(k+1)}$ with $a \parallel c$.

We distinguish the cases $(q, c, x, r) \in \Delta^{(k)}$ and $(q, c, x, r) \in \Delta^{(k+1)} \setminus \Delta^{(k)}$.

Case 1. Suppose $(q, c, x, r) \in \Delta^{(k)}$.

If, in addition, $(p, a, w, q) \in \Delta^{(k)}$, then the induction hypothesis yields a state q' with $(p, c, x, q'), (q', a, w, r) \in \Delta^{(k)} \subseteq \Delta^{(k+1)}$.

So suppose $(p, a, w, q) \in \Delta^{(k+1)} \setminus \Delta^{(k)}$ (cf. Fig. 6a for a visualization of the proof where all arrows belong to $\Delta^{(k+1)}$ and the dashed ones even to $\Delta^{(k)}$). By the definition of $\Delta^{(k+1)}$, there are words $u, v \in A^*$ with $w = \mathrm{lnf}(uv)$, a letter $b \in A$ with $u \parallel b$, and a state $s \in Q$ such that $(p, a, ubv, s), (s, b, \varepsilon, q) \in \Delta^{(k)}$.

Note that $(s, b, \varepsilon, q), (q, c, x, r) \in \Delta^{(k)}$. We know $D(b) \subseteq D(ubv) \subseteq D(a)$ where the last inclusion follows from $(p, a, ubv, s) \in \Delta^{(k)}$ since $\mathfrak{P}^{(k)}$ satisfies (P1). From $a \parallel c$, we obtain $c \notin D(a) \supseteq D(b)$, i.e., $b \parallel c$. Since the tPDS $\mathfrak{P}^{(k)}$ satisfies (P2), we obtain a state $q' \in Q$ with transitions $(s, c, x, q'), (q', b, \varepsilon, r) \in \Delta^{(k)}$.

Note further that $(p, a, ubv, s), (s, c, x, q') \in \Delta^{(k)}$ and $a \parallel c$. Hence the diamond property (P2) in $\mathfrak{P}^{(k)}$ implies the existence of a state $s' \in Q$ with $(p, c, x, s'), (s', a, ubv, q') \in \Delta^{(k)}$.

Finally note that $(s', a, ubv, q'), (q', b, \varepsilon, r) \in \Delta^{(k)}$. Since $b \parallel u$ and $w = \mathrm{lnf}(uv)$, the construction ensures $(s', a, w, r) \in \Delta^{(k+1)}$.

In summary, we found a state $s' \in Q$ with $(p, c, x, s') \in \Delta^{(k)}$ and $(s', a, w, r) \in \Delta^{(k+1)}$. Since $\Delta^{(k)} \subseteq \Delta^{(k+1)}$, this finishes the verification of (P2) in case 1.

Case 2. Suppose $(q, c, x, r) \in \Delta^{(k+1)} \setminus \Delta^{(k)}$ (cf. Fig. 6b for a visualization of the proof).

By construction of $\Delta^{(k+1)}$, there are words $y, z \in A^*$ with $x = \mathrm{lnf}(yz)$, a letter $d \in A$ with $y \parallel d$, and a state $s \in Q$ such that $(q, c, ydz, s), (s, d, \varepsilon, r) \in \Delta^{(k)}$.

Note that $(p, a, w, q) \in \Delta^{(k+1)}$ and $(q, c, ydz, s) \in \Delta^{(k)}$ with $a \parallel c$. Hence, by case 1, there is a state $q' \in Q$ with $(p, c, xdy, q') \in \Delta^{(k)}$ and $(q', a, w, s) \in \Delta^{(k+1)}$.

Note further that $(q', a, w, s) \in \Delta^{(k+1)}$ and $(s, d, \varepsilon, r) \in \Delta^{(k)}$. We have $D(d) \subseteq D(ydz) \subseteq D(c)$ where the last inclusion follows from $(q, c, ydz, s) \in \Delta^{(k)}$ since $\mathfrak{P}^{(k)}$ satisfies (P1) by the induction hypothesis. From $a \parallel c$, we obtain $a \notin D(c) \supseteq D(d)$ and therefore $a \parallel d$. Hence, again by case 1, there is a state s' with $(q', d, \varepsilon, s') \in \Delta^{(k)}$ and $(s', a, w, r) \in \Delta^{(k+1)}$.

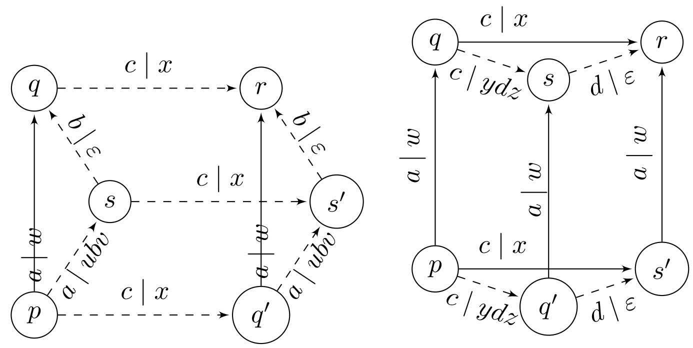
Finally note that $(p, c, ydz, q'), (q', d, \varepsilon, s') \in \Delta^{(k)}$. Since $d \parallel y$ and $\mathrm{lnf}(x) = yz$, the construction of $\Delta^{(k+1)}$ yields $(p, c, x, s') \in \Delta^{(k+1)}$.

Thus, we found a state $s' \in Q$ with $(p, c, x, s'), (s', a, w, r) \in \Delta^{(k+1)}$ which finishes the verification of (P2) in case 2.

As the above two cases cover all possibilities, we showed that $\mathfrak{P}^{(k+1)}$ satisfies (P2). □

Since $\mathfrak{P}$ and $\mathfrak{P}^{(\infty)}$ have the same alphabet and the same set of states, the sets of configurations of these two trace-pushdown systems coincide. The following lemma states that they also agree in their reachability relations.

Lemma 5.16. *For any two configurations $(p, [u])$ and $(q, [v])$, we have $(p, [u]) \vdash^*_{\mathfrak{P}} (q, [v])$ if, and only if, $(p, [u]) \vdash^*_{\mathfrak{P}^{(\infty)}} (q, [v])$.*



(a) The case $(q, c, x, r) \in \Delta^{(k)}$.

(b) The case $(p, c, w_2, r_2) \in \Delta^{(k+1)} \setminus \Delta^{(k)}$.

Figure 6: Visualization of the proof of the diamond properties (cf. proof of Lemma 5.15). All arrows vizualize transitions from $\Delta^{(k+1)}$ and dashed arrows transitions that even belong to $\Delta^{(k)}$).

Proof. We first demonstrate the implication "⇒". For this, it suffices to observe that $(p, [u]) \vdash_{\mathfrak{P}} (q, [v])$ implies $(p, [u]) \vdash_{\mathfrak{P}^{(\infty)}} (q, [v])$. But this is trivial since the transitions $(p, a, w, q) \in \Delta$ and $(p, a, \mathrm{lnf}(w), q) \in \Delta^{(0)} \subseteq \Delta^{(\infty)}$ have the same effect on any configuration. For the other implication, it suffices to show for any natural number k that $(p, [u]) \vdash_{\mathfrak{P}^{(k+1)}} (r, [v])$ implies $(p, [u]) \vdash^*_{\mathfrak{P}^{(k)}} (r, [v])$, i.e., that every single step of $\mathfrak{P}^{(k+1)}$ can be simulated by a sequence of steps of $\mathfrak{P}^{(k)}$.

Since $(p, [u]) \vdash_{\mathfrak{P}^{(k+1)}} (r, [v])$, there is a transition $(p, a, w, r) \in \Delta^{(k+1)}$ and a word u' such that $u \sim au'$ and $v \sim wu'$. If the transition (p, a, w, r) belongs to $\Delta^{(k)}$, we immediately get $(p, [u]) \vdash_{\mathfrak{P}^{(k)}} (r, [v])$. Otherwise, there are words w_1 and w_2 with $w = \mathrm{lnf}(w_1 w_2)$ such that, for some letter $b \in A$ and some state $q \in Q$, we have $(p, a, w_1 b w_2, q), (q, b, \varepsilon, r) \in \Delta^{(k)}$ and $w_1 \parallel b$. But then $w_1 b w_2 \sim b w_1 w_2$ implying

$$\begin{aligned}(p, [u]) = (p, [au']) &\vdash_{\mathfrak{P}^{(k)}} (q, [w_1 b w_2 u']) = (q, [b w_1 w_2 u']) \\ &\vdash_{\mathfrak{P}^{(k)}} (r, [w_1 w_2 u']) = (r, [v])\,.\end{aligned}$$

Hence, also the implication "⇐" of the claim holds. □

So far, we proved that the reachability relation of all saturated trace-pushdown systems are lc-rational and that all trace-pushdown system can be saturated, i.e., the reachability relation of all trace-pushdown systems are lc-rational. Our main result Theorem 5.1 also claims that a transducer for this relation can be constructed in polynomial time (assuming the dependence alphabet $\mathcal{D}$ to be fixed). Since Theorem 5.12 contains a corresponding statement, it remains to be shown that the above saturation procedure can be carried out in polynomial time (and therefore leads to a system of polynomial size).

Recall that the independence number $\alpha(\mathcal{D})$ is the maximal size of a set of letters that are mutually independent.

Lemma 5.17. *Let $\mathcal{D} = (A, D)$ be a dependence alphabet and $\mathfrak{P}$ be a trace-pushdown system. The number of transitions in $\Delta^{(\infty)}$ is polynomial in $\|\mathfrak{P}\|^{\alpha(\mathcal{D})}$ and this set can be computed in time polynomial in $\|\mathfrak{P}\|^{\alpha(\mathcal{D})}$.*

Proof. For $k \geq 0$, let $W_k = \{w \in A^* \mid (p, a, w, r) \in \Delta^{(k)}\}$ denote the set of words written by some transition from $\Delta^{(k)}$.

By induction on k, we show that, for any word $w \in W_k$, there exists a word $v \in W_0$ such that the trace $[w]$ is a *suffix* of the trace $[v]$ (i.e., $v \sim uw$ for some word $u \in A^*$). Setting $v := w$, this is trivial for $k = 0$. Now let $k \geq 0$ and $w \in W_{k+1} \setminus W_k$. Then there is a transition $(p, a, w, r) \in \Delta^{(k+1)} \setminus \Delta^{(k)}$ that is added in step $k + 1$ of our construction. Hence there is some transition $(p, a, ubv, q) \in \Delta^{(k)}$ with $u \parallel b$ and $w = \mathrm{lnf}(uv)$. Hence

$$\begin{aligned} ubv &\sim b\,uv && \text{(since } u \parallel b) \\ &\sim b\,w && \text{(since } w = \mathrm{lnf}(uv)) \end{aligned}$$

implying that $[w]$ is a suffix of $[ubv]$ and $ubv \in W_k$. By the induction hypothesis, there is some $x \in W_0$ such that $[ubv]$ is a suffix of $[x]$. Now the transitivity of the suffix relation implies the claim for w which completes the inductive proof.

Let N be the maximal length of a word from W_0. From [34], we obtain that the set of *traces*

$$W'_k = \{[w] \in A^* \mid (p, a, w, r) \in \Delta^{(k)}\}$$

contains at most $|W_0| \cdot N^{\alpha(\mathcal{D})}$ elements since all traces from W'_k are suffixes of traces from W'_0. Since, for any word $w \in A^*$, there is a unique word v in lexicographic normal form with $w \sim v$, we get

$$|W_k| = |W'_k| \leq |W_0| \cdot N^{\alpha(\mathcal{D})}\,.$$

It follows that

$$|\Delta^{(k)}| \leq |Q \times A \times Q| \cdot |\Delta| \cdot N^{\alpha(\mathcal{D})}$$

which is polynomial in $\|\mathfrak{P}\|^{\alpha(\mathcal{D})}$. □

The main properties of the above saturation procedure can be summarized as follows:

Proposition 5.18. *Let $\mathcal{D} = (A, D)$ be a dependence alphabet and $\mathfrak{P}$ a trace-pushdown system. Then, in time polynomial in $\|\mathfrak{P}\|^{\alpha(\mathcal{D})}$, one can construct an equivalent saturated system $\mathfrak{P}^{(\infty)}$, i.e., a saturated trace-pushdown system with $\vdash^*_{\mathfrak{P}} = \vdash^*_{\mathfrak{P}^{(\infty)}}$.* □

5.4. Proof of Theorem 5.1

We finally summarize our proof of the fact that the reachability relation of trace-pushdown systems is efficiently lc-rational.

So let $\mathcal{D} = (A, D)$ be a dependence alphabet and $\mathfrak{P}$ a trace-pushdown system.

1. Using Prop. 5.18, we can construct an equivalent saturated system $\mathfrak{P}^{(\infty)}$ in time polynomial in $\|\mathfrak{P}\|^{\alpha(\mathcal{D})}$. Since the independence number $\alpha(\mathcal{D})$ is bounded by the twin index $\mathrm{TI}(\mathcal{D})$, the construction of $\mathfrak{P}^{(\infty)}$ is possible in time polynomial in $\|\mathfrak{P}\|^{\mathrm{TI}(\mathcal{D})}$

2. Now the claim follows from Theorem 5.12. □

6. Closure Properties of the Reachability Sets of tPDS

From [4, 5], we know that classical pushdown systems efficiently preserve the regularity of sets of configurations under forwards and under backwards reachability. More formally, suppose $\mathfrak{P}$ is a pushdown system (i.e., a trace-pushdown system over the trace monoid A^*), p, q, r are three states, and $\mathfrak{B}$ is an NFA over the alphabet A. Let $C = \{q\} \times L(\mathfrak{B})$ denote the set of configurations with state q whose pushdown content belongs to the language $L(\mathfrak{B})$. Then $\{u \mid (p, u) \vdash_{\mathfrak{P}}^* C\}$ is the set of pushdown contents that allow to reach some pushdown content from $L(\mathfrak{B})$ (while changing state from p to q). Caucal's result says that this language is efficiently regular, i.e., that we can compute in polynomial time an NFA $\mathfrak{A}$ accepting this set. Symmetrically, $\{w \mid C \vdash_{\mathfrak{P}}^* (r, w)\}$ is the set of pushdown contents that can be reached from some pushdown content from $L(\mathfrak{B})$ (while changing state from q to r). Caucal's result implies that also this language is efficiently regular, i.e., that we can compute in polynomial time an NFA $\mathfrak{C}$ accepting this set.

In this section, we ask to what extent these two results generalize to trace-pushdown systems. Since now, we are concerned with sets of traces as opposed to sets of words, we can consider rational or recognizable such sets. Accordingly, we ask whether the rationality or the recognizability is preserved under forwards and backwards reachability in a trace-pushdown system.

In [22], we consider cooperating multi-pushdown sytems. Any such system is a trace-pushdown system, but not conversely. There, we present a cooperating multi-pushdown system and a rational set C of configurations such that the set of configurations backwards reachable from C is not even decidable, let alone rational, cf. [22, Prop. 17].

In [22], we also show that the backwards reachability of cooperating multi-pushdown systems preserves the recognizability of a set of configurations. Next, we generalize this result to tPDS and we prove, in addition, that tPDS preserve the rationality under forwards reachability.

Theorem 6.1. *Let $\mathcal{D} = (A, D)$ be a dependence alphabet and $\mathfrak{P} = (Q, \Delta)$ a trace-pushdown system. Then the following two statements hold:*

(1) The backwards reachability relation preserves recognizability. More precisely, from $\mathcal{D}$, $\mathfrak{P}$, $p, q \in Q$, and from a closed NFA $\mathfrak{B}$*, one can compute a* closed NFA $\mathfrak{A}$ *such that*

$$[L(\mathfrak{A})] = \{[u] \mid (p, [u]) \vdash_{\mathfrak{P}}^* \{q\} \times [L(\mathfrak{B})]\} ;$$

this computation can be carried out in time polynomial in $\|\mathfrak{P}\|^{O(\mathrm{TI}(\mathcal{D}))} + \|\mathfrak{B}\|$.

(2) *The forwards reachability relation preserves rationality. More precisely, from* $\mathcal{D}$, $\mathfrak{P}$, $q, r \in Q$, *and from an* NFA $\mathfrak{B}$, *one can compute an* NFA $\mathfrak{C}$ *such that*

$$[L(\mathfrak{C})] = \{[w] \mid \{q\} \times [L(\mathfrak{B})] \vdash_{\mathfrak{P}}^{*} (r, [w])\} ;$$

this computation can be carried out in time polynomial in $\|\mathfrak{P}\|^{O(\mathrm{TI}(\mathcal{D}))} + \|\mathfrak{B}\|$.

Proof. We first prove statement (1). Let $\mathcal{L} = \{[u] \mid (p, [u]) \vdash_{\mathfrak{P}}^{*} \{q\} \times [L(\mathfrak{B})]\}$ such that we have to construct a closed NFA $\mathfrak{A}$ with $[L(\mathfrak{A})] = \mathcal{L}$.

Recall that the relation $\mathcal{R} := \mathrm{Reach}_{p,q}(\mathfrak{P})$ is the set of pairs $([u], [v])$ such that $(p, [u]) \vdash_{\mathfrak{P}}^{*} (q, [v])$. Consequently, $\mathcal{L} = {}^{\mathcal{R}}[L(\mathfrak{B})]$.

By Theorem 5.1, the relation $\mathcal{R}$ is lc-rational. Since the NFA $\mathfrak{B}$ is closed, Theorem 4.16(i) implies that the set $\mathcal{L}$ is recognizable, i.e., that there exists a closed NFA $\mathfrak{A}$ with $[L(\mathfrak{A})] = \mathcal{L}$.

To verify the upper time bound, recall that a left-closed transducer $\mathfrak{T}$ for $\mathcal{R}$ can be computed in time polynomial in $\|\mathfrak{P}\|^{O(\mathrm{TI}(\mathcal{D}))}$ by Theorem 5.1. From this left-closed transducer $\mathfrak{T}$ and the closed NFA $\mathfrak{B}$, we can construct the closed NFA $\mathfrak{A}$ in time polynomial in $\|\mathcal{D}\| + \|\mathfrak{T}\| + \|\mathfrak{B}\| \leq \|\mathfrak{T}\| + \|\mathfrak{B}\|$. Consequently, $\mathfrak{A}$ can be constructed in time polynomial in $\|\mathfrak{P}\|^{O(\mathrm{TI}(\mathcal{D}))} + \|\mathfrak{B}\|$.

Statement (2) can be shown similarly, using Theorem 4.16(ii) instead of Theorem 4.16(i). ☐

It remains to consider the question whether a tPDS preserves the recognizability under forwards reachability. The answer to this question is negative: every rational trace language $\mathcal{L}$ is the set of configurations reachable from some single configuration. Since $\mathcal{L}$ can be non-recognizable, this implies that the recognizability is not preserved under forwards reachability:

Proposition 6.2. *Let* $\mathcal{D} = (A, D)$ *be a dependence alphabet and* $\mathcal{L} \subseteq \mathbb{M}(\mathcal{D})$ *be rational. Then there are a dependence alphabet* $\mathcal{D}' = (A', D')$ *with* $A \subseteq A'$ *and* $D = D' \cap (A \times A)$, *a trace-pushdown system* $\mathfrak{P} = (Q, \Delta)$ *over* $\mathcal{D}'$, $c \in \mathrm{Conf}_{\mathfrak{P}}$, *and* $q \in Q$ *such that* $\mathcal{L} = \{[w] \mid c \vdash^{*} (q, [w])\}$.

Proof. Since $\mathcal{L}$ is rational, there is an NFA $\mathfrak{A} = (S, A, I, T, F)$ with $[L(\mathfrak{A})] = \mathcal{L}$. Without loss of generality, we can assume that $S \cap A = \emptyset$ holds. We want to simulate the runs of $\mathfrak{A}$ in inverse direction with the help of a (stateless) trace-pushdown system $\mathfrak{P}$. To this end, our system initially writes a final state $f \in F$ on top of the pushdown of $\mathfrak{P}$. Then we simulate an edge $(p, a, q) \in T$ by replacing the state q at the top of the pushdown by the word pa. We are done once $\mathfrak{A}$ is in an initial state $\iota \in I$. In this case, we simply pop the state ι from the pushdown.

For this construction we first have to extend our dependence alphabet $\mathcal{D}$ to $\mathcal{D}' = (A', D')$ which is also able to handle states of $\mathfrak{A}$:

- $A' := A \cup S \cup \{\#\}$ where $\# \notin A \cup S$ and

- D' is the symmetric closure of $D \cup ((S \cup \{\#\}) \times A')$, i.e., the new letters from $S \cup \{\#\}$ are dependent from all letters in A'.

Then we construct a trace-pushdown system $\mathfrak{P} = (Q, \Delta')$ with

- $Q := \{\top\}$ and
- Δ' consists of the following transitions:
  - $(\top, \#, f, \top)$ for each $f \in F$,
  - $(\top, p, qa, \top)$ for each transition $(q, a, p) \in \Delta$, and
  - $(\top, \iota, \varepsilon, \top)$ for each $\iota \in I$.

As mentioned before, $\mathfrak{P}$ simulates computations of $\mathfrak{A}$ backwards. Underneath the top symbol $q \in S$ of our pushdown, we find a trace $[w] \in \mathbb{M}(\mathcal{D})$ such that $\mathfrak{A}$ has a v-labeled run (with $v \sim w$) from q to F. Formally, for all $p, q \in Q$ and $w \in A^*$, we have $(\top, [p]) \vdash_{\mathfrak{P}}^* (\top, [qw])$ if, and only if, there is a word $v \in A^*$ with $q \xrightarrow{v}_{\mathfrak{A}} p$ and $v \sim w$. This implies

$$(\top, [\#]) \vdash_{\mathfrak{P}}^* (\top, [w]) \iff [w] \in [L(\mathfrak{A})]$$

for each word $w \in A^*$. Hence, $\mathcal{L} = [L(\mathfrak{A})] = \{[w] \mid (\top, [\#]) \vdash_{\mathfrak{P}}^* (\top, [w])\}$. □

7. Conclusion

In this paper, we investigate to what extend Caucal's preservation results for pushdown systems can be extended to trace-pushdown systems. Under a natural restriction of the transition relation, this turns out to be possible, but requires nontrivial extensions of his ideas: rational relations have to be restricted to lc-rational relations (whose theory is developed in this paper), the two distinct generalizations of regular languages to trace languages play different roles, and the saturation of a system does not allow to replace every path by a path with at most two phases, but only by a path with a bounded number of phases where the bound depends on the dependence alphabet.

We show that the reachability relation is decidable in polynomial time. This result complements Zetzsche's work on valence automata over loop-free graph monoids: while he imposes conditions on the monoid, we restrict the transition relation. Recently, the first-order theory of the reachability relation has been considered in [35]: it is undecidable in general, but (using the theory of automatic structures [36, 37, 38]) decidable provided the trace-pushdown system is saturated and puts a connected trace onto the stack along every loop. These results complement those from [20] analogously to the complementation of the results from [18, 19] by those of the current paper: they concern the transition structure and not the algebraic properties of the trace monoid.

It is not known whether the recurrent and the alternating reachability problems for trace-pushdown systems are decidable. Nor do we have a good understanding of the expressive power; Example 3.3 demonstrates that it exceeds the class of context-free languages.

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Appendix

In Example 4.4, we saw that the subword relation is not left-closed, and we claimed that its inverse $\succeq$ (the superword relation) is left-closed. We now prove this claim. Suppose $u = x\,ab\,y$ with $a \parallel b$ and $x\,ba\,y = u' \succeq v'$. If $v' = \varepsilon$, we get $u \succeq \varepsilon \sim v'$. So suppose $v' \neq \varepsilon$. Then there are $n \in \mathbb{N}$, $u_1 \in A^*$, $u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \in A^+$, and $u_{n+1} \in A^*$ such that $v_1 v_2 \cdots v_n = v'$ and $u_1 v_1 u_2 v_2 \cdots u_n v_n u_{n+1} = u' = x\,ba\,y$. This gives two decompositions of the word u' into the blocks u_i and v_i on the one hand, and into the factors x, ba, and y on the other hand. The factor ba from the second factorization can be covered by some factor u_i, by some factor v_i, or it belongs to some consecutive factors u_i and v_i or v_i and u_{i+1} of the former factorization—in any of these cases, we construct a word v with $u \succeq v \sim v'$:

- Suppose $x = u_1 v_1 \cdots u_{i-1} v_{i-1} x'$, $u_i = x' bay'$, and $y = y' v_i u_{i+1} v_{i+1} \cdots u_{n+1}$. Then

  $$\begin{aligned} u &= u_1 v_1 u_2 \cdots u_{i-1} v_{i-1}\, x' aby'\, v_i u_{i+1} v_{i+1} \cdots u_{n+1} \\ &\succeq v_1 v_2 \cdots v_n = v' \,. \end{aligned}$$

  Thus, setting $v := v'$ yields $u \succeq v \sim v'$.

- Suppose $x = u_1 v_1 \cdots u_{i-1} v_{i-1} u_i x'$, $v_i = x' bay'$, and $y = y' u_{i+1} v_{i+1} \cdots u_{n+1}$. Then

  $$\begin{aligned} u &= u_1 v_1 \cdots u_{i-1} v_{i-1} u_i\, x' aby'\, u_{i+1} v_{i+1} \cdots v_n u_{n+1} \\ &\succeq v_1 \cdots v_{i-1}\, x' aby'\, v_{i+1} \cdots v_n \\ &\sim v_1 \cdots v_{i-1}\, v_i\, v_{i+1} \cdots v_n = v' \,. \end{aligned}$$

  Thus, setting $v := v_1 \cdots v_{i-1}\, x' aby'\, v_{i+1} \cdots v_n$ yields $u \succeq v \sim v'$.

- Suppose $x = u_1 v_1 \cdots u_{i-1} v_{i-1} x'$, $u_i = x' b$, $v_i = ay'$, and $y = y' u_{i+1} v_{i+1} \cdots u_{n+1}$. Then

  $$\begin{aligned} u &= u_1 v_1 \cdots u_{i-1} v_{i-1}\, x' b\, ay'\, u_{i+1} v_{i+1} \cdots u_{n+1} \\ &\succeq v_1 \cdots v_{i-1}\, ay'\, v_{i+1} \cdots v_n = v' \,. \end{aligned}$$

  Thus, setting $v := v'$ yields $u \succeq v \sim v'$.

- Finally, suppose $x = u_1 v_1 \cdots u_i x'$, $v_i = x' b$, $u_{i+1} = ay'$, and $y = y' v_{i+1} u_{i+2} v_{i+2} \cdots u_{n+1}$. Then

  $$\begin{aligned} u &= u_1 v_1 \cdots u_i\, x' b\, ay'\, v_{i+1} u_{i+2} v_{i+2} \cdots u_{n+1} \\ &\succeq v_1 \cdots v_{i-1}\, x' b\, v_{i+1} \cdots v_{n+1} = v \,. \end{aligned}$$

  Thus, setting $v := v'$ yields $u \succeq v \sim v'$.

Since $\sim$ is the least equivalence relation identifying $xaby$ with $xbay$ for $a \parallel b$, this proves that the superword-relation $\succeq$ is left-closed.