THE COUNTING POWER OF TRANSFORMERS

Marco Sälzer
RPTU Kaiserslautern-Landau
Kaiserslautern, Germany
marco.saelzer@rptu.de

Chris Köcher
MPI-SWS
Kaiserslautern, Germany
ckoecher@mpi-sws.org

Alexander Kozachinskiy
Centro Nacional de Inteligencia Artificial
Santiago, Chile
alexander.kozachinskyi@cenia.cl

Georg Zetzsche
MPI-SWS
Kaiserslautern, Germany
georg@mpi-sws.org

Anthony Widjaja Lin
MPI-SWS and RPTU Kaiserslautern-Landau
Kaiserslautern, Germany
awlin@mpi-sws.org

ABSTRACT

Counting properties (e.g. determining whether certain tokens occur more than other tokens in a given input text) have played a significant role in the study of expressiveness of transformers. In this paper, we provide a formal framework for investigating the counting power of transformers. We argue that all existing results demonstrate transformers' expressivity only for (semi-)linear counting properties, i.e., which are expressible as a boolean combination of linear inequalities. Our main result is that transformers can express counting properties that are highly nonlinear. More precisely, we prove that transformers can capture all semialgebraic counting properties, i.e., expressible as a boolean combination of arbitrary multivariate polynomials (of any degree). Among others, these generalize the counting properties that can be captured by C-RASP softmax transformers, which capture only linear counting properties. To complement this result, we exhibit a natural subclass of (softmax) transformers that completely characterizes semialgebraic counting properties. Through connections with the Hilbert's tenth problem, this expressivity of transformers also yields a new undecidability result for analyzing an extremely simple transformer model — surprisingly with neither positional encodings (i.e. NoPE-transformers) nor masking. We also experimentally validate trainability of such counting properties.

1 INTRODUCTION

Transformers Vaswani et al. (2017) have emerged in recent years as a powerful model with a plethora of successful applications including (among others) natural language processing, computer vision, and speech recognition. Despite the success of transformers, the question of what transformers can express is still not well-understood and has in recent years featured in a rich body of research works (e.g. Strobl et al. (2024); Hahn (2020); Pérez et al. (2021); Hao et al. (2022)). In particular, formal language theory provides a formal framework in understanding expressivity issues for sequential models like transformers and Recurrent Neural Networks (RNNs).

One recurring theme when studying the expressibility of transformers is the *counting power* of transformers. Intuitively, counting amounts to asserting an arithmetic relationship between the numbers of occurrences of various tokens in a given text. Counting properties are essentially the class of properties for textual data under consideration in the well-known *Vector Space Model (VSM)* Salton et al. (1975); Wong et al. (1985) (also see Chapter 10 in the excellent book Shawe-Taylor & Cristianini (2004)), or the similar *Bag-of-Words (BoW)* model Harris (1954). A simple example of a counting property can be found in a sentiment analysis application[1]: the number of positive words exceeds the number of negative words in a text. In the formal language theory, such a counting property can be formalized as the following language

$$\mathsf{MAJ} := \{w \in \{a,b\}^* : |w|_a > |w|_b\}, \tag{1}$$

which is often referred to as *majority*. Here, $|w|_a$ (resp. $|w|_b$) refers to the number of occurrences of a (resp. b) in the string w. For example, `aab` $\in$ MAJ but `abb` $\notin$ MAJ. Note that "tokens" in NLP are synonymous to "letters" in formal language theory. Another counting property that plays an important role in the theory of expressibility of transformers is *parity* language:

$$\mathsf{PARITY} := \{w \in \{a,b\}^* : a \text{ occurs an even number of times in } w\}. \tag{2}$$

Multiple theoretical and empirical results (e.g. Hahn & Rofin (2024); Chiang & Cholak (2022); Huang et al. (2025); Hahn (2020); Hao et al. (2022); Bhattamishra et al. (2020); Anil et al. (2022); Delétang et al. (2023)) have shown that, while transformers can be efficiently trained for MAJ, this is not the case for PARITY. Several theoretical explanations have been offered, e.g., *sensitivity* by Hahn & Rofin (2024) and length generalization admitted by limit transformers by Huang et al. (2025)).

Thus far, existing results have touched only upon *semilinear* counting properties. For example, defining MAJ requires only a linear inequality (i.e. $|w|_a > |w|_b$). In fact, logical languages, which were devised in Barceló et al. (2024); Yang & Chiang (2024); Huang et al. (2025) epitomizing languages expressible by transformers, permit only linear expressions (e.g. $|w|_a + |w|_b > 2.|w|_c$). However, polynomial expressions (cf. Shawe-Taylor & Cristianini (2004)) are also used to express *co-occurrence* of terms/tokens in a text. For example, using a *higher-degree* monomial such as

$$\#(\text{nvidia}) \times \#(\text{intel}) \times \#(\text{deal}),$$

where $\#(w)$ counts the number of occurrences of a word w in the text, one can emphasize the co-occurrence of "nvidia", "intel" and "deal" in a text. This motivates the following question:

Research Question. *What counting properties are expressible on transformers? Can they express nonlinear counting properties?*

The main contribution of this paper is the following result.

Theorem 1.1. *Transformers can capture all semialgebraic counting properties, i.e., those expressible as a boolean combination of inequalities between multivariate polynomials, where each variable counts the number of occurrences of a specific token in the text.*

This means that transformers can capture expressions involving higher-degree polynomials like:

$$7\#(\text{nvidia}) \times \#(\text{intel}) \times \#(\text{deal}) + 2\#(\text{shares}) - 8\#(\text{war}) > 10,$$

or boolean combinations (i.e. unions/intersections) of similar polynomial expressions. We provide a rigorous proof of this result (using softmax transformers) and experimentally validate this claim. In particular, *our proof requires the use of neither positional encodings nor positional masking.*

The next question we address is an attempt to better understand the expressivity of softmax transformers for capturing counting properties: *which class of softmax transformers capture semialgebraic counting properties?* To this end, we provide a rather surprising characterization involving *average hard attention* Hao et al. (2022); Pérez et al. (2021), which was devised to "approximate" soft attention by attending to all positions with maximum attention score and forwarding their average. In particular, Average Hard Attention Transformers (AHATs) with only *uniform layers* (written AHAT[U]) — that is, where maximum attention score is achieved at every position — immediately form a subclass of SoftMax Attention Transformers (SMAT). In the sequel, we write NoPE-AHAT (resp. NoPE-AHAT[U]) to mean AHAT (resp. AHAT[U]) that do not use Positional Encodings (PEs) (also no positional masking).

Theorem 1.2. NoPE-AHAT *and* NoPE-AHAT[U] *capture precisely semialgebraic counting properties. In particular, as far as expressing counting properties,* NoPE-AHAT *is a subset of* SMAT.

This theorem is a surprising result, especially because it is still a major open problem whether AHAT can be captured by SMAT Yang & Chiang (2024); Hahn (2020); Yang et al. (2024b) for general (not necessarily) counting properties.

[1]https://medium.com/data-science/sentiment-analysis-with-text-mining-13dd2b33de27

A corollary of Theorem 1.1, combined with Matiyasevich's celebrated solution to the notorious Hilbert's 10th Problem Matiyasevich (1993), is a kind of *universality* (i.e. Turing-completeness) of transformers. More precisely, any recursively enumerable counting property $P \subseteq \Sigma^*$ can be represented in terms of a program that, given an input string $w \in \Sigma^*$, feeds each string wv (where $v \in \Gamma^*$, for some $\Gamma \cap \Sigma = \emptyset$) into a transformer T and accepts if T accepts some wv. In this case, we say that P is a *projection* of the language accepted by T. In fact, we show that transformers T with only two attention layers are sufficient and necessary to achieve this result:

Theorem 1.3. *Every recursively enumerable counting property is a projection of a language recognized by a* $\mathsf{NoPE\text{-}AHAT[U]}$*, and thus by an* SMAT*. Here, two attention layers in* $\mathsf{NoPE\text{-}AHAT[U]}$ *and* SMAT *are sufficient.*

Similarly, our results also imply a new undecidability result for analyzing an extremely simple transformer model—surprisingly with neither positional encodings (i.e. NoPE-transformers) nor masking:

Theorem 1.4. *Given a* $\mathsf{NoPE\text{-}AHAT[U]}$ *or* SMAT *(with just two attention layers), it is undecidable whether its language is empty.*

Recent results on this, established by Sälzer et al. (2025), require a substantially more complex architecture, including non-trivial, idealised components, to achieve such an undecidability result, i.e., with powerful positional encoding and average hard attention.

Finally, *how do general transformers compare with other machine learning models as far as capturing counting properties?* To this end, let us discuss two models. First is the class of polynomial separators that can be generated by mapping to a higher dimension and look for a linear separator in this higher dimension. This is a standard technique in classical machine learning literature, where one can apply techniques like Support Vector Machines (SVM) (e.g. using polynomial kernel) in the *Vector Space Model (VSM)* Salton et al. (1975); Wong et al. (1985) (also see Chapter 10 of Shawe-Taylor & Cristianini (2004)). Our result shows that transformers generalize such counting properties, in that not only polynomial counting properties can be captured, but also *boolean combinations* thereof. Second is the model called C-RASP Huang et al. (2025), which is a simple declarative language that formalizes the so-called *RASP-L conjecture* Zhou et al. (2024) capturing "efficiently learnable" properties on transformers. In particular, C-RASP allows only inequalities over linear counting terms. We prove that C-RASP can capture *only* linear counting properties. Our experiments supporting Theorem 1.1 reveals that counting properties like

$$L_k := \{w \in \{a, b\}^+ : |w|_a^k \geq |w|_b\}$$

are also efficiently learnable for $k \geq 2$. This suggests that C-RASP provides only a partial characterization of efficiently learnable properties.

Organization. We recall transformer models and define our framework for studying counting properties in Section 2. We then show how to capture semialgebraic counting properties using transformers in Section 3. In Section 4, we provide a natural subclass of softmax transformers that completely characterizes semialgebraic counting properties. In Section 5, we show applications of our semialgebraic results for a better understanding of expressiveness of transformers, e.g., universality/undecidability and comparison to work on C-RASP transformers. We report our experimental results in Section 6 and conclude in Section 7. Some details have been relegated into the Appendix.

2 Framework: Transformers and Counting Properties

In this section, we define a formal framework for investigating the expressive power of transformers for counting properties.

Formal language theory primer We assume some basic understanding of formal language theory (at the level of a standard undergraduate textbook by Sipser (2013)) and will only fix some notation.

For an alphabet $\Sigma = \{\mathtt{a}_1, \ldots, \mathtt{a}_m\}$. A *language* is a set of strings over Σ. We Σ^* (resp. Σ^+) to mean the set of all strings (resp. all nonempty strings) over Σ. We write $|w|$ to denote the length of w. For each $a \in \Sigma$, we write $|w|_a$ to mean the number of occurrences of a in w. A language $K \subseteq \Sigma^*$ is a *projection* of a language $L \subseteq \Sigma^*$ if there is a subalphabet $\Gamma \subseteq \Sigma$ such that K is obtained from L by deleting all occurrences of letters in Γ from words in L. For a class $\mathcal{C}$ of languages, by $\mathsf{Proj}(\mathcal{C})$, we denote the class of projections of languages in $\mathcal{C}$.

We will touch upon regular languages and recursively enumerable languages (see Sipser (2013) for details). In summary, regular languages are languages that can be described by regular expressions. *Recursively enumerable* languages are those that are recognized by (possibly nonterminating) Turing machines. The class of such languages is denoted RE. In particular, a machine model is said to be *Turing-complete* if it can capture all recursively enumerable languages.

For an alphabet $\Sigma = \{\mathtt{a}_1, \ldots, \mathtt{a}_m\}$, we define the *Parikh image* (a.k.a. *Parikh map* as the function $\Psi\colon \Sigma^* \to \mathbb{N}^m$, where $\Psi(w)[i] := |w|_{\mathtt{a}_i}$ is the number of $\mathtt{a}_i$'s in w. Intuitively, Parikh image of a word w provides the letter counts in w, e.g., over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, we have $\Psi(abaa) = (3, 1)$. The Parikh map can also be extended to a language L; that is, $\Psi(L) = \{\Psi(w) : w \in L\} \subseteq \mathbb{N}^{|\Sigma|}$. For example, if $L = \{\mathtt{a}^n\mathtt{b}^n\mathtt{a}^n : n \geq 0\}$ is a language over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, we have $\Psi(L) = \{(2n, n) : n \geq 0\}$.

2.1 Transformers

We now recall the formal definition of transformers. Loosely speaking, a transformer is a composition of finitely many attention layers, each converting a sequence σ of $\mathbb{R}^d$-vectors into another sequence σ' of $\mathbb{R}^k$-vectors, for some d and k. To turn a transformer T into a language recognizer, we have to embed any letter in the finite alphabet Σ as a $\mathbb{R}^d$-vector, where d is smaller than the dimension of the first attention layer. For example, $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, and the *one-hot* embeddings of $\mathtt{a}$, $\mathtt{b}$, $\mathtt{c}$ are (respectively) $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Finally, to determine acceptance, we simply run T on the embeddings of the input string w into a sequence of vectors (possibly expanded with positional information) and check if the last vector $\boldsymbol{v}$ satisfies that the dot product $\boldsymbol{v}.\boldsymbol{t}$ is greater than 0 (for some pre-defined vector $\boldsymbol{t}$ of weights). In particular, w is accepted by T iff $\boldsymbol{v}.\boldsymbol{t} > 0$.

Example. *Suppose we are given the input string $w = \mathtt{abac}$. Additionally, suppose we use the positional embedding $p\colon n \mapsto 1/n$. Then, checking whether T accepts w amounts to running T on the sequence σ:*

$$(1, 0, 0, 1)(0, 1, 0, 1/2)(1, 0, 0, 1/3)(0, 0, 0, 1/4).$$

After running T on σ, the resulting sequence is of the form $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$. Determining whether T accepts w amounts to checking whether $\boldsymbol{t}.\boldsymbol{v}_1 > 0$. For example, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ could be:

$$(1, 1, 7, 1, 1)(2, 3, 1, 10, 1/2)(1, 8, 0, 8, 1/3)(0, 0, 1, -1, 1/4)$$

which will be accepted, whenever $t = (1, 0, 0, 1, 0)$.

Next we formalize the definition of transformers by defining how each attention layer functions.

ReLU networks. We first define ReLU networks, which are used inside an attention layer. A *ReLU node* v is a function $\mathbb{Q}^m \to \mathbb{Q}$, where $m \in \mathbb{N}$ is referred to as the input dimension, and is defined as $v(x_1, \ldots, x_n) = \max(0, b + \sum_{i=1}^n w_i x_i)$, where $w_i \in \mathbb{Q}$ are the *weights*, and $b \in \mathbb{Q}$ is the *bias*. A *ReLU layer* ℓ is a tuple of ReLU nodes $(v_1, \ldots, v_n)$, all having the same input dimensionality, computing a function $\mathbb{R}^m \to \mathbb{R}^n$, where $n \in \mathbb{N}$ is referred to as the output dimension. Finally, a *ReLU network* $\mathcal{N}$ is a tuple of ReLU layers $(\ell_1, \ldots, \ell_k)$, such that the input dimension of ℓ_{i+1} is equal to the output dimension of ℓ_i. It computes a function $\mathbb{Q}^{m_1} \to \mathbb{Q}^{n_k}$, given by $\mathcal{N}(x_1, \ldots, x_{m_1}) = \ell_k(\cdots \ell_1(x_1, \ldots, x_{m_1}) \cdots)$.

Attention layers Each attention layer involves a *weight normalizer* $\mathtt{wt} : \mathbb{R}^* \to \mathbb{R}^*$, which turns any d-sequence of weights into another such d-sequence. Two widely used weight normalizers are:

1. The softmax normalizer softmax. That is, given a sequence $\sigma = x_1, \ldots, x_n \in \mathbb{R}$, define $\mathrm{softmax}(\sigma) := y_1, \ldots, y_n$, where
$$y_i := \frac{e^{x_i}}{\sum_{j=1}^n e^{x_j}}.$$
2. The averaging hard attention normalizer aha. We define $\mathsf{aha}(\sigma) :=:= y_1, \ldots, y_n$, where
$$y_i := \begin{cases} 1/|P| & \text{if } x_i = \max(\sigma), \\ 0 & \text{or else.} \end{cases}$$

where P consists of positions i in σ such that x_i is maximum in σ. That is, aha behaves like $\mathrm{softmax}$ but maps all non-maximum weights to 0, and all maximum weights to $1/|P|$.

One can also allow a temperature scaling $\tau > 0$ to softmax, i.e., $\mathrm{softmax}_\tau(\sigma) = y_1, \ldots, y_n$ and set $y_i := \frac{e^{x_i/\tau}}{\sum_{j=1}^{n} e^{x_j\tau}}$. This is not so relevant in our paper since our proof works for *any* $\tau > 0$.

An *attention layer* is a function $\lambda\colon (\mathbb{R}^d)^* \to (\mathbb{R}^e)^*$, given by affine maps $Q, K\colon \mathbb{R}^d \to \mathbb{R}^m$, $V\colon \mathbb{R}^d \to \mathbb{R}^k$ (query, key, and value matrices) and a ReLU neural net $\mathcal{N}\colon \mathbb{Q}^{d+k} \to \mathbb{Q}^e$. Given an input sequence $x = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in (\mathbb{Q}^d)^n$, the output sequence $y = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n) \in (\mathbb{Q}^d)^n$ is computed as follows. First, one computes the sequences of key, query, and value vectors: $\boldsymbol{k}_i = K\boldsymbol{x}_i,\ \boldsymbol{q}_i = Q\boldsymbol{x}_i,\ \boldsymbol{v}_i = V\boldsymbol{x}_i$, for each $i = 1, \ldots, n$, then we define

$$\boldsymbol{y}_i = \mathcal{N}(\boldsymbol{x}_i, \boldsymbol{a}_i)$$

with $\boldsymbol{a}_i = \sum_{j=1}^{n} \boldsymbol{w}(j)\boldsymbol{v}_j$, where $\boldsymbol{w} = \mathtt{wt}(\{\langle \boldsymbol{k}_i, \boldsymbol{q}_j\rangle\}_{j=1}^{n})$.

We say that λ is a *softmax* (resp. *aha*) layer if $\mathtt{wt} = \mathrm{softmax}$ (resp. aha). We say that it is a *uniform*-aha layer if it is an aha layer such that $K\boldsymbol{x} = Q\boldsymbol{x} = \boldsymbol{0}$ for all $\boldsymbol{x}$, i.e., $\langle K\boldsymbol{x}, Q\boldsymbol{y}\rangle = 0$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$. Note that a uniform-aha is both an aha layer and a $\mathrm{softmax}$ layer since noting that

$$\mathrm{softmax}(s_1, \ldots, s_n) = \mathrm{softmax}_\tau(s_1, \ldots, s_n) = \mathsf{aha}(s_1, \ldots, s_n) = [1/n, \cdots, 1/n],$$

whenever $s_1 = \cdots = s_n$, which can be guaranteed for uniform aha layers. This holds for *all* $\tau > 0$.

Remark. Some papers (e.g. Yang et al. (2024a); Huang et al. (2025); Yang & Chiang (2024)) apply *strict future masking*, which means that attention is only applied to positions up to the current position i. Our work does not apply masking.

Defining transformers. To define a transformer and its language, we first extend the finite alphabet Σ with an *end marker* $\$ \notin \Sigma$. That is, $\Gamma := \Sigma \cup \{\$\}$. An *transformer* with ℓ layers over a finite alphabet Σ is then a function $T\colon \Sigma^+ \to \{0, 1\}$, given by: (i) the "input embedding" function $\iota\colon \Gamma \to \mathbb{Q}^{d_1}$, (ii) the positional encoding $p\colon \mathbb{N}^2 \to \mathbb{R}^{d_1}$, and (iii) a sequence of layers $\lambda_1\colon (\mathbb{R}^{d_1})^* \to (\mathbb{R}^{d_2})^*, \ldots, \lambda_\ell\colon (\mathbb{R}^{d_\ell})^* \to (\mathbb{R}^{d_{\ell+1}})^*$. Given an input word $w = a_1 \cdots a_n \in \Sigma^n$, the output $T(w)$ is computed as follows. First, we set $\boldsymbol{x}_1 = \iota(a_1) + p(n+1, 1),\ \ldots,\ \boldsymbol{x}_n = \iota(a_n) + p(n+1, n),\ \boldsymbol{x}_{n+1} = \iota(\$) + p(n+1, n)$. Then we compute $(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{n+1}) = \lambda_\ell(\lambda_{\ell-1}(\cdots\lambda_1(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+1})\cdots))$, and we set $T(w) = 1$ if and only if $\boldsymbol{y}_n[1] > 0$, and $T(w) = 0$ otherwise. The language $L(T)$ accepted by T is defined as $\{w \in \Sigma^*\colon T(w) = 1\}$. We say that T has *no positional encoding* (NoPE) if the positional encoding is a constant function.

A *SoftMax Averaging Transformer (*SMAT*)* uses only softmax layers, whereas an *AHA Transformer (*AHAT*)* (resp. AHA[U] Transformers (AHAT[U]) uses only aha (resp. aha$[U]$) layers. We will mainly use SMAT and AHAT[U], and the subclasses NoPE-SMAT and NoPE-AHAT[U].

2.2 Counting Properties

We now define a framework for studying the counting ability of transformers. Intuitively, our framework focuses on "counting properties". As we shall see below, we can build many interesting formal languages with the help of purely counting properties.

Given a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and a string $w = w_1 \cdots w_n$ of length n, the string $\pi(w) := w_{\pi(1)} \cdots w_{\pi(n)}$ is obtained by permuting the letters in w according to π.

Definition 2.1. A *counting property* over the alphabet Σ is a permutation-closed language L, i.e., for each $w \in \Sigma^*$, it is the case that $w \in L$ iff $\pi(w) \in L$ for *each* permutation π over $\{1, \ldots, |w|\}$.

Examples of counting properties are MAJ and PARITY (see (1), (2)). We often identify a counting property L with its set $\Psi(w) \subseteq \mathbb{N}^{|\Sigma|}$ of letter counts (i.e. Parikh image). By PI, we denote the class of counting properties over Σ. Counting properties are also called *permutation-invariant* or "proportion-invariant" languages, e.g., see Pérez et al. (2021); Barceló et al. (2024).

Why counting properties? Certainly, many languages of interests have both a "counting component" and an "order component". Take, for example, the language $L_1 = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n : n \geq 0\}$. Our framework focuses on *purely* counting properties for two reasons. Firstly, it abstracts away non-counting components that cannot be captured by the model. Secondly, many formal languages L of interests can be constructed by taking intersection of a counting property P and an order (and counting-insensitive) language L'. For example, L_1 above can be written as $P \cap L'$, where $P = \{w \in \Sigma^* : |w|_{\mathtt{a}} = |w|_{\mathtt{b}} = |w|_{\mathtt{c}}\}$ and $L' = \mathtt{a}^*\mathtt{b}^*\mathtt{c}^*$. Finally, multiple key languages in the literature on the expressivity of transformers are in fact counting properties (e.g. MAJ and PARITY).

3 Capturing Semialgebraic Counting Properties

A subset $S \subseteq \mathbb{N}^m$ is *semi-algebraic* if it is a Boolean combination of sets of the form $S_p = \{\boldsymbol{x} \in \mathbb{N}^m \mid p(\boldsymbol{x}) > 0\}$ for some polynomial $p \in \mathbb{Z}[X_1, \ldots, X_m]$. A language $L \subseteq \Sigma^*$ is *semi-algebraic* if there is a semi-algebraic set $S \subseteq \mathbb{N}^m$ and $\Sigma = \{\mathtt{a}_1, \ldots, \mathtt{a}_m\}$ such that $L = \{w \in \{\mathtt{a}_1, \ldots, \mathtt{a}_m\}^* \mid \Psi(w) \in S\}$. Let $\mathsf{SemiAlg}$ denote the class of semi-algebraic languages. An example is

$$\mathsf{SQRT} = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid |w|_{\mathtt{a}} < |w|/\sqrt{2}\}, , \tag{3}$$

since $|w|_{\mathtt{a}} < |w|/\sqrt{2}$ if and only if $2|w|_{\mathtt{a}}^2 < |w|^2$. Likewise, extending the coefficients of our polynomials to rational numbers does not increase the expressiveness of semialgebraic sets, e.g., $\frac{7}{3}xy + y^2 > 8x - 3$ can be rewritten as $7xy + 3y^2 > 24x - 9$. Note that for every $p \in \mathbb{Z}[X_1, \ldots, X_m]$, the set $\{\boldsymbol{x} \in \mathbb{N}^m \mid p(\boldsymbol{x}) = 0\}$ is semi-algebraic, because $p(\boldsymbol{x}) = 0$ if and only if $-p(\boldsymbol{x})^2 + 1 > 0$. Thus, every solution set to polynomial equations is also semi-algebraic.

We show Theorem 1.1. Since $\mathsf{AHAT}[\mathsf{U}] \subseteq \mathsf{SMAT}$, it suffices to construct a $\mathsf{AHAT}[\mathsf{U}]$. We will even construct a $\mathsf{NoPE\text{-}AHAT}[\mathsf{U}]$. The key ingredient is:

Proposition 3.1. *For every polynomial $p \in \mathbb{Z}[X_1, \ldots, X_m]$, the language $L_{p>0} = \{w \in \{\mathtt{a}_1, \ldots, \mathtt{a}_m\}^* \mid p(\Psi(w)) > 0\}$ belongs to* $\mathsf{NoPE\text{-}AHAT}[\mathsf{U}]$. *Thus, $L_{p>0}$ is in SMAT.*

Let us see why Proposition 3.1 implies $\mathsf{SemiAlg} \subseteq \mathsf{NoPE\text{-}AHAT}[\mathsf{U}]$. First, the complement of each language $L_{p>0}$ can be obtained, because $p(\boldsymbol{x}) > 0$ is violated if and only if $-p(\boldsymbol{x}) + 1 > 0$. Moreover, $\mathsf{NoPE\text{-}AHAT}$ is closed under union and intersection (we prove a stronger fact in Section A.2). We can thus accept all Boolean combinations of languages of the form $L_{p>0}$, and hence $\mathsf{SemiAlg}$.

To show Proposition 3.1, we will use polynomials that are *homogeneous*, meaning all monomials have the same degree. Note that given an arbitrary polynomial $p \in \mathbb{Z}[X_1, \ldots, X_m]$ of degree d, we can consider the polynomial $q \in \mathbb{Z}[X_0, \ldots, X_m]$ with $q = X_0^d p(\frac{X_1}{X_0}, \ldots, \frac{X_m}{X_0})$, which is homogeneous. It has the property that $p(x_1, \ldots, x_m) > 0$ if and only if $q(1, x_1, \ldots, x_m) > 0$. Therefore, from now on, we assume that we have a homogeneous polynomial $q \in \mathbb{Z}[X_0, \ldots, X_m]$ and want to construct an AHAT[U] for the language $K_q = \{w \in \{\mathtt{a}_1, \ldots, \mathtt{a}_m\}^* \mid q(1, \boldsymbol{x}) > 0 \text{ for } \boldsymbol{x} = \Psi(w)\}$.

To simplify notation, we denote the end marker by $\mathtt{a}_0$. Thus, the input will be a string $w \in \{\mathtt{a}_0, \ldots, \mathtt{a}_m\}^+$ that contains $\mathtt{a}_0$ exactly once, at the end. Since $|w|_{\mathtt{a}_0} = 1$ is satisfied automatically, our AHAT[U] only has to check that $q(x_0, \ldots, x_m) > 0$, where $x_i = |w|_{\mathtt{a}_i}$. The input encoding is the map $\{\mathtt{a}_0, \ldots, \mathtt{a}_m\}^* \to \mathbb{Q}^m$ with $\mathtt{a}_i \mapsto \boldsymbol{e}_i$, where $\boldsymbol{e}_i \in \mathbb{Q}^m$ is the i-th unit vector.

Step I: Compute frequencies Our AHAT[U] first uses an attention layer to compute $m + 1$ new components, where i-th component holds $\frac{x_i}{n+1}$, where $n + 1$ is the length of the input (including the end marker). This is easily done by attending to all positions and computing the averages of the first $m + 1$ components. To simplify notation, we will index vectors starting with index 0.

Step II: Multiplication gadgets Second, we have a sequence of gadgets (each consisting of two layers). Each gadget introduces a new component, and does not change the existing components. Between gadget executions, the following additional invariants are upheld: (i) Overall, a gadget does not change existing components: it introduces one new component. (ii) The components $\{0, \ldots, m\}$ are called the *initial* components. (iii) All other components are *uniform*, i.e. they are the same across all positions. (iv) The uniform components carry values in $[0, 1]$. Thus, we will call components $0, \ldots, m$ the *initial* components; and we call components $> m$ the *uniform* components.

Our gadgets do the following. Suppose we have already produced ℓ additional components. For each initial component $i \in [0, m]$ and uniform component $j \in [m + 1, m + 1 + \ell]$, gadget $\mathsf{omult}(\ell, i, j)$, which introduces a new component, will carry the value $\frac{x_i \cdot y_j}{n+1}$, where y_j is the value in component j of all vectors. Recall that we use x_i to denote the number of $\mathtt{a}_i$ occurrences in the input for $i \in [0, m]$.

We implement the gadget $\mathsf{omult}(\ell, i, j)$ using some ReLU layers and an attention layer. Suppose that before, we have the vector $\boldsymbol{u}_p \in \mathbb{Q}^{m+1+\ell}$ in position p. First, using ReLU layers, we introduce a new component that in position p has the value $\boldsymbol{u}_p[i] \cdot \boldsymbol{u}_p[j]$. This can be achieved since $\boldsymbol{u}_p[i]$ is in $\{0, 1\}$ and $\boldsymbol{u}_p[j] \in [0, 1]$: Notice that $\boldsymbol{u}_p[i] \cdot \boldsymbol{u}_p[j] = \mathrm{ReLU}(\boldsymbol{u}_p[j] - (1 - \boldsymbol{u}_p[i]))$. Indeed, if $\boldsymbol{u}_p[i] = 1$, then this evaluates to $\boldsymbol{u}_p[j]$; if $\boldsymbol{u}_p[i] = 0$, then we get $\mathrm{ReLU}(\boldsymbol{u}_p[j] - 1) = 0$. We then use uniform attention to compute the average of this new $\boldsymbol{u}_p[i] \cdot \boldsymbol{u}_p[j]$-component across all vectors. Since there are $n + 1$ vectors, exactly x_i of them have $\boldsymbol{u}_p[i] = 1$, and also $\boldsymbol{u}_p[j] = y_j$, we get the desired $\frac{x_i \cdot y_j}{n+1}$.

Step III: Computing the polynomial We now use our gadgets to compute the value of the polynomial. For each monomial of q, say $X_{i_1} \cdots X_{i_d}$, we use $d-1$ gadgets to compute $x_{i_1} \cdots x_{i_d}/(n+1)^d$: The frequency computation in the beginning yields $x_{i_1}/(n + 1)$, and then we use gadgets to compute $x_{i_1}x_{i_2}/(n+1)^2$, $x_{i_1}x_{i_2}x_{i_3}/(n+1)^3$, etc. until $x_{i_1} \cdots x_{i_d}/(n+1)^d$. Finally, we use a ReLU layer to multiply each monomial with a rational coefficient, and compute the sum of all the monomials. Thus, we have computed $q(x_0, \ldots, x_m)/(n + 1)^d$. We accept if and only if $q(x_0, \ldots, x_m)/(n + 1)^d > 0$. Note that this is the case if and only if $q(x_0, \ldots, x_m) > 0$.

This completes Proposition 3.1 and thus $\mathsf{SemiAlg} \subseteq \mathsf{NoPE\text{-}AHAT[U]}$.

4 CHARACTERIZING SEMI-ALGEBRAIC COUNTING PROPERTIES

We have shown that $\mathsf{NoPE\text{-}AHAT[U]} \subseteq \mathsf{SMAT}$ can capture semi-algebraic counting properties. We now prove that the subclass $\mathsf{NoPE\text{-}AHAT[U]}$ precisely characterizes $\mathsf{SemiAlg}$.

Proposition 4.1. $\mathsf{NoPE\text{-}AHAT} \subseteq \mathsf{SemiAlg}$.

Proof. Suppose that $\Sigma = \{\mathtt{a}_1, \ldots, \mathtt{a}_m\}$ is our alphabet, $\mathtt{a}_0$ the end marker, and $x_i \in \mathbb{N}$ the number of occurrences of $\mathtt{a}_i$ in the input. We say that a position p is an $\mathtt{a}_i$*-position* if the input holds $\mathtt{a}_i$ at position p. Notice that an AHAT without positional encoding cannot distinguish vectors that come from the same input letter. This means, in any layer, any two $\mathtt{a}_i$-positions will hold the same vector. Thus, the vector sequence on layer ℓ is described by rational vectors $\boldsymbol{u}_{\ell,0}, \ldots, \boldsymbol{u}_{\ell,m}$, where $\boldsymbol{u}_{\ell,i}$ is the vector at all the $\mathtt{a}_i$-positions on layer ℓ. Moreover, for each i, the set of positions maximizing an attention score also either contains all $\mathtt{a}_i$-positions, or none of them. Therefore, if the AHAT has a attention layers, there are at most $((2^{m+1})^{m+1})^a = 2^{(m+1)^2 a}$ possible ways to choose the positions of maximal score: On each attention layer, and for each $i \in [0, m]$, we select a subset of the $m + 1$ letters. For each ReLU node and each i, there are two ways its expression $\mathrm{ReLU}(v)$ can be evaluated: as 0 or as v. Thus, if there are r ReLU nodes, then there are 2^r ways to evaluate all those nodes.

For each of these $2^{r+(m+1)^2 a}$ choices, we construct a conjunction of polynomial inequalities that verify that (i) this choice actually maximized scores, (ii) the resulting vector at the right-most position in the last layer satisfies the accepting condition. This is easy to do by building, for each layer ℓ and each i, expressions in $x_1, \ldots, x_m$ for the vectors $\boldsymbol{u}_{\ell,i}$, assuming our choice above. These expressions have the form $p(x_1, \ldots, x_m)/q(x_1, \ldots, x_m)$ (averaging can introduce denominators). Here, once we have expressions for $\boldsymbol{u}_{\ell,i}$, we can use them to build expressions for $\boldsymbol{u}_{\ell+1,i}$ by following the definition of AHAT. Checking (i) and (ii) is then also easy, because inequalities involving quotients $p(x_1, \ldots, x_m)/q(x_1, \ldots, x_m)$ can be turned into polynomial inequalities by multiplying with common denominators. Finally, we take a disjunction over all $2^{r+(m+1)a}$ conjunctions. $\square$

Inexpressibility of PARITY. Our characterization of $\mathsf{NoPE\text{-}AHAT}$ (i.e. Proposition 4.1) implies an interesting inexpressibility result regarding PARITY (see (2)):

Corollary 4.2. PARITY *does not belong to* $\mathsf{NoPE\text{-}AHAT}$.

PARITY is known to be accepted by AHAT Barceló et al. (2024) and by SMAT Chiang & Cholak (2022) (with PE). Inexpressibility of PARITY in a length-generalizable subclass of SMAT and AHAT (with struct future masking and positional encodings) is known (Huang et al. (2025)). Similarly, PARITY is not expressible by SMAT with strict future masking Hahn (2020). Corollary 4.2 complements these results and is an easy corollary of Proposition 4.1 (see Section A.3).

5 Applications

In this section, we apply our results connecting transformers and semialgebraic counting properties.

5.1 Universality and undecidability of transformers

Let us discuss why universality/undecidability (i.e. Theorems 1.3 and 1.4) follow from Theorem 1.2. First, by the well-known theorem "MRDP" theorem (Matiyasevich, 1993) due to Matiyasevich, Robinson, Davis, and Putnam, every language in $\mathsf{RE} \cap \mathsf{PI}$ is a projection of a language of the form $L_p = \{w \in \{\mathtt{a}_1, \dots, \mathtt{a}_m\}^* \mid p(\Psi(w)) = 0\}$, where $p \in \mathbb{Z}[X_1, \dots, X_m]$ is a polynomial. Since L_p belongs to $\mathsf{NoPE\text{-}AHAT}[\mathsf{U}]$, we thus obtain Theorem 1.3. Furthermore, since our translation from polynomials to $\mathsf{NoPE\text{-}AHAT}[\mathsf{U}]$ (and thus SMAT) is effective, this also implies Theorem 1.4: By the MRDP theorem (which is also effective), it is undecidable whether a given polynomial $p \in \mathbb{Z}[X_1, \dots, X_m]$ has a solution. Using our translations, we can turn such a p into a $\mathsf{NoPE\text{-}AHAT}$ (or SMAT) that is non-empty if and only if p has a solution.

Using only two layers In fact, in Theorems 1.3 and 1.4, we even claim that two layers suffice for universality and undecidability. Let us sketch this here. First, our construction above yields a $\mathsf{NoPE\text{-}AHAT}[\mathsf{U}]$ of at most ℓ layers, provided that the polynomials in the semialgebraic set all have degree $\leq \ell$ (see Section A). In particular, we show that for each ℓ, $\mathsf{NoPE\text{-}AHAT}[\ell, \mathsf{U}]$ is closed under union and intersection (see Section A.2). Furthermore, we rely on the well-known fact that the set of solutions of a polynomial equation $p = 0$ can always be written as the projection of the set of solutions of a *system of quadratic equations*. Since by our stronger version of Theorem 1.2, intersections of solution sets of quadratic equations only require a $\mathsf{NoPE\text{-}AHAT}[\mathsf{U}]$ with ≤ 2 layers, this yields the stronger versions of Theorems 1.3 and 1.4. See Section B for details (where we also show that with just one layer, Theorems 1.3 and 1.4 do not hold).

5.2 Comparison with C-RASP and LTL with Counting

C-RASP (Huang et al., 2025; Yang & Chiang, 2024) is a simple programming language that can be converted into softmax transformers. In particular, it is a subset of the so-called *LTL with Counting* (Yang & Chiang, 2024; Barceló et al., 2024). For example, $\{w \in \{a, b\}^* : |w|_a = |w|_b\}$ can be written as the following formula in LTL with Counting: $\overrightarrow{\#a} = \overrightarrow{\#b}$. In particular, only linear expressions can be constructed in such formulas. We show in the appendix that LTL with Counting (and therefore C-RASP) only capture (semi)linear counting properties, i.e., boolean combinations of linear inequalities (and modulo arithmetics).

Proposition 5.1. *LTL with Counting can define only (semi)linear counting properties.*

In particular, languages like $L_k := \{w \in \{a, b\} : |w|_a^k \geq |w|_b\}$ are in SMAT, for $k \geq 2$, but are not definable in LTL with Counting.

6 Experiments

In this section, we experimentally validate our main result (cf. Theorem 1.1) that transformers can capture solutions of polynomial equations of higher degree. In particular, our results suggest that softmax transformers should be able to learn languages encoding solutions of polynomial equations.

We test our hypothesis on extensions of MAJ with polynomial inequalities. That is, we define the language L_k is defined by $L_k = \{w \in \{a, b\}^+ \mid |w|_b \leq (|w|_a)^k\}$, representing the set of solutions for the simple equation $y \geq x^k$.

Do softmax transformer classifiers perform well on language L_k? Additionally, can we observe tendencies of length-generalization?

In other words, the task of the transformer is a binary classification such that $T(w)$ accepts if $w \in L_k$ and it does not if $w \notin L_k$. For details on the precise experimental setup see Appendix D.

Figure 1 presents the outcome of our experiments. The table on the left-hand side demonstrates the best observed performance on the validation dataset (first column), a balanced test dataset derived

k	Val. Perf.	Test Perf.	Gen. Perf.
1	0.015	0.016/0.99	0.301/0.95
2	0.024	0.033/0.99	0.324/0.94
3	0.023	0.021/0.99	0.299/0.96
4	0.019	0.020/0.99	0.099/0.97
5	0.020	0.024/0.99	0.107/0.96

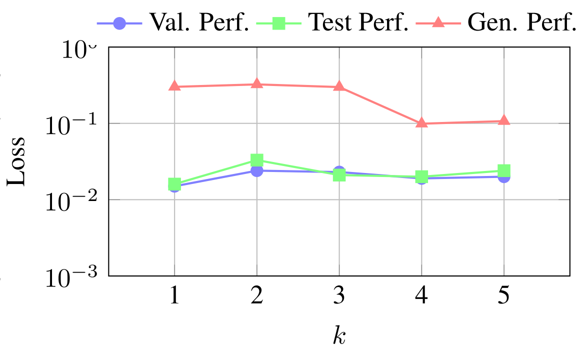

Figure 1: Performance of softmax transformer classifiers for L_k ($k = 1$ to 5). **Val. Perf.**: BCEWithLogitsLoss on validation data. **Test Perf.**: BCEWithLogitsLoss/Accuracy on test data. **Gen. Perf.**: BCEWithLogitsLoss/Accuracy on generalization test set. The y-axis uses a logarithmic scale to accommodate the different orders of magnitude in the results. This implies that longer bars reflect smaller values and vice-versa.

from the same distribution as the training and validation data (second column). This specifically implies that this dataset also only includes words of length up to 500. The final column represents another balanced test dataset encompassing words from length 501 to 1000, used to unveil some length generalization performance. The plot on the right visualizes the same results.

Generally, we observe very high performance with an accuracy of ≥ 0.99 on the in-distribution test dataset. Additionally, while the performance on the test dataset with longer words decreases, it remains relatively high, with an accuracy of ≥ 0.94 in all instances[2]. Especially, it is to be assumed that with a more extensive experimental setup, this gap in performance will decrease. Therefore, we infer that our trained softmax encoders perform well and that length generalization is supported by our experiments, indicating that the model can capture the semantics of L_k. In conclusion, the experiments conducted here highlight that our theoretical results, such as Proposition 3.1, can imply observable practical performance of transformers.

7 Concluding Remarks

Related Work. Lots of work have been done in recent years on the expressiveness of transformers for general (not necessarily counting) properties (cf. see the recent survey by Strobl et al. (2024)). Despite these, counting properties have played a central role in understanding the expressivity of transformers, e.g., the languages PARITY and MAJ, which have frequently featured in transformers expressivity research. Various theoretical transformer models have been used in the literature employing different assumptions on the attention mechanisms (hardmax attention vs. softmax attention), positional encodings, etc. For example, a large proportion of results use hardmax attention, which is not used by practical transformers (which instead use softmax attention). In addition, some works (e.g. Pérez et al. (2021); Barceló et al. (2024)) employ extremely complex positional encodings with no restrictions. That said, several recent works have adopted more practical models. In particular, the works of (Yang & Chiang, 2024; Huang et al., 2025; Yang et al., 2024b) employ softmax attention transformers and simple classes of positional encodings (causal masking, local, etc.). Our results also employ a similar model (AHAT[U] and SMAT); in fact, we proved that semialgebraic counting properties can be captured by transformers without any positonal encodings.

Future Work. We mention several open problems. Firstly, can softmax attention transformers with casual masking capture counting properties beyond semialgebraic sets? Secondly, our work has identified a gap in the formalization of the RASP-L conjecture by Huang et al. (2025). That is, transformers can capture and efficiently learn semialgebraic counting properties, which are beyond

[2]We observe slight variations in the performance on the length generalization dataset. However, it is reasonable to assume that these are due to the limits of our experimental setup or hyperparameter exploration and do not imply any differences in expressive power.

the language C-RASP. It is open whether the extension of C-RASP with inequalities over *nonlinear* polynomials can still be captured by softmax transformers.

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A TRANSLATING SEMIALGEBRAIC SETS TO NoPE-AHAT

A.1 FINE-GRAINED ANALYSIS OF POLYNOMIAL DEGREE VS. DEPTH

In this subsection, we show the inclusion $\mathsf{SemiAlg} \subseteq \mathsf{NoPE\text{-}AHAT}[\mathsf{U}]$. In fact, we show a stronger statement (Proposition A.1), which requires some notation. By $\mathsf{SemiAlg}[\leq \ell]$ we denote the restriction of the class $\mathsf{SemiAlg}$ to the semi-algebraic languages $L \subseteq \Sigma^*$ such that the underlying semi-algebraic set $S \subseteq \mathbb{N}^m$ is a Boolean combination of sets S_p where $p \in \mathbb{Z}[X_1, \ldots, X_m]$ are polynomials of degree $\leq \ell$. In particular, we have $\mathsf{SemiAlg}[\leq 1] = \mathsf{QFPA}$. Our construction for $\mathsf{SemiAlg} \subseteq \mathsf{NoPE\text{-}AHAT}[\mathsf{U}]$ actually shows the following:

Proposition A.1. *For each $\ell > 0$ we have $\mathsf{SemiAlg}[\leq \ell] \subseteq \mathsf{NoPE\text{-}AHAT}[\leq \ell, \mathsf{U}]$.*

For showing Proposition A.1, we need some more technical definitions. Let T be an AHAT with input embedding $\iota \colon \Sigma \to \mathbb{Q}^{d_1}$ and layers $\lambda_1 \colon (\mathbb{Q}^{d_1})^* \to (\mathbb{Q}^{d_2})^*, \ldots, \lambda_\ell \colon (\mathbb{Q}^{d_\ell})^* \to (\mathbb{Q}^{d_{\ell+1}})^*$. We define the function $f_T \colon \Sigma^+ \to \mathbb{Q}$ as follows: for a word $w = a_1 a_2 \ldots a_n \in \Sigma^+$, if $\lambda_1 \circ \cdots \circ \lambda_\ell(\iota(a_1), \ldots, \iota(a_n)) = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)$, then $f_T(w) = \boldsymbol{y}_n[1]$. In other words, we have $f_T(w) > 0$ iff $T(w) = 1$.

Proposition A.2. *For every polynomial $p \in \mathbb{Z}[X_1, \ldots, X_m]$ of degree ℓ, the language $L_{p>0} = \{w \in \{\mathsf{a}_1, \ldots, \mathsf{a}_m\}^* \mid p(\Psi(w)) > 0\}$ belongs to $\mathsf{NoPE\text{-}AHAT}[\leq \ell, \mathsf{U}]$.*

To show Proposition 3.1, we will use polynomials that are *homogeneous*, meaning all monomials have the same degree. Note that given an arbitrary polynomial $p \in \mathbb{Z}[X_1, \ldots, X_m]$ of degree ℓ, we can consider the polynomial $q \in \mathbb{Z}[X_0, \ldots, X_m]$ with $q = X_0^d p(\frac{X_1}{X_0}, \ldots, \frac{X_m}{X_0})$, which is homogeneous. It has the property that $p(x_1, \ldots, x_m) > 0$ if and only if $q(1, x_1, \ldots, x_m) > 0$. Therefore, from now on, we assume that we have a homogeneous polynomial $q \in \mathbb{Z}[X_0, \ldots, X_m]$ and want to construct an AHAT for the language $K_q = \{w \in \{\mathsf{a}_1, \ldots, \mathsf{a}_m\}^* \mid q(1, \boldsymbol{x}) > 0 \text{ for } \boldsymbol{x} = \Psi(w)\}$.

To simplify notation, we denote the end marker \$ by a_0. Thus, the input will be a string $w \in \{\mathsf{a}_0, \ldots, \mathsf{a}_m\}^+$ that contains a_0 exactly once, at the end. Since $|w|_{\mathsf{a}_0} = 1$ is satisfied automatically, our AHAT only has to check that $q(x_0, \ldots, x_m) > 0$, where $x_i = |w|_{\mathsf{a}_i}$. The input encoding is the map $\{\mathsf{a}_0, \ldots, \mathsf{a}_m\}^* \to \mathbb{Q}^m$ with $\mathsf{a}_i \mapsto \boldsymbol{e}_i$, where $\boldsymbol{e}_i \in \mathbb{Q}^m$ is the i-th unit vector.

In a first lemma we show that each monomial of q can be computed by a NoPE-AHAT with ℓ uniform attention layers.

Lemma A.3. *For every monomial $r \in \mathbb{Z}[X_0, X_1, \ldots, X_m]$ of degree ℓ, there is a NoPE-AHAT T with ℓ uniform attention layers such that*

$$f_T(w) = \frac{r(\Psi(w))}{|w|^\ell}$$

for each word $w \in \Sigma^*$. In particular, we have $f_T(w\$) > 0$ if and only if $r(\Psi(w)) > 0$.

Proof. We use the word embedding $\iota \colon \Sigma \to \mathbb{Q}^{m+1}$ with $\iota(\mathsf{a}_i) = \boldsymbol{e}_i$ for each $i \in [0, m]$.

Step I: Compute frequencies Our AHAT first uses an attention layer to compute $m+1$ new components, where i-th component holds $\frac{x_i}{n+1}$, where $n+1$ is the length of the input (including the end marker). This is easily done by attending to all positions and computing the averages of the first $m+1$ components. To simplify notation, we will index vectors starting with index 0.

Step II: Multiplication gadgets Second, we have a sequence of gadgets (each consisting of one uniform layer and several ReLU layers). Each gadget introduces a new component, and does not change the existing components. Between gadget executions, the following additional invariants are upheld: (i) Overall, a gadget does not change existing components: it introduces one new component. (ii) The components $\{0,\ldots,m\}$ are called the *initial* components. (iii) All other components are *uniform*, i.e. they are the same across all positions. (iv) The uniform components carry values in $[0,1]$. Thus, we will call components $0,\ldots,m$ the *initial* components; and we call components $> m$ the *uniform* components.

Our gadgets do the following. Suppose we have already produced k additional components. For each initial component $i \in [0,m]$ and uniform component $j \in [m+1, m+1+k]$, gadget $\mathsf{omult}(k,i,j)$, which introduces a new component, will carry the value $\frac{x_i \cdot y_j}{n+1}$, where y_j is the value in component j of all vectors. Recall that we use x_i to denote the number of a_i occurrences in the input for $i \in [0,m]$.

We implement the gadget $\mathsf{omult}(k,i,j)$ using some ReLU layers and an attention layer. Suppose that before, we have the vector $\boldsymbol{u}_p \in \mathbb{Q}^{m+1+k}$ in position p. First, using ReLU layers, we introduce a new component that in position p has the value $\boldsymbol{u}_p[i] \cdot \boldsymbol{u}_p[j]$. This can be achieved since $\boldsymbol{u}_p[i]$ is in $\{0,1\}$ and $\boldsymbol{u}_p[j] \in [0,1]$: Notice that $\boldsymbol{u}_p[i] \cdot \boldsymbol{u}_p[j] = \mathrm{ReLU}(\boldsymbol{u}_p[j] - (1 - \boldsymbol{u}_p[i]))$. Indeed, if $\boldsymbol{u}_p[i] = 1$, then this evaluates to $\boldsymbol{u}_p[j]$; if $\boldsymbol{u}_p[i] = 0$, then we get $\mathrm{ReLU}(\boldsymbol{u}_p[j] - 1) = 0$. We then use uniform attention to compute the average of this new $\boldsymbol{u}_p[i] \cdot \boldsymbol{u}_p[j]$-component across all vectors. Since there are $n+1$ vectors, exactly x_i of them have $\boldsymbol{u}_p[i] = 1$, and also $\boldsymbol{u}_p[j] = y_j$, we get the desired $\frac{x_i \cdot y_j}{n+1}$.

Step III: Computing the monomial We now use our gadgets to compute the value of the monomial. Let $r(X_0,\ldots,X_m) = \alpha \cdot X_{i_1} \cdots X_{i_\ell}$. We use $\ell - 1$ gadgets to compute $x_{i_1} \cdots x_{i_\ell}/(n+1)^\ell$: The frequency computation in the beginning yields $x_{i_1}/(n+1)$, and then we use gadgets to compute $x_{i_1}x_{i_2}/(n+1)^2$, $x_{i_1}x_{i_2}x_{i_3}/(n+1)^3$, etc. until $x_{i_1} \cdots x_{i_\ell}/(n+1)^\ell$. Finally, we use a ReLU layer to multiply $x_{i_1} \cdots x_{i_\ell}/(n+1)^\ell$ with α. Thus, we have computed $r(x_0,\ldots,x_m)/(n+1)^\ell$. $\square$

A.2 COMBINING NoPE-AHAT[U] WITHOUT ADDITIONAL LAYERS

The following lemma states that two NoPE-AHAT with only uniform attention layers can be parallelized resulting in a NoPE-AHAT with the same number of uniform layers. Their outputs can also be combined via a ReLU neural network. In particular, $\mathsf{NoPE\text{-}AHAT}[\leq \ell, \mathsf{U}]$ is closed under union and intersection.

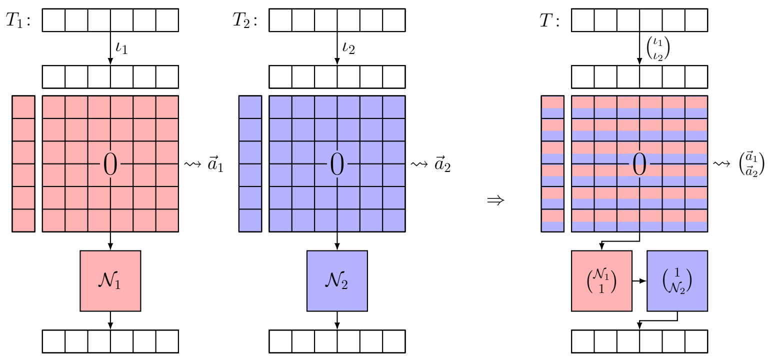
Lemma A.4. *Let T_1, T_2 be two NoPE-AHAT with ℓ uniform attention layers and let $\mathcal{N}$ be a ReLU neural network computing a function $\mathcal{N}\colon \mathbb{Q}^2 \to \mathbb{Q}$. Then there is a NoPE-AHAT $T_\mathcal{N}$ with ℓ uniform attention layers computing $f_{T_\mathcal{N}}(w\$) = \mathcal{N}(f_{T_1}(w\$), f_{T_2}(w\$))$.*

Proof. The idea of $T_\mathcal{N}$ is, that it concatenates the components from T_1 with those of T_2 and keeps the sets of components always disjoint. By uniformity we are able to apply the attention layers of T_1 and T_2 in parallel. In the last attention layer we can simply apply $\mathcal{N}$ to the first components of T_1 and T_2.

By $\iota_i\colon \Sigma \to \mathbb{Q}^{d_{1,i}}$ we denote the word embedding of T_i. From this we construct a new word embedding $\iota\colon \Sigma \to \mathbb{Q}^{d_{1,1}+d_{1,2}}$ with $\iota(\mathsf{a}_j) = (\iota_1(\mathsf{a}_j), \iota_2(\mathsf{a}_j))$ for each $j \in [0,m]$.

Now, let $\lambda_{k,i}\colon \mathbb{Q}^{d_{k,i}} \to \mathbb{Q}^{d_{k+1,i}}$ be the kth layer of T_i for $1 \leq k \leq \ell$. By K_i, Q_i, V_i, and $\mathcal{N}_i$ we denote the parameters of $\lambda_{k,i}$. Since $\lambda_{k,i}$ is uniform, the key and query maps K_i and Q_i are constantly mapping to zero. We now construct a uniform layer $\lambda_k\colon \mathbb{Q}^{d_{k,1}+d_{k,2}} \to \mathbb{Q}^{d_{k+1,1}+d_{k+1,2}}$ composed of $\lambda_{k,1}$ and $\lambda_{k,2}$: the key and query maps K and Q still map to zero. If $V_i(\boldsymbol{x}_i) = A_i\boldsymbol{x} + \boldsymbol{b}_i$ then we define the new value map V by

$$V\begin{pmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{pmatrix} = \begin{pmatrix}A_1 & \mathbf{0}\\ \mathbf{0} & A_2\end{pmatrix}\begin{pmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{pmatrix} + \begin{pmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{pmatrix} = \begin{pmatrix}A_1\boldsymbol{x}_1 + \boldsymbol{b}_1\\ A_2\boldsymbol{x}_2 + \boldsymbol{b}_2\end{pmatrix} = \begin{pmatrix}V_1(\boldsymbol{x}_1)\\ V_2(\boldsymbol{x}_2)\end{pmatrix}.$$



T_1:
ι_1
0
$\rightsquigarrow \vec{a}_1$
$\mathcal{N}_1$
T_2:
ι_2
0
$\rightsquigarrow \vec{a}_2$
$\mathcal{N}_2$
$\Rightarrow$
T:
$\binom{\iota_1}{\iota_2}$
0
$\rightsquigarrow \binom{\vec{a}_1}{\vec{a}_2}$
$\binom{\mathcal{N}_1}{1}$
$\binom{1}{\mathcal{N}_2}$

Figure 2: Visualization of the proof of Lemma A.4.

By this definition we obtain that the attention vectors $\boldsymbol{a}_j$ in λ_k are the concatenation of the attention vectors $\boldsymbol{a}_{j,1}$ and $\boldsymbol{a}_{j,2}$ in $\lambda_{k,1}$ resp. $\lambda_{k,2}$. Similarly, we build the composition of $\mathcal{N}_1$ and $\mathcal{N}_2$ resulting in an FFN computing $\binom{\mathcal{N}_1(\boldsymbol{x}_{j,1},\boldsymbol{a}_{j,1})}{\mathcal{N}_2(\boldsymbol{x}_{j,2},\boldsymbol{a}_{j,2})}$.

Finally, in the last layer, we add the FFN $\mathcal{N}'$ that takes the first components of the output of $\mathcal{N}_i(\boldsymbol{x}_{j,i}, \boldsymbol{a}_{j,i})$ and simulates $\mathcal{N}$ on these two numbers. $\square$

Recall that from a polynomial $p \in \mathbb{Z}[X_1, \ldots, X_m]$ we constructed a homogeneous polynomial $q \in \mathbb{Z}[X_0, X_1, \ldots, X_m]$ such that $p(\boldsymbol{x}) > 0$ if and only if $q(1, \boldsymbol{x}) > 0$ holds for all vectors $\boldsymbol{x} \in \mathbb{Q}^m$. Let $r_1, \ldots, r_k \in \mathbb{Z}[X_0, X_1, \ldots, X_m]$ be the monomials in q. Since q is homogeneous, all monomials have the same degree ℓ. Lemma A.3 yields NoPE-AHATs $T_1, \ldots, T_k$ that are computing the monomials r_i. Each of these AHATs has exactly ℓ uniform attention layers. Finally, we can apply Lemma A.4 to construct a NoPE-AHAT T with ℓ uniform layers computing $f_T(w\$) = \frac{q(\Psi(w\$))}{|w\$|^\ell}$ (since addition is an affine map). Then T accepts w iff $\frac{q(\Psi(w\$))}{|w\$|^\ell} > 0$ iff $q(\Psi(w\$)) > 0$ iff $p(\Psi(w)) > 0$. In other words, T accepts the language $L_{p>0}$.

A.3 Inexpressibility of PARITY

Let us now show Corollary 4.2. By Theorem 1.2, it suffices to show that PARITY is not semi-algebraic. Suppose it is. Then there is a disjunction of conjunctions of polynomial inequalities that characterizes PARITY. The polynomials are over $\mathbb{Z}[X, Y]$, where X is the variable for a's and Y is the variable for b's. By plugging in $Y = 0$, we conclude that the set of even numbers is semi-algebraic. Hence, there is a disjunction $\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} p_{i,j}(X) > 0$ of conjunctions that is satisfied exactly for the even numbers. This implies that for some i, there are infinitely many even numbers k such that $\bigwedge_{j=1}^{m} p_{i,j}(k) > 0$. Therefore, for every $j \in [1, m]$, the leading coefficient of $p_{i,j}$ must be positive. But then, $\bigwedge_{j=1}^{m} p_{i,j}(k) > 0$ must hold for all sufficiently large k, not just the even ones, a contradiction.

B Parametric analysis

In this section, we study how the expressive power of NoPE-AHAT[U] and SMAT depends on the number of attention layers. In particular, we show that Theorems 1.3 and 1.4 hold already in the case of two layers. The main insight of this proof is that the number of layers needed to express a semialgebraic set depends on the degrees of the involved polynomials (see Proposition A.1): Note that our sketch of an NoPE-AHAT for $L_{p>0}$ in Section 3 directly yields a NoPE-AHAT with ℓ layers, where ℓ is the degree of p. For Proposition A.1, one then has to show that Boolean combinations of such sets can be expressed without growing the number of attention layers. See Section A for details.

Capturing RE with two layers From Proposition A.1, we can now deduce the two-attention-layer version of Theorems 1.3 and 1.4. The first ingredient is the following version of the MRDP theorem on Diophantine sets Matiyasevich (1993):

Theorem B.1. *Let $\Sigma = \{\mathtt{a}_1, \ldots, \mathtt{a}_m\}$. A language $L \subseteq \Sigma^*$ belongs to $\mathsf{RE} \cap \mathsf{PI}$ if and only if there is a $k \in \mathbb{N}$ and a polynomial $p \in \mathbb{Z}[X_1, \ldots, X_{m+k}]$ such that $L = \pi_{\mathtt{a}_1,\ldots,\mathtt{a}_m}(K)$, where*

$$K = \{w \in \{\mathtt{a}_1, \ldots, \mathtt{a}_{m+k}\}^* \mid p(\Psi(w)) = 0\}.$$

In other words, every language in $\mathsf{RE} \cap \mathsf{PI}$ is a projection of a language of the form $L_p = \{w \in \{\mathtt{a}_1, \ldots, \mathtt{a}_m\}^* \mid p(\Psi(w)) = 0\}$, where $p \in \mathbb{Z}[X_1, \ldots, X_m]$ is a polynomial. Thus, it suffices to place L_p in $\mathsf{Proj}(\mathsf{NoPE\text{-}AHAT}[\leq 2, \mathsf{U}])$. First observe that in Theorem 1.2, we use one attention layer for each multiplication, so this avenue is closed if we want to stay within two attention layers. Instead, we use that for every polynomial $p \in \mathbb{Z}[X_1, \ldots, X_m]$, there are *quadratic* (i.e. degree ≤ 2) polynomials $q_1, \ldots, q_r \in \mathbb{Z}[X_1, \ldots, X_{m+k}]$ for some $r, k \geq 0$ such that for $\boldsymbol{x} \in \mathbb{N}^m$, we have $p(\boldsymbol{x}) = 0$ if and only if there is some $\boldsymbol{y} \in \mathbb{N}^k$ with $q_1(\boldsymbol{x}, \boldsymbol{y}) = 0, \ldots, q_r(\boldsymbol{x}, \boldsymbol{y}) = 0$: Just introduce a fresh variable for each multiplication in p and use the q_i to assign these fresh variables. Since the language $K := \{w \in \{\mathtt{a}_1, \ldots, \mathtt{a}_{m+k}\}^* \mid q_1(\Psi(w)) = \cdots = q_r(\Psi(w))\}$ belongs to $\mathsf{SemiAlg}[\leq 2]$ (since the q_i have degree ≤ 2) and L_p is a projection of K, this means L_p belongs to $\mathsf{Proj}(\mathsf{SemiAlg}[\leq 2])$. By Proposition A.1, $\mathsf{Proj}(\mathsf{SemiAlg}[\leq 2]) \subseteq \mathsf{Proj}(\mathsf{NoPE\text{-}AHAT}[\leq 2, \mathsf{U}])$.

NoPE AHAT with a single layer The fact that two layers suffice for universality among counting properties raises the question of whether this is even possible with a single attention layer. We show here that this is not the case:

Theorem B.2. $\mathsf{NoPE\text{-}AHAT}[\leq 1] = \mathsf{NoPE\text{-}AHAT}[\leq 1, \mathsf{U}] = \mathsf{QFPA}$.

This means, with a single attention layer, NoPE-AHAT can recognize precisely those counting properties expressible using quantifier-free Presburger formulas. Since satisfiability of Presburger arithmetic is well-known to be decidable Haase (2018); Chistikov (2024), this implies that universality and undecidability of NoPE-AHAT (as we have shown for two attention layers), do not hold with just one attention layer. However, we leave open whether SMAT with one attention layer have a decidable emptiness problem.

Before going into details, let us sketch the proof of Theorem B.2. For the inclusion $\mathsf{NoPE\text{-}AHAT}[\leq 1] \subseteq \mathsf{QFPA}$, we proceed similarly to Proposition 4.1, while observing that the inequalities we have to verify are all linear inequalities: This is because a single attention layer averages only once. Conversely, for the inclusion $\mathsf{QFPA} \subseteq \mathsf{NoPE\text{-}AHAT}[\leq 1, \mathsf{U}]$ follows easily from Proposition A.1.

Proof of Theorem B.2. We begin by proving that $\mathsf{NoPE\text{-}AHAT}[\leq 1] \subseteq \mathsf{QFPA}$. Let T be an AHAT with input embedding $\iota : \Sigma \cup \{\$\} \to \mathbb{Q}^d$, a single AHA layer λ utilising affine maps $Q, K \in \mathbb{Q}^{m \times d}$, $V \in \mathbb{Q}^{k \times d}$, given as matrices, and the ReLU network $\mathcal{N} : \mathbb{Q}^{d+k} \to \mathbb{Q}^e$. Our goal is to construct a quantifier-free PA formula φ_T with variables x_i for $i \in \{1, \ldots, |\Sigma|\}$ such that $\Psi^{-1}(\llbracket \varphi \rrbracket) = \{w \in \Sigma^* \mid T \text{ accepts } w\$\}$. In the following, we assume $\Sigma = \{a_1, \ldots, a_m\}$ and denote $\Sigma \cup \{\$\}$ by Σ'.

First, we observe that for all words $w \in \Sigma^*$, the output of T given $w\$$ is computed by

$$\mathcal{N}\left(\iota(\$), \frac{1}{|w\$|_{a_{i_1}} + \cdots + |w\$|_{a_{i_h}}} \sum_{j=1}^{h} |w\$|_{a_{i_j}} V\iota(a_{i_j})\right),$$

where $\Gamma = \{a_{i_1}, \ldots, a_{i_h}\} \subseteq \Sigma'$ is exactly the subset of symbols a_{i_j} occurring in $w\$$ that maximise $\langle Q\iota(a_{i_j}), K\iota(\$)\rangle$. We construct φ_T such that it mirrors exactly this computational structure. We have $\varphi_T = \bigvee_{\Gamma \subseteq \Sigma'} \varphi_\Gamma$, where $\bigvee$ ranges over those subsets Γ where $\langle Q\iota(a_{i_j}), K\iota(\$)\rangle$ is maximal for precisely the $a_{i_j} \in \Gamma$. The subformula φ_Γ is defined as follows. For now, we assume that $\$ \notin \Gamma$ and introduce some auxiliary formulas. Throughout the following construction steps, we assume that atomic formulas are normalised to the form $c_1 x_1 + \cdots + c_n x_n \leq b$.

Given the ReLU network $\mathcal{N}$, it is straightforward to construct a quantifier-free PA formula $\varphi^{\mathcal{N}}$ such that $\llbracket \varphi^{\mathcal{N}} \rrbracket$ exactly includes those $x_1, \ldots, x_{d+k} \in \mathbb{N}^{d+k}$ satisfying $\mathcal{N}(x_1, \ldots, x_{d+k})_1 > 0$, where $\mathcal{N}(\cdot)_1$ denotes the first output dimension of $\mathcal{N}$. The key idea here is that the computation of a single ReLU node $v(x_1, \ldots, x_{d+k}) = y$, with weights c_i and bias b of $\mathcal{N}$, is described by the quantifier-free PA formula: $(c_1x_1 + \cdots + c_{d+k}x_{d+k} + b \leq 0 \wedge 0 = y) \vee (c_1x_1 + \cdots + c_{d+k}x_{d+k} + b > 0 \wedge c_1x_1 + \cdots + c_{d+k}x_{d+k} + b = y)$. Then, by nesting this construction iteratively from the last layer to the first layer of $\mathcal{N}$, and finally replacing $= y$ with > 0 in the atomic formulas related to the first output dimension of $\mathcal{N}$, we achieve the construction of $\varphi^{\mathcal{N}}$. This nesting and replacement also ensures that $\varphi^{\mathcal{N}}$ includes only the variables $x_1, \ldots, x_{d+k}$.

Let $\Gamma \subseteq \Sigma$ such that $\Gamma = \{a_{i_1}, \ldots, a_{i_h}\}$. Consider the ReLU network $\mathcal{N}$, the value matrix V, and the embedding ι. We construct a quantifier-free PA formula $\varphi_\Gamma^{\mathcal{N},V}$ such that $[\![\varphi_\Gamma^{\mathcal{N},V}]\!]$ exactly includes those $(x_{i_1}, \ldots, x_{i_h}) \in \mathbb{N}^h$ satisfying $\mathcal{N}(\iota(\$), \frac{1}{x_{i_1} + \cdots + x_{i_h}} \sum_{j=1}^h x_{i_j} V\iota(a_{i_j}))_1 > 0$. To do so, we adjust the formula $\varphi^{\mathcal{N}}$ as described in the following. To account for the fixed input $\iota(\$)$, we replace each occurrence of x_1 to x_d in $\varphi^{\mathcal{N}}$ by the respective entry of $\iota(\$)$. Furthermore, to handle the specific form of the input $\frac{1}{x_{i_1} + \cdots + x_{i_h}} \sum_{j=1}^h x_{i_j} V\iota(a_{i_j})$, we first replace each occurrence of x_{d+l} with $l \in \{1, \ldots, k\}$ in the already modified $\varphi^{\mathcal{N}}$ by:

$$(v_{l1}\iota(a_{i_1})_1 + \cdots + v_{ld}\iota(a_{i_1})_d)x_{i_1} + \cdots + (v_{l1}\iota(a_{i_h})_1 + \cdots + v_{ld}\iota(a_{i_h})_d)x_{i_h},$$

where v_{lj} are the respective entries of V. Lastly, we replace each atomic constraint $c_1x_{i_1} + \cdots + c_hx_{i_h} \leq b$ in the adjusted formula with $(c_1 - b)x_{i_1} + \cdots + (c_h - b)x_{i_h} \leq 0$ to adjust for the factor $\frac{1}{x_{i_1} + \cdots + x_{i_h}}$ present in the input.

Now, we define φ_Γ as $\varphi_\Gamma^{\mathcal{N},V,\iota}$. If $\$ \in \Gamma$, we adjust $\varphi_\Gamma^{\mathcal{N},V,\iota}$ slightly. Assuming $\$ = a_{i_j} \in \Gamma$, we replace the variable x_{i_j} with the constant 1 in $\varphi_\Gamma^{\mathcal{N},V,\iota}$. Given this construction, it is clear that $\Psi^{-1}([\![\varphi_T]\!]) = \{w \in \Sigma^+ \mid T \text{ accepts } w\$\}$, as φ_T mimics the computation of T for all possible attention situations Γ.

For the inclusion $\mathsf{QFPA} \subseteq \mathsf{NoPE\text{-}AHAT}[\leq, \mathsf{U}]$, we observe that $\mathsf{QFPA} \subseteq \mathsf{SemiAlg}[\leq 1]$, and thus the inclusion follows from Proposition A.1. $\square$

C Counting properties expressible by other models

C.1 Semilinear counting properties

A counting property $P \subseteq \mathbb{N}^d$ is said to be *semilinear* if can be defined as a boolean combination of inequalities over linear arithmetic expressions (over variables $x_1, \ldots, x_d$ and integer constants) and modulo arithmetic expressions of the form $x_i \equiv a \pmod{b}$, where $a, b \in \mathbb{N}$ are fixed constants. In particular, semilinear counting properties cannot define semialgebraic counting properties involving polynomials of degrees 2 or above.

It is also convenient to use quantifiers when defining semilinear sets. In particular, they do not increase the expressive power since they can be eliminated. This results in the logic called *Presburger arithmetic (PA)*, which refers to the first-order theory of the structure $\langle \mathbb{N}; +, 0, 1, < \rangle$; see (Haase, 2018; Chistikov, 2024).

C.2 Permutation-invariant languages of LTL with counting

$\mathsf{LTL}[\mathsf{Count}]$ has the following syntax:

$$\phi ::= a \mid t \leq t \mid \neg\phi \mid \phi \vee \phi \mid \mathrm{X}\,\phi \mid \phi\,\mathrm{U}\,\phi$$
$$t ::= k \mid k \cdot \overleftarrow{\#\phi} \mid k \cdot \overrightarrow{\#\phi} \mid t + t$$

where $a \in \Sigma$ and $k \in \mathbb{Z}$. Next we define the semantics of $\mathsf{LTL}[\mathsf{Count}]$. For any word $w = a_1a_2 \cdots a_\ell \in \Sigma^*$ with $a_1, a_2, \ldots, a_\ell \in \Sigma$, for each $1 \leq i \leq \ell$, and each formula $\phi \in \mathsf{LTL}[\mathsf{Count}]$ we write $w, i \models \phi$ if the formula ϕ is satisfied in w at position i. Formally, this relation is defined inductively as follows:

- $w, i \models a$ (for $a \in \Sigma$) iff $a_i = a$,
- $w, i \models \neg\phi$ iff $w, i \not\models \phi$,

- $w, i \models \phi \vee \psi$ iff $w, i \models \phi$ or $w, i \models \psi$,
- $w, i \models \mathrm{X}\,\phi$ iff $i < \ell$ and $w, i+1 \models \phi$,
- $w, i \models \phi \,\mathrm{U}\, \psi$ iff there is $i \leq j \leq k$ with $w, j \models \psi$ and for all $i \leq k < j$ we have $w, k \models \phi$,
- $w, i \models t_1 \leq t_2$ iff $[\![t_1]\!](w,i) \leq [\![t_2]\!](w,i)$ where the semantics $[\![t]\!]\colon \Sigma^* \times \mathbb{N} \to \mathbb{Z}$ of a term t is defined as follows: $[\![k]\!](w,i) = k$, $[\![t_1 + t_2]\!](w,i) = [\![t_1]\!](w,i) + [\![t_2]\!](w,i)$, $[\![k \cdot \overleftarrow{\#\phi}]\!] = k \cdot |\{1 \leq j < i \mid w, j \models \phi\}|$, and $[\![k \cdot \overrightarrow{\#\phi}]\!] = k \cdot |\{i \leq j \leq \ell \mid w, j \models \phi\}|$.

Our main result on $\mathsf{LTL}[\mathsf{Count}]$ is the following:

Theorem C.1. *Every permutation-invariant language definable in* $\mathsf{LTL}[\mathsf{Count}]$ *has a semilinear Parikh image.*

Before we can prove Theorem C.1, we need a few more definitions. For an alphabet Σ write Σ_ε for the set $\Sigma \cup \{\varepsilon\}$. A ($d$-dimensional) *Parikh automaton* is a tuple $\mathfrak{A} = (Q, \Sigma, \iota, \Delta, (C_q)_{q \in Q})$ where Q is a finite set of *states*, Σ is the *input alphabet*, $\iota \in Q$ is an *initial* state, $\Delta \subseteq Q \times \Sigma_\varepsilon \times \mathbb{N}^d \times Q$ is a finite *transition* relation, and $C_q \subseteq \mathbb{N}^d$ are semilinear *target* sets. A word $w \in \Sigma^*$ is *accepted* by $\mathfrak{A}$ if there are $a_1, a_2, \ldots, a_\ell \in \Sigma_\varepsilon$, states $q_0, q_1, \ldots, q_\ell \in Q$, and vectors $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \in \mathbb{N}^d$ such that (i) $q_0 = \iota$ and $\boldsymbol{v}_0 = \boldsymbol{0}$, (ii) for each $0 \leq i < \ell$ there is a transition $(q_i, a_i, \boldsymbol{x}_i, q_{i+1}) \in \Delta$ with $\boldsymbol{v}_{i+1} = \boldsymbol{v}_i + \boldsymbol{x}_i$, and (iii) $\boldsymbol{v}_\ell \in C_{q_\ell}$. The accepted language $L(\mathfrak{A})$ of $\mathfrak{A}$ is the set of all words accepted by $\mathfrak{A}$. It is a well-known fact that for each Parikh automaton $\mathfrak{A}$ the accepted language $L(\mathfrak{A})$ has a semilinear Parikh image. Observe that 0-dimensional Parikh automata are essentially NFA and, hence, accept exactly the regular languages.

A *Parikh transducer* is a Parikh automaton with input alphabet $\Sigma_\varepsilon \times \Gamma_\varepsilon$ where Σ and Γ are two alphabets. The accepted language $L(\mathfrak{A}) \subseteq \Sigma^* \times \Gamma^*$ of a Parikh transducer can also be seen as a map: if $(v, w) \in L(\mathfrak{A})$ then we can see v as the input and w as the output of the transducer. Formally, for an input language $L \subseteq \Sigma^*$ a Parikh transducer computes the output $T_{\mathfrak{A}}(L) = \{w \in \Gamma^* \mid \exists v \in L\colon (v, w) \in L(\mathfrak{A})\}$. If L is accepted by a Parikh automaton then $T_{\mathfrak{A}}(L)$ is also accepted by a Parikh automaton. To see this, we can take the synchronized product of the Parikh automaton $\mathfrak{B}$ accepting L and $\mathfrak{A}$ (i.e., $\mathfrak{B}$ reads the same letter from the input as $\mathfrak{A}$ in its first component). Accordingly, cascading of Parikh transducers is also possible, i.e., if $\mathfrak{A}$ and $\mathfrak{B}$ are Parikh transducers over $\Sigma_\varepsilon \times \Gamma_\varepsilon$ and $\Gamma_\varepsilon \times \Pi_\varepsilon$, we can also construct a Parikh transducer $\mathfrak{C}$ over $\Sigma_\varepsilon \times \Pi_\varepsilon$ computing $T_{\mathfrak{C}} = T_{\mathfrak{B}} \circ T_{\mathfrak{A}}$.

With the definition of Parikh automata and Parikh transducers we are now able to prove Theorem C.1.

Proof. Let $\phi \in \mathsf{LTL}[\mathsf{Count}]$ be a formula such that the described language $L(\phi)$ is permutation-invariant. We will prove by induction on the structure of ϕ that the Parikh image of $L(\phi)$ (or actually a *bounded* subset of $L(\phi)$) is semilinear. Here, a language $L \subseteq \Sigma^*$ is *bounded* if there are letters $a_1, a_2, \ldots, a_n \in \Sigma$ with $L \subseteq a_1^* a_2^* \cdots a_n^*$. So, let $a_1, a_2, \ldots, a_n \in \Sigma$ be distinct letters with $\Sigma = \{a_1, a_2, \ldots, a_n\}$. Then $L(\phi) \cap a_1^* a_2^* \cdots a_n^*$ is clearly bounded and has the same Parikh image as $L(\phi)$.

For each subformula ψ of ϕ we construct a Parikh transducer that labels each position satisfying ψ. In the base case, we decorate each letter a by $\boldsymbol{b} \in \{0,1\}^n$ where $\boldsymbol{b}[i] = 1$ iff $a_i = a$. Note that this transducer handles all atomic formulas $a \in \Sigma$ at once. For $\psi = \chi_1 \vee \chi_2$ we add the decoration $b \in \{0,1\}$ to each letter where $b = 1$ iff one of the decorations corresponding to χ_1 and χ_2 is 1. There are similar transducers (which do not introduce counters) for the cases $\psi = \neg\chi$, $\psi = \mathrm{X}\,\chi$, and $\psi = \chi_1 \,\mathrm{U}\, \chi_2$. Note that applying these transducers to a bounded language always yields another bounded language.

Now, consider a counting subformula, i.e. $\psi = \sum_{i=1}^{\ell_1} k_i \cdot \overleftarrow{\#\chi_i} + \sum_{i=\ell_1+1}^{\ell_2} k_i \cdot \overrightarrow{\#\chi_i} \leq k$. Observe that the set of positions satisfying ψ is convex in the set of positions satisfying any χ_i. This is true since we consider only a bounded input language. Hence, we can split the input word into three (possibly empty) intervals: (i) the positions at the beginning of the input that do not satisfy ψ, (ii) the positions where all positions satisfying a χ_i also satisfy ψ, and (iii) the positions at the end of the input that do not satisfy ψ. We describe in the following a Parikh transducer with $3 \cdot \ell_2$ many counters - one for each of these three intervals and each formula χ_i. The transducer guesses the three intervals (note that this is non-deterministic), counts positions satisfying a χ_i accordingly, decorates only the positions in the second interval labeled with a χ_i with 1 (and everything else with a 0), and validates in the end our choice of the intervals (via appropriate semilinear target sets ensuring that the equation in ϕ is not satisfied in the first and third interval and is satisfied in the second interval). Clearly, this all can be done in one (non-deterministic) Parikh transducer.

Finally, we have a cascade of (Parikh) transducers decorating each position in a bounded input word with a Boolean value indicating whether ϕ holds in that position. If we use $a_1^* a_2^* \cdots a_n^*$ as input language for our transducers (note that this language is regular) and intersect the output with all words decorated with a 1 in the first position, we obtain a Parikh automaton accepting exactly the language $L(\phi) \cap a_1^* a_2^* \cdots a_n^*$. Since Parikh automata accept only languages with semilinear Parikh image, we infer that $L(\phi) \cap a_1^* a_2^* \cdots a_n^*$ and, hence, $L(\phi)$ have a semilinear Parikh image. $\square$

D DETAILS OF EXPERIMENTAL SETUP

We train softmax encoders without positional encoding and otherwise in line with the vanilla model, introduced by Vaswani et al. (2017), as binary classifiers using components offered by Pytorch's `nn.Module` based on a balanced dataset of $5 \cdot 10^5$ data points sampled from L_k for $k = 1, \ldots, 5$ of words up to length 500[3]. In all experiments, we conduct a single epoch and choosed the best model conducting early stopping based on the binary-cross entropy loss combined with softmax, offered by Pytorch's `nn.Module` in form of `BCEWithLogitsLoss` , on a validation dataset sampled from the same distribution and of the same size as the training dataset. To partially explore the hyperparameter space, we conduct a grid search over number of layers 1 to 5, number of heads per layer 1, 2 or 4. In all experiments, we fixed the input features to 32, the feedforward dimension to 64, the dropout rate to 0.3, and optimized using the AdamW optimizer with a learning rate of 10^-4 and weight decay of 0.01 as, again, offered by Pytorch's `optim` package.

[3]In contrast to the recent study Huang et al. (2025), examining length generalisation behaviour of transformers on formal languages, we train on comparatively large word lengths. The reason for this is that L_k for increasing k demands certain lengths to ensure that enough positive samples are included in the dataset.