Mean Time To Failure

Lower-Bounding the MTTF for systems with (m,k) constraints and IID iteration failure probabilities

Independent and Identically Distributed

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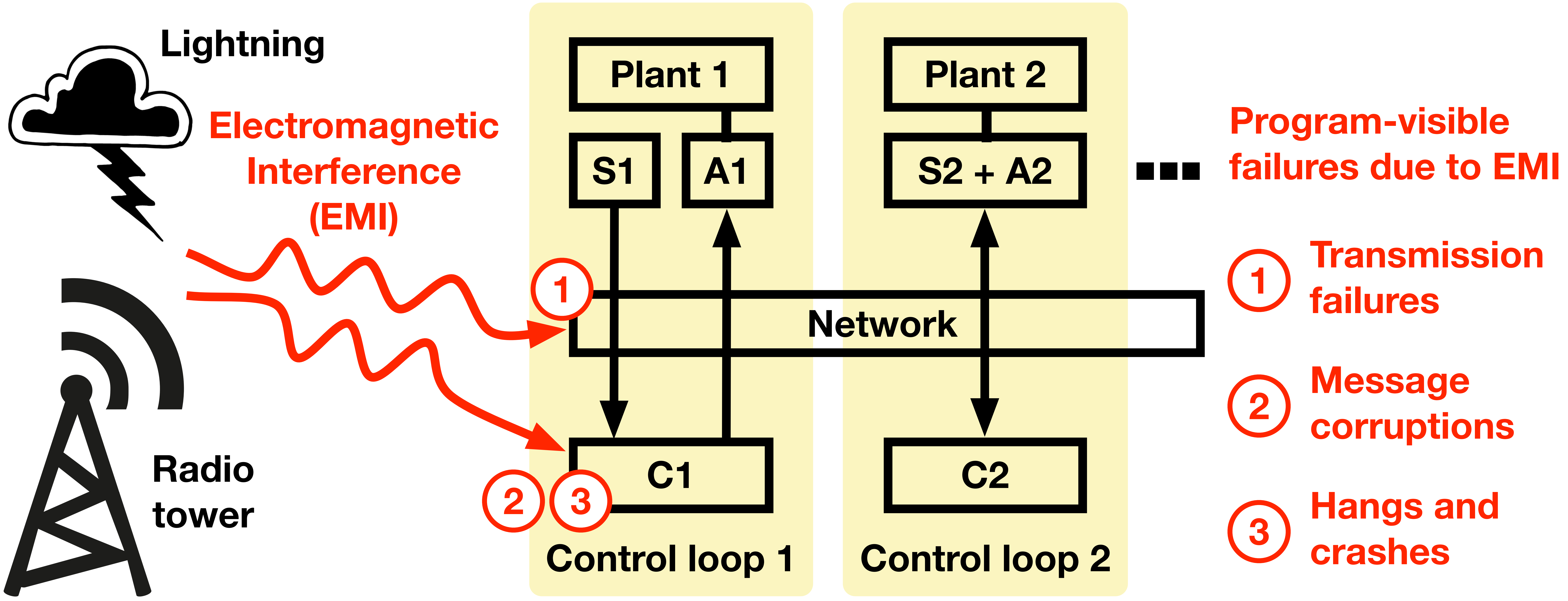
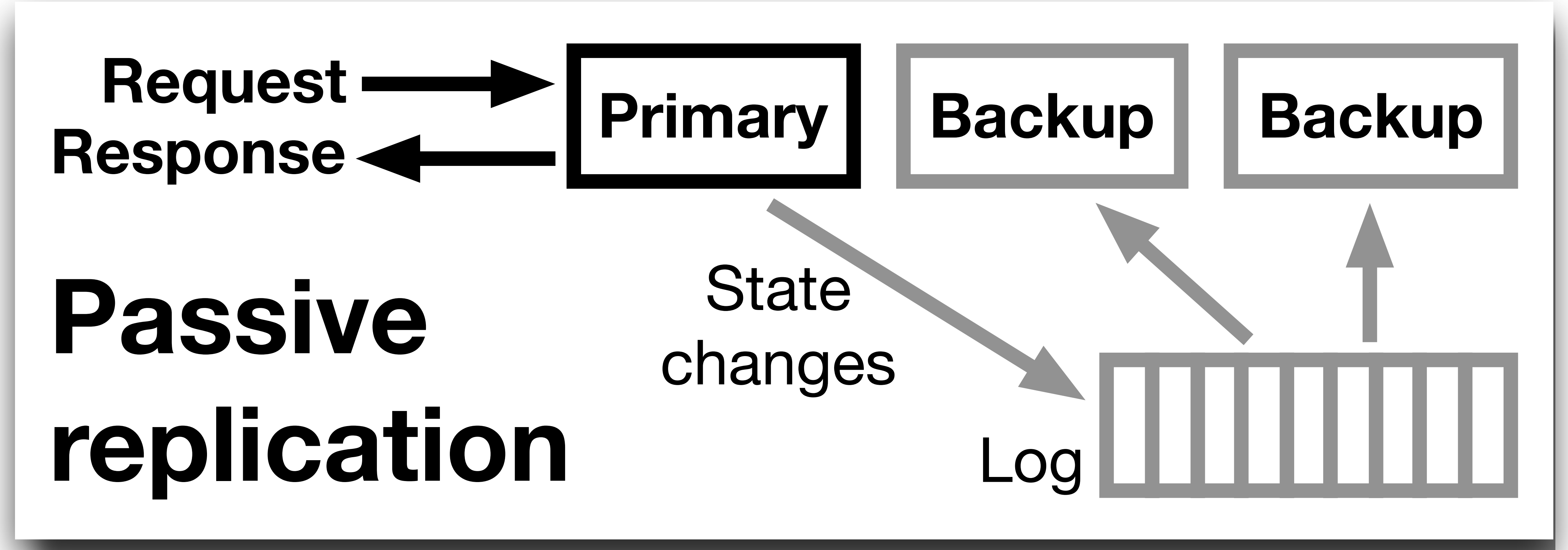
Reliability analysis of Networked Control Systems (NCS)

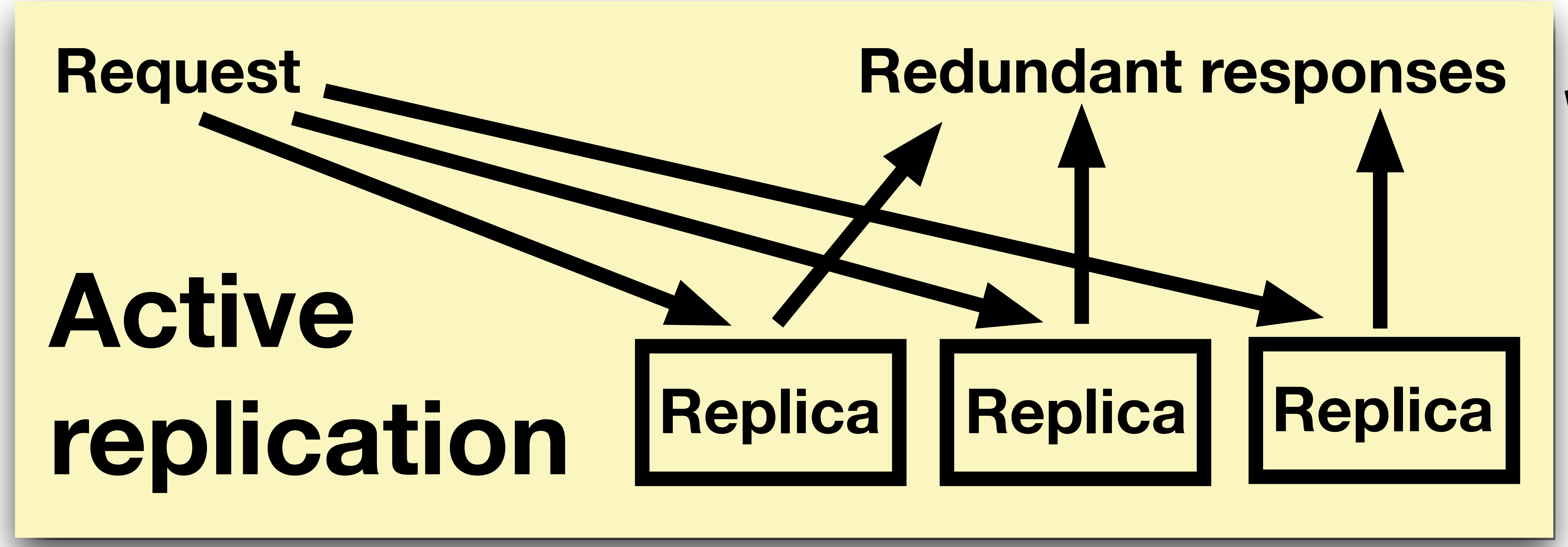
= multiple feedback control loops + distributed hosts
+ shared communication network



Safety-critical NCS must be fail-operational

i.e., continue functioning despite EMI-induced failures





Active replication is often used because...

A. NCSs are time-sensitive

B. they contain high-frequency control loops

Problem

What is a good active replication scheme?

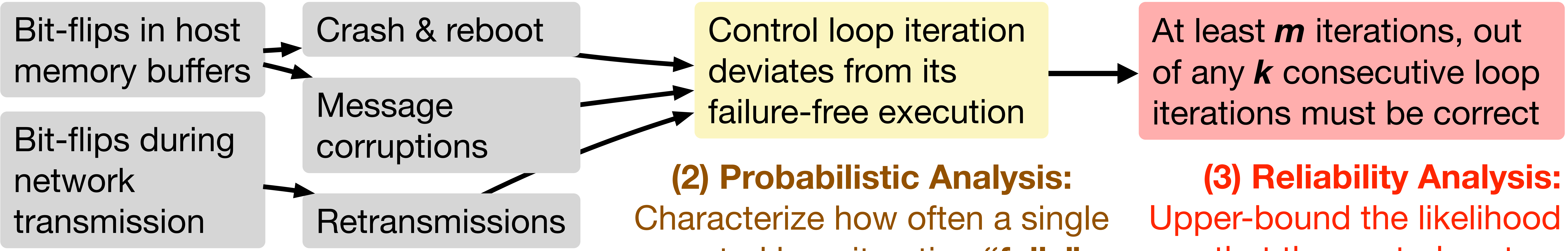
Constraints: size, weight, power, and cost

Objective: meet the dependability requirements

Opportunity: controller inherently robust to occasional disturbances

Quantifying NCS resiliency to EMI-induced transient faults

… to provide engineers with an objective metric for comparing different active replication schemes



(1) Fault Modeling: Transient faults modeled using **Poisson** distribution, empirically-derived **peak EMI** rates

(2) Probabilistic Analysis: Characterize how often a single control loop iteration **"fails"**

(3) Reliability Analysis: Upper-bound the likelihood that the control system **"fails beyond recovery"**

This probability is upper-bounded by **F**, which satisfies the **IID** property w.r.t. each iteration

(under submission)

Quantifying NCS resiliency to EMI-induced transient faults

… to provide engineers with an objective metric for comparing different active replication schemes

At least ***m*** iterations, out of any ***k*** consecutive loop iterations must be correct

violation of the *(m,k)* constraint

Given F, lower-bound the Mean Time To Failure (MTTF)

(3) Reliability Analysis:

Upper-bound the likelihood that the control system **"fails beyond recovery"**

Given F, lower-bound the mean time to failure (MTTF)

Outline

1 Discrete probability density function (dPDF)

g(n) = P(first (m,k) violation in the n^{th} *iteration)*

2 Probability density function (PDF)

f(t) = P(first (m,k) violation at time t)

3 Mean time to failure (MTTF)

MTTF = E [system lifetime] = $\int_0^\infty t f(t)\, dt$

4 Evaluation

Failure = Violation of the (m,k) constraint:

At least ***m*** iterations, out of any ***k*** consecutive loop iterations must be correct

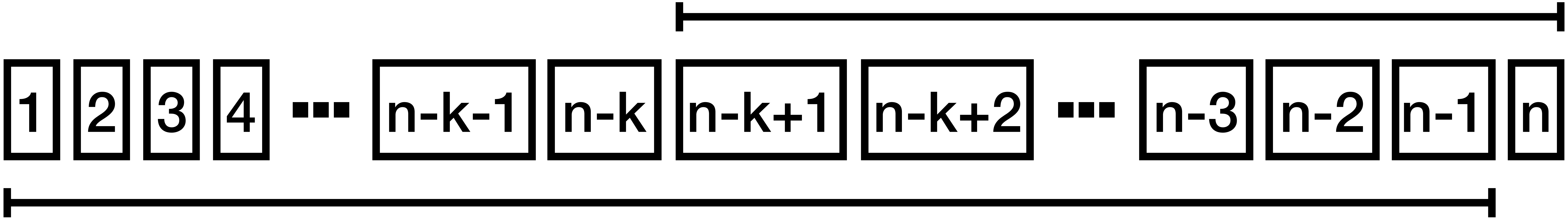
Lower-bounding dPDF (1/3)

g(n) = P(first (m,k) violation in the n^{th} iteration)

At least ***m*** iterations, out of any ***k*** consecutive loop iterations must be correct

$$P(C1) = \binom{k-1}{k-m} F^{(k-m+1)}(1-F)^{m-1}$$

C1: Less than *m* correct iterations out of last *k* iterations



C2: *(m,k)* constraints not violated any time before the *n^{th}* iteration

Computationally challenging $P(C2) = ?$

Requires evaluating all possible combinations of failed and successful iterations among the first *n – 1* iterations.

Lower-bounding dPDF (2/3)

1 | 2 | 3 | 4 | ··· | n-k-1 | n-k | n-k+1 | n-k+2 | ··· | n-3 | n-2 | n-1 | n

C2: *(m,k)* constraints not violated any time before the n^{th} iteration

Computationally challenging

$$P(C2) = ?$$

Requires evaluating all possible combinations of failed and successful iterations among the first n – 1 iterations.

modeled as

a-within-consecutive-b-out-of-c:F system

- consists of c (c ≥ a) linearly ordered components,
- fails iff at least a (a ≤ b) components fail among any b consecutive components.

Sfakianakis et al. (1992)

$$P(C2) >= R_{abc}(k - m + 1,\ k,\ n - 1)$$

Lower-bounding dPDF (3/3)

$$P(C1) = \binom{k-1}{k-m} F^{(k-m+1)}(1-F)^{m-1}$$

$$P(C2) >= R_{abc}(k-m+1,\ k,\ n-1)$$

$$g(n) \geq g_{LB}(n) = \binom{k-1}{k-m} F^{(k-m+1)}(1-F)^{m-1} R_{abc}(k-m+1,\ k,\ n-1)$$

Given F, lower-bound the mean time to failure (MTTF)

Failure = Violation of the (m,k) constraint:

At least ***m*** iterations, out of any ***k*** consecutive loop iterations must be correct

Outline

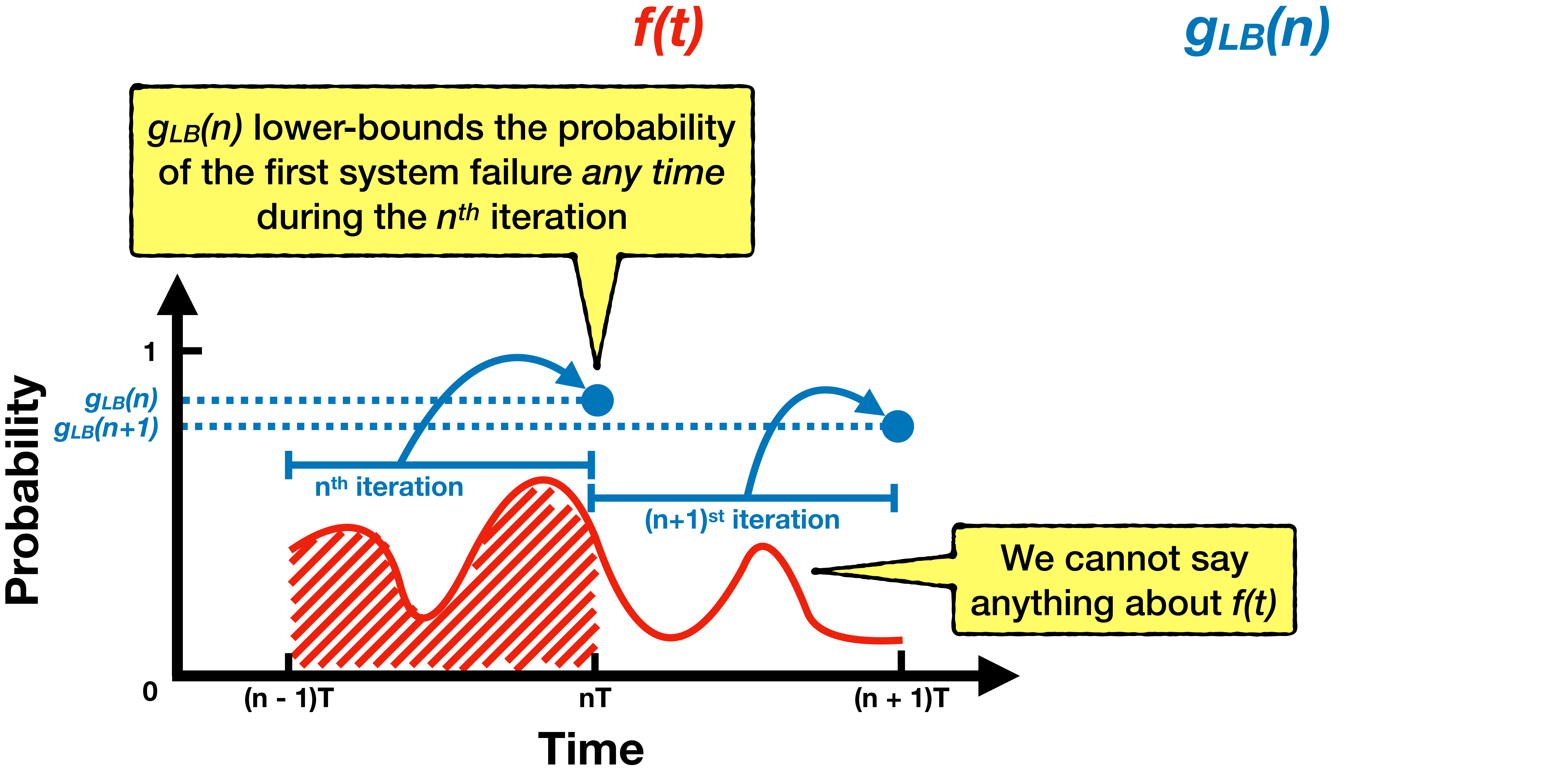
1. Discrete probability density function (dPDF)
   g(n) = P(first (m,k) violation in the n^{th} iteration)

2. Probability density function (PDF)
   f(t) = P(first (m,k) violation at time t)

3. Mean time to failure (MTTF)
   MTTF = E [system lifetime] = $\int_0^\infty t f(t)\, dt$

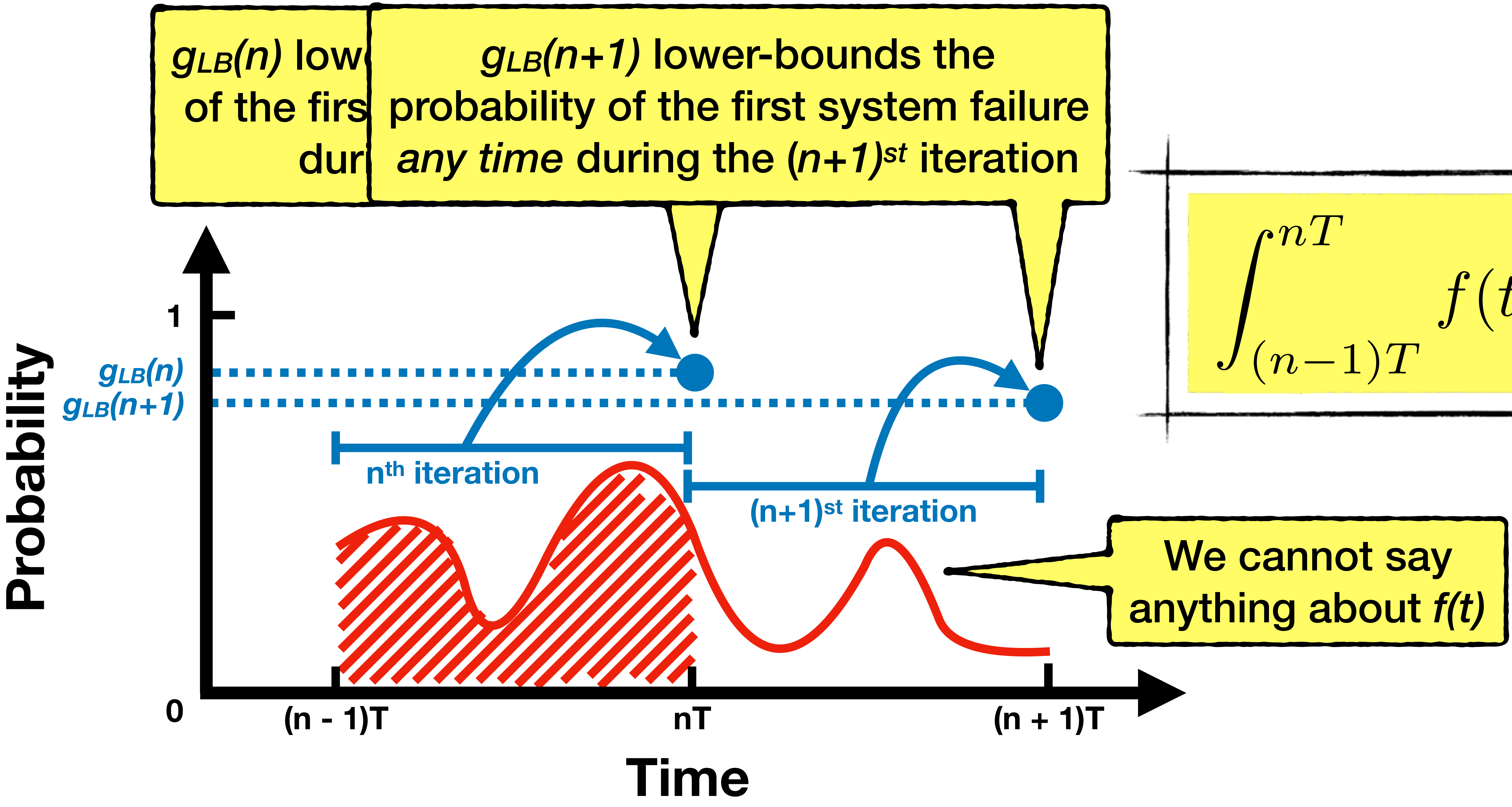
4. Evaluation

Lower-bounding PDF using dPDF lower bound



Lower-bounding PDF using dPDF lower bound

f(t) — *$g_{LB}(n)$*



$$\int_{(n-1)T}^{nT} f(t) \geq g_{LB}(n)$$

Given F, lower-bound the mean time to failure (MTTF)

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4 Evaluation

Challenges

$$g(n) \geq g_{LB}(n) = \binom{k-1}{k-m} F^{(k-m+1)}(1-F)^{m-1} R_{abc}(k-m+1,\ k,\ n-1)$$

Problem

- Complex definition
- Multiple sub-cases
- Recursive expressions

$$\int_{(n-1)T}^{nT} f(t) \geq g_{LB}(n)$$

$$MTTF = \int_0^\infty t f(t)\, dt$$

Challenges

#	Case	Definition	Type	Source
1	$a = 0$	$R_1(a,b,c) = 0$	Exact	–
2	$a = 1$	$R_2(a,b,c) = P_S^c$	Exact	–
3	$a = 2 \wedge c \leq 4b$	$R_3(a,b,c) = \sum_{i=0}^{\lfloor \frac{c+b-1}{b} \rfloor} \binom{c-(i-1)(b-1)}{i} P_F^i P_S^{c-i}$	Exact	[12, §11.4.1] (Eqs. 11.9 and 11.10)
4	$a = 2 \wedge c > 4b$	$R_4(a,b,c) = R_3(a,b,b+t-1)(R_3(a,b,b+3))^u$ where $t = (c-b+1) \bmod 4$ and $u = \lfloor \frac{c-b+1}{4} \rfloor$	LB	[12, §11.4.1] (Eq. 11.16)
5	$a > 2 \wedge c \leq 2b \wedge a = b$	$R_5(a,b,c) = \begin{cases} 1 & 0 \leq c < a \\ 1 - P_F^a - (c-k)P_F^a P_S & a \leq c \leq 2a \end{cases}$	Exact	[12, §9.1.1] (Eqs. 9.2, 9.9, and 9.20)
6	$a > 2 \wedge c \leq 2b \wedge a \neq b \wedge c \leq b$	$R_6(a,b,c) = \sum_{i=c-a+1}^{c} \binom{c}{i} P_S^i P_F^{c-i}$	Exact	[12, §7.1.1] (Eq. 7.2)
7	$a > 2 \wedge c \leq 2b \wedge a \neq b \wedge c > b$	$R_7(a,b,c) = \sum_{i=0}^{a-1} \binom{b-s}{i} P_F^i P_S^{b-s-i} M(a',s,2s)$ where $s = c-b$ and $a' = a-i$, and $M(a',s,2s) = \begin{cases} 1 & a' > s \\ R_2(a',s,2s) & a' = 1 \\ R_3(a',s,2s) & a' = 2 \\ R_5(a',s,2s) & a' > 2 \wedge a' = s \\ R_7(a',s,2s) & a' > 2 \wedge a' \neq s \end{cases}$	Exact	[12, §11.4.1] (Eq. 11.14)
8	$a > 2 \wedge c > 2b$	$R_8(a,b,c) = R_\phi(a,b,b+t-1)(R_\phi(a,b,b+3))^u$ where $t = (c-b+1) \bmod 4$ and $u = \lfloor \frac{c-b+1}{4} \rfloor$, and $R_\phi(a,b,c) = \begin{cases} R_5(a,b,c) & a = b \\ R_6(a,b,c) & a \neq b \wedge a \leq b \\ R_7(a,b,c) & a \neq b \wedge a > b \end{cases}$	LB	[12, §11.4.1] (Eq. 11.16)

TABLE I. **Type** indicates whether the reliability definition for that respective case is an exact value or a lower bound.

$$R_{abc}(k - m + 1,\ k,\ n - 1)$$

Problem

- Complex definition
- Multiple sub-cases
- Recursive expressions

Symbolic integration not an option!

Numeric, but sound, method to lower-bound the MTTF

$$g(n) \geq \underbrace{g_{LB}(n)} = \binom{k-1}{k-m} F^{(k-m+1)} (1-F)^{m-1} R_{abc}(k-m+1,\ k,\ n-1)$$

Computing $g_{LB}(n)$ for a given < *m, k, n, F* > is easy

- *m, k, F* are constants for a given system

But what about *n*?

- n varies from 0 to ∞

$$\int_{(n-1)T}^{nT} f(t) \geq g_{LB}(n)$$

$$MTTF = \int_0^{\infty} t f(t)\, dt$$

Compute $g_{LB}(n)$ at L + 1 distinct points $d_0, d_1, \ldots, d_L$

$g_{LB}(d_0)$

$g_{LB}(d_1)$

$g_{LB}(d_2)$

⋮

$g_{LB}(d_{L-1})$

$g_{LB}(d_L)$

Paper

$$MTTF = \int_0^{\infty} t \times f(t)\,dt$$

{splitting $(0, \infty)$ into a finite number of subintervals $(0, d_0T]$, $(d_0T, d_1T]$, …, $(d_{D-1}T, d_DT]$, and (d_DT, ∞); and dropping the integrals for subintervals $(0, d_0T]$ and (d_DT, ∞) since we are interested in lower-bounding the MTTF}

$$\geq \sum_{i=0}^{D-1} \int_{d_iT}^{d_{i+1}T} t \times f(t)\,dt$$

{since for all $t \in (d_iT, d_{i+1}T]$, $t \geq d_iT$}

$$\geq \sum_{i=0}^{D-1} \left(d_iT \times \int_{d_iT}^{d_{i+1}T} f(t)\,dt \right)$$

{splitting each subinterval $(d_iT, d_{i+1}T]$ into multiple subintervals $(d_iT, (d_i+1)T]$, $((d_i+1)T, (d_i+2)T]$, …, $((d_{i+1}-1)T, (d_{i+1})T]$, each of length T}

$$= \sum_{i=0}^{D-1} \left(d_iT \times \left(\sum_{j=0}^{d_{i+1}-d_i-1} \int_{(d_i+j)T}^{(d_i+j+1)T} f(t)\,dt \right) \right)$$

{since $\int_{(d_i+j)T}^{(d_i+j+1)T} f(t)\,dt \geq g_{LB}(d_i+j+1)$ (from Eq. 2)}

$$\geq \sum_{i=0}^{D-1} \left(d_iT \times \left(\sum_{j=0}^{d_{i+1}-d_i-1} g_{LB}(d_i+j+1) \right) \right)$$

{since $g_{LB}(n)$ is decreasing with increasing n, for each integer j in the interval $[0, d_{i+1}-d_i-1]$, $g_{LB}(d_i+j+1) \geq g_{LB}(d_i + d_{i+1} - d_i - 1 + 1) = g_{LB}(d_{i+1})$}

$$\geq \sum_{i=0}^{D-1} \left(d_iT \times \left(\sum_{j=0}^{d_{i+1}-d_i-1} g_{LB}(d_{i+1}) \right) \right)$$

{simplifying the innermost summation}

$$= \sum_{i=0}^{D-1} \left(d_iT \times g_{LB}(d_{i+1}) \times (d_{i+1} - d_i) \right) \quad \square$$

starting with

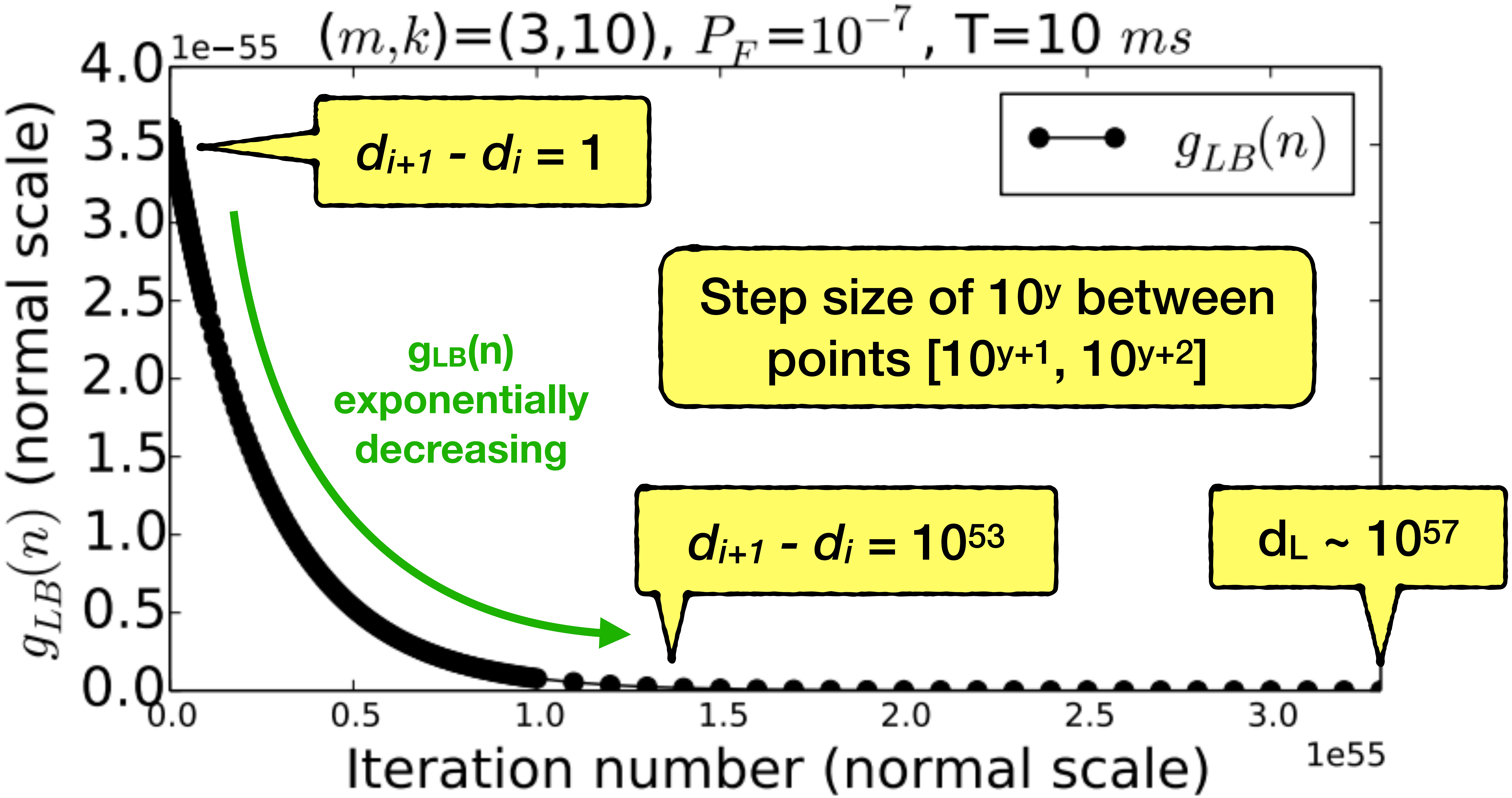
$$MTTF = \int_0^{\infty} t f(t)\,dt$$

using the relation between PDF and dPDF

$$\int_{(n-1)T}^{nT} f(t) \geq g_{LB}(n)$$

$$MTTF \geq \sum_{i=0}^{L-1} \left(d_i \cdot g_{LB}(d_{i+1}) \cdot (d_{i+1} - d_i) \cdot T \right)$$

Choosing points d_0, d_1, ..., d_L



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4 Evaluation

Approximating MTTF using simulation

Biased-coin toss experiment

Tails with probability *F*
- system iteration is incorrect

Heads with probability *1 - F*
- system iteration is correct

Each trial

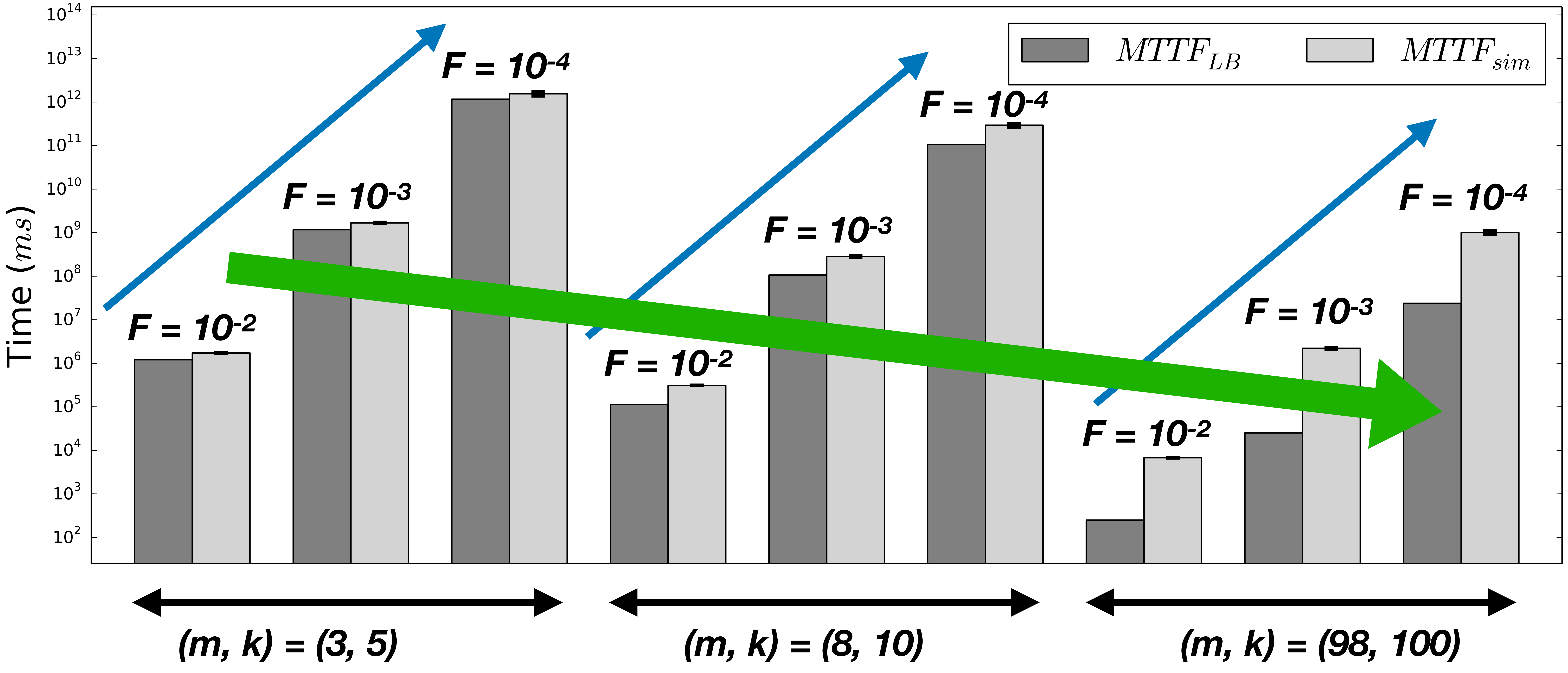
Repeat coin toss until the (m,k) constraint is violated

$MTTF_{sim}$ = Average tosses per trial x control period

Comparing $MTTF_{LB}$ and $MTTF_{sim}$

MTTF increases when *F* decreased from 10^{-2} to 10^{-4}

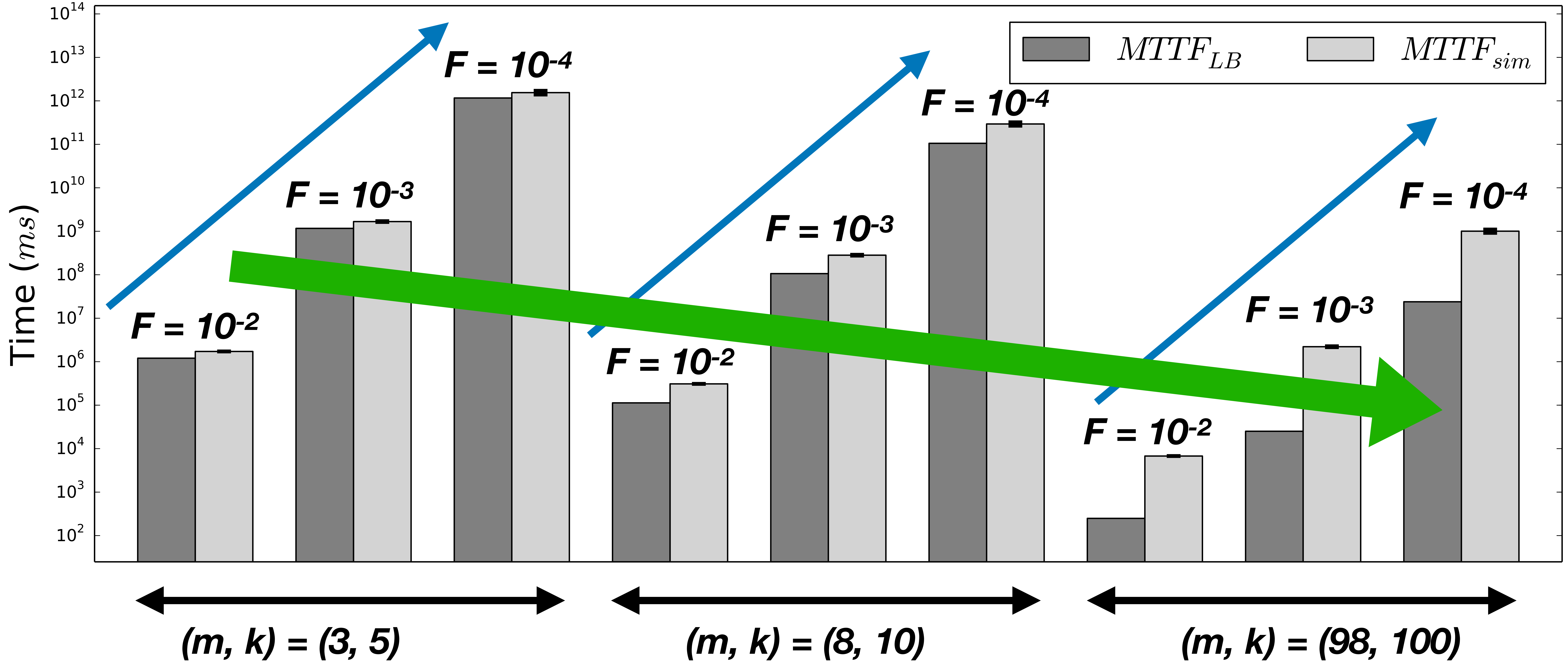
MTTF decreases when *m/k* increased from 3/5 to 98/100



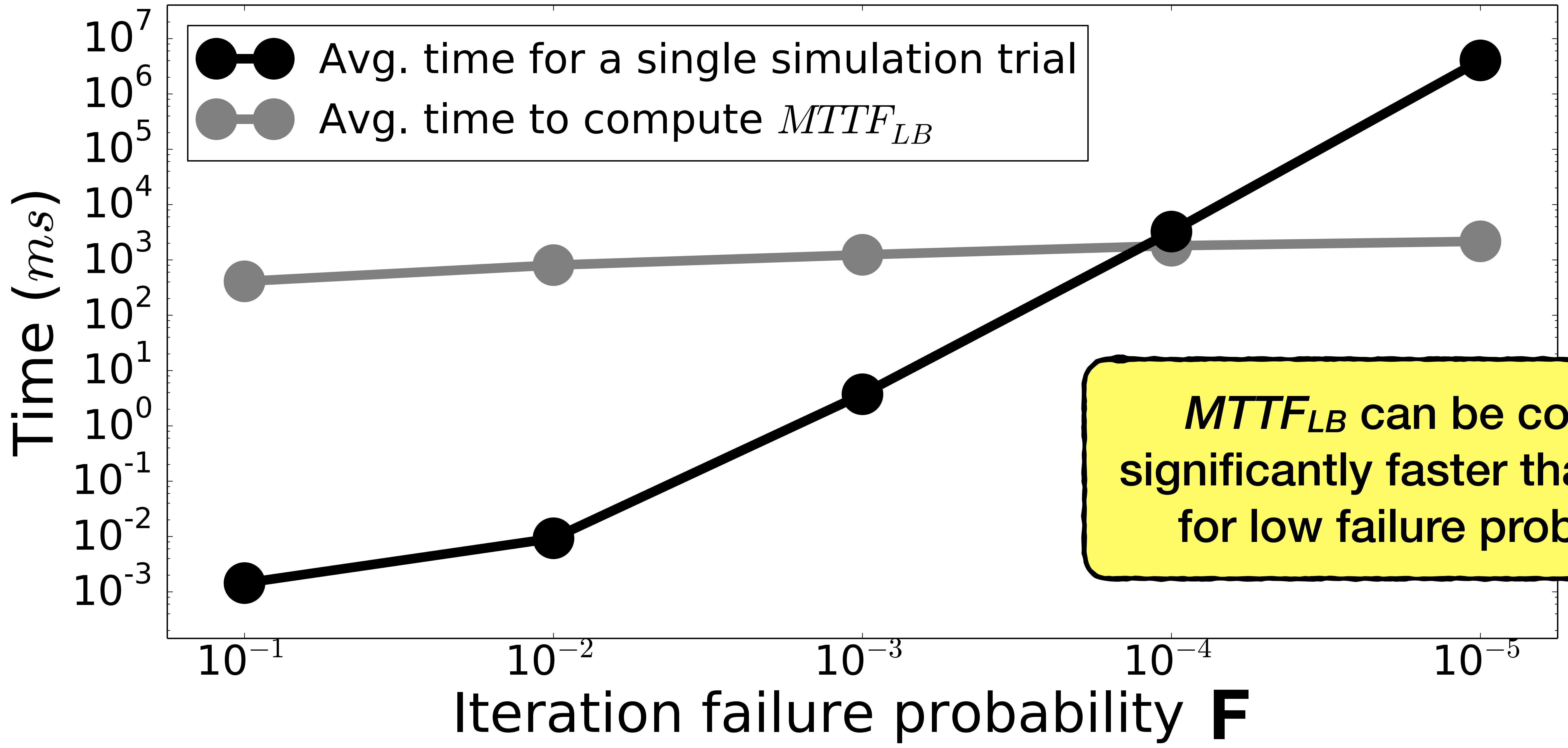
Comparing $MTTF_{LB}$ and $MTTF_{sim}$

$MTTF_{LB}$ is always less than $MTTF_{sim}$

In all cases, $MTTF_{LB}$ and $MTTF_{sim}$ are roughly of the same orders of magnitude

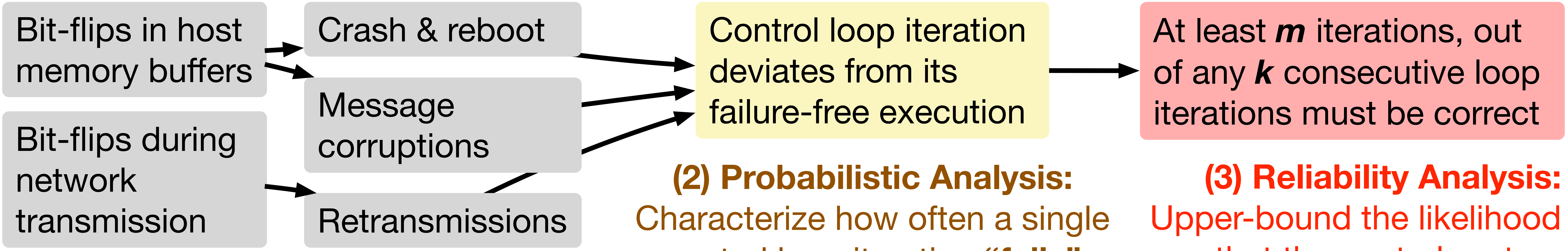


Comparing time to compute $MTTF_{LB}$ and $MTTF_{sim}$



$MTTF_{LB}$ can be computed significantly faster than *$MTTF_{sim}$* for low failure probabilities

Summary



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(2) Probabilistic Analysis: Characterize how often a single control loop iteration **"fails"**

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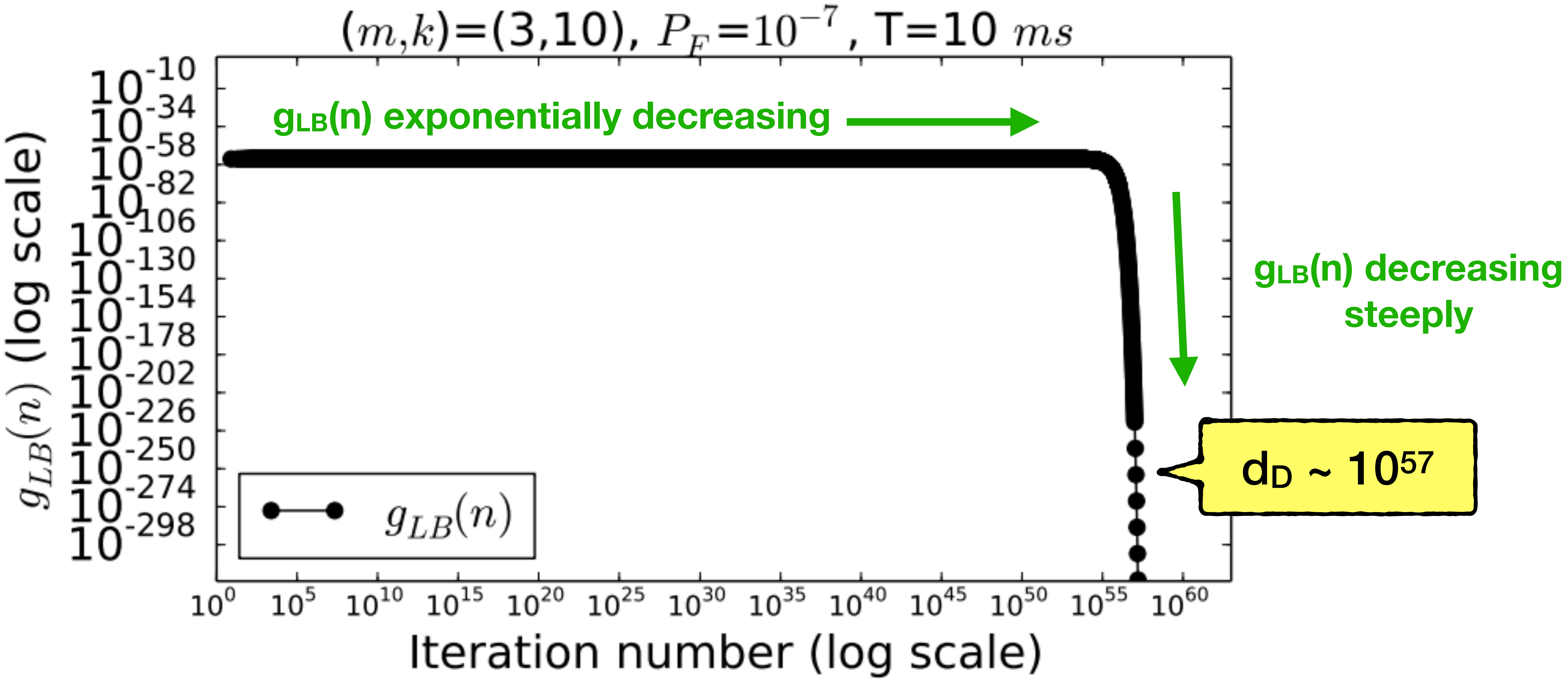
Given a bound *F* on the iteration failure probability, also satisfying the IID property

→

Safe lower bound on the system MTTF for systems with *(m, k)* constraints

Thank you. Questions?

Backup



(m,k)=(3,10), P_F=10^-7, T=10 ms
g_LB(n) exponentially decreasing
g_LB(n) decreasing steeply
d_D ~ 10^57
g_LB(n)
g_LB(n) (log scale)
10^-10
10^-34
10^-58
10^-82
10^-106
10^-130
10^-154
10^-178
10^-202
10^-226
10^-250
10^-274
10^-298
10^0
10^5
10^10
10^15
10^20
10^25
10^30
10^35
10^40
10^45
10^50
10^55
10^60
Iteration number (log scale)